

AN ϵ -APPROXIMATION SCHEME FOR MINIMUM VARIANCE PROBLEMS

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(Received January 23, 1989; Revised May 22, 1989)

Abstract Minimum variance problem may arise when we want to allocate a given amount of resource as fairly as possible to a finite set of activities under certain constraints. More formally, it is described as follows. Given a finite set E , a subset S of R^E , and a function $h_e(x(e))$ from a certain domain to R for each $e \in E$ (which represents the profit resulting from allocating $x(e)$ amount of resource to activity e), the problem seeks to find $x = \{x(e) : e \in E\} \in S$ that minimizes the variance of the vector $\{h_e(x(e)) : e \in E\}$. Here the variance of $\{h_e(x(e)) : e \in E\}$ is defined as the summation over $e \in E$ of the square of difference between the profit $h_e(x(e))$ and the mean value of profits of all activities. Such problem is called minimum variance problem.

This paper first presents a parametric characterization of optimal solutions. Based on this, for a class of problems satisfying certain assumptions, we shall develop an ϵ -approximation scheme which requires to solve the corresponding parametric problem a number of times polynomial in the input length and $1/\epsilon$. We shall then present three special cases for which such ϵ -approximation scheme becomes a fully polynomial time approximation scheme. The first case is that h_e is linear and increasing for each e , and the feasible set S is described by the set of linear equalities and/or inequalities containing the constraint such that the sum of $x(e)$ over all $e \in E$ is a fixed constant, and the second one is that h_e is linear and increasing for each e , and the feasible set S is the set of integral or real bases of submodular systems. The third one is that h_e is a certain nonlinear function and the feasible set S is the set of integral or real bases of a polymatroid. Finally we shall give a pseudopolynomial time algorithm if $x(e)$ is an integer with lower and upper bounds on it, and the sum of $x(e)$ over all $e \in E$ is a fixed constant.

1 Introduction

The problem of allocating a limited resource to relevant activities in a fair manner on the basis of a certain general objective function has recently been considered by Katoh, Ibaraki and Mine [19]. Fujishige, Katoh and Ichimori [7] extended this result to the one with submodular constraints. The problem considered by [7] is written as follows.

$$\begin{aligned} (1) \quad \text{FAIR :} \quad & \text{minimize } g(\max_{e \in E} h_e(x(e)), \min_{e \in E} h_e(x(e))) \\ (2) \quad & \text{subject to } x \in IB \end{aligned}$$

Here E is a finite set of activities, IB is the set of integer bases associated with the underlying submodular constraints, g is a function from R^2 to R such that $g(u, v)$ is

monotone nondecreasing in u and monotone nonincreasing in v , and $h_e, e \in E$, are nondecreasing functions from Z to R , where Z and R denote the sets of integers and reals, respectively. $h_e(x(e))$ denotes the profit resulting from allocating $x(e)$ amount of resource to activity e .

This problem arises whenever the distribution of a given amount of integer resource to a given set of activities is required so that the profit differences among activities are minimized. The fairness of the allocation is measured by the function g in problem FAIR. Zeitlin [28] and Burt and Harris [1] considered the special case of FAIR such as $g(u, v) = u - v$, and gave a finite algorithm. [19] and [7] gave polynomial time algorithms for the general case. [19] studied this problem under simpler constraints, i.e., the total amount of resource to be allocated to all activities is fixed and the lower and upper bounds on the amount of resource to be allocated to each activity is given. The case in which the feasible set in problem FAIR is the set of real bases of a submodular system is investigated by [6].

The fairness of the allocation may be measured alternatively by the variance among the profits resulting from the allocation. Letting x be a feasible allocation, the variance among profits is defined by

$$(3) \quad \text{var}(x) \equiv \sum_{e \in E} [h_e(x(e)) - \frac{1}{|E|} \sum_{e \in E} h_e(x(e))]^2.$$

The aim of this paper is to investigate the problem with such objective function under the constraints which are more general than those discussed above, and is to propose an efficient ϵ -approximation scheme for a class of problems satisfying certain assumptions. Namely, given a finite set E (representing the set of activities), the feasible set S (representing the set of feasible allocations $\{x(e) : e \in E\}$) and a function h_e for each $e \in E$ (representing the profit function for activity e), the problem we consider, which we call the *minimum variance problem*, is described by

$$(4) \quad P : \text{minimize } \{\text{var}(x) \mid x \in S\}.$$

Here h_e is a function from a certain domain (e.g., R or Z depending on the cases). The evaluation of $h_e(x(e))$ for each $x(e)$ in such domain is assumed to be done in constant time. It is also assumed in this paper that the basic arithmetic operations such as addition, subtraction, multiplication, division and comparison of two numbers in R can be done in constant time.

We first give a parametric characterization stating that an optimal solution of the parametric problem $P(\lambda)$ defined below provides an optimal solution of P , if an appropriate number λ is chosen.

$$(5) \quad P(\lambda) : z(\lambda) \equiv \text{minimize} \left\{ \sum_{e \in E} (\{h_e(x(e))\}^2 - \lambda h_e(x(e))) \mid x \in S \right\}.$$

Thus, solving P is reduced to find a $\lambda = \lambda^*$ with which an optimal solution to $P(\lambda^*)$ is also optimal to P . Such characterizations can be obtained in the same manner as was done by Katoh [17] (Sniedovich [25], [26] and Katoh and Ibaraki [18] treat more general cases). [21] and [20] also gave the similar result for variance constrained Markov decision process and for minimum variance Markov decision process, respectively.

This characterization, however, does not tell how to find such λ^* . The straightforward approach for finding λ^* is to compute optimal solutions of $P(\lambda)$ over the entire range of λ .

The number of optimal solutions of $P(\lambda)$ generated over the entire range of λ does not seem to be polynomially bounded in most cases (see Chapter 10 of Ibaraki and Katoh [15] for a special case of S). In addition, $P(\lambda)$ for a given λ does not seem to be solved efficiently unless $\{h_e(x(e))\}^2 - \lambda h_e(x(e))$ is convex. Notice that $\{h_e(x(e))\}^2 - \lambda h_e(x(e))$ is not convex in general even if $h_e(x(e))$ is convex. Therefore it seems to be difficult to develop polynomial time algorithms, and we then focus on approximation schemes in this paper. A solution is said to be an ϵ -approximate solution if its relative error is bounded above by ϵ . An ϵ -approximation scheme is an algorithm containing $\epsilon > 0$ as a parameter such that, for any given ϵ , it can provide an ϵ -approximate solution. If it runs in time polynomial in the input size of each problem instance and in $1/\epsilon$, the scheme is called a *fully polynomial time approximation scheme* (FPAS) [10], [24].

Under a technical assumption as well as an assumption that three related problems defined in (7), (8) and (9) in Section 2, which are in general easier to solve than P , have all finite optimal values and are solvable by certain algorithms, we shall present an ϵ -approximation scheme for P that requires to solve each of these three problems once, and to solve $P(\lambda)$ a number of times polynomial in the input size and $1/\epsilon$. The idea to achieve an ϵ -approximation scheme is to systematically generate a polynomial number of λ 's so that among such λ 's there exists a λ such that the relative error of the variance of the obtained solution of $P(\lambda)$ to the minimum variance (optimal objective value of P) is within ϵ .

We shall then show three special cases for which this ϵ -approximation scheme becomes an FPAS. The first case is that h_e is linear and increasing for each e , and the feasible set S is described by the set of linear equalities and/or inequalities containing the constraint such that the sum of $x(e)$ over all $e \in E$ is a fixed constant. In this case, it is shown that all three problems of (7), (8) and (9) are solvable in polynomial time and have finite optimal values, and that $P(\lambda)$ can be solved in polynomial time. As a result, the proposed ϵ -approximation scheme becomes an FPAS. Minimum variance Markov decision process studied by Kawai [20] can be viewed as a special class of this case. [20] proposed a parametric approach for this problem based on the parametric characterization which is essentially the same as the one shown in this paper. Its running time cannot be, however, bounded above by a certain polynomial for the same reason as indicated above. Therefore our approximation scheme provides a new and efficient approach for solving a minimum variance decision process.

The second case is that h_e is linear and increasing for each e , and S is the set of integral or real bases of submodular systems. We shall also show that in this case the assumptions made in Section 2 are satisfied and $P(\lambda)$ can be solved in polynomial time. As a result, in this case, problem P has an FPAS. The last case is that h_e is a certain nonlinear function and S is the set of integral or real bases of a polymatroid. For this case, we shall also show that problem P has an FPAS. Finally, we shall present a *pseudopolynomial* algorithm for P (see [10] for the definition of a pseudopolynomial algorithm) if S is the feasible set of the so-called *resource allocation problems* studied in the literature (see [15]), i.e., S is described by

$$(6) \quad S = \{x \mid x \in Z^E, \sum_{e \in E} x(e) = c, l \leq x \leq u\},$$

where $c \in Z$, Z^E is the set of all the vectors $x = (x(e) : e \in E)$ with coordinates indexed by E and $x(e) \in Z$, $e \in E$, and $l, u \in Z^E$ such that $l \leq u$ and $\sum_{e \in E} l(e) \leq c \leq \sum_{e \in E} u(e)$.

We should mention here relationships between this paper and related papers [17], [18]. Recently, Katoh [17] studied the minimum variance combinatorial problems with 0–1 variables and gave an FPAS under the assumption that the corresponding minimum sum problem can be solved in polynomial time. The idea employed in this paper is based on this idea. An FPAS for the problems similar to P has been proposed by Katoh and Ibaraki [18]. Though the techniques employed therein are similar to those developed in this paper, our problem P does not belong to the class of problems for which they developed an FPAS (especially the condition (A5) given in Section 5 of [18] does not hold for P). Based on this idea, we shall present a *pseudopolynomial* algorithm for P .

This paper is organized as follows. Section 2 gives the assumptions made throughout this paper and explains basic properties on submodular systems and base polyhedra. Section 3 gives the relationship between P and $P(\lambda)$. Section 4 gives an outline of an ϵ -approximation scheme for P . Section 5 describes an ϵ -approximation scheme for P and analyzes its running time. Section 6 presents three special cases satisfying the assumptions in Section 2, and shows that in these cases, problem P has an FPAS. Section 7 proposes a pseudopolynomial time algorithm for S expressed by (6).

2 Assumptions and Basic Concepts

We shall give assumptions based on which an ϵ -approximation scheme is developed for P in the succeeding sections. Define the following problems.

$$(7) \quad P_{MINIMAX} : \text{minimize} \{ \max_{e \in E} h_e(x(e)) \mid x \in S \}$$

$$(8) \quad P_{MAXIMIN} : \text{maximize} \{ \min_{e \in E} h_e(x(e)) \mid x \in S \}$$

$$(9) \quad P_{FAIR} : \text{minimize} \{ \max_{e \in E} h_e(x(e)) - \min_{e \in E} h_e(x(e)) \mid x \in S \}.$$

By letting $g(u, v) = u - v$, the objective function of P_{FAIR} can be regarded as a special case of the one considered in problem FAIR of (1). We assume the following throughout this paper.

(A1) There exists a finite algorithm for solving each of three problems $P_{MINIMAX}$, $P_{MAXIMIN}$ and P_{FAIR} . In addition, all of these problems have finite optimal values.

(A2) Let $v_{MINIMAX}$ (resp. $v_{MAXIMIN}$) denote the optimal objective value of $P_{MINIMAX}$ (resp. $P_{MAXIMIN}$). Then we have

$$(10) \quad v_{MAXIMIN} \leq v_{MINIMAX}.$$

Section 6 gives important subclasses of problems satisfying these assumptions that include submodular constraints.

Next we give basic concepts related to submodular systems that are necessary for the discussion in Section 6. Let \mathcal{D} be a collection of subsets of E closed with respect to set union and intersection. Such \mathcal{D} is called a *distributive lattice* with set union and intersection as the lattice operations, join and meet. A function f from \mathcal{D} to R is called a *submodular function* on \mathcal{D} if for each pair of $X, Y \in \mathcal{D}$

$$(11) \quad f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y).$$

We assume throughout the paper that $\emptyset, E \in \mathcal{D}$ and $f(\emptyset) = 0$. We call the pair (\mathcal{D}, f) a *submodular system* on E and f the *rank function* of (\mathcal{D}, f) . For a submodular system (\mathcal{D}, f) , we define the *base polyhedron* and *integer base polyhedron* as

$$(12) \quad B = \{x \mid x \in R^E, \forall X \in \mathcal{D} : x(X) \leq f(X), x(E) = f(E)\},$$

$$(13) \quad IB = \{x \mid x \in Z^E, \forall X \in \mathcal{D} : x(X) \leq f(X), x(E) = f(E)\},$$

respectively, where R^E is the set of all vectors $x = (x(e) : e \in E)$ with coordinates indexed by E , and $x(X)$ is defined by

$$(14) \quad x(X) = \sum_{e \in X} x(e).$$

A vector in B (resp. IB) is called a *base* (resp. *integer base*) of (\mathcal{D}, f) . When we speak of IB , f is assumed to be integer-valued. We define an operation on base polyhedron B (see [5]). For a vector $a \in (R \cup \{-\infty, +\infty\})^E$, we define

$$(15) \quad B_a = \{x \mid x \in B, \forall e \in E : x(e) \geq a(e)\},$$

If B_a is nonempty, then B_a is called the *reduction* of B by vector a . Especially, B_0 is called a *polymatroid*, where 0 denotes a zero vector. A reduction with respect to an integer vector a is similarly defined for IB , and IB_0 is called an integer polymatroid.

3 Relationship between P and $P(\lambda)$

Katoh and Ibaraki [18] and Sniedovich [25], [26] considered the following problem Q .

$$(16) \quad Q : \text{minimize } \{r(q_1(x), q_2(x)) \mid x \in \mathcal{S}\},$$

where x denotes an n -dimensional decision vector and \mathcal{S} denotes a feasible set not necessarily satisfying assumption (A1) or (A2). $q_i, i = 1, 2$, are real-valued functions and $r(u_1, u_2)$ is quasiconcave over an appropriate convex region in \mathbb{R}^2 and differentiable in $u_i, i = 1, 2$. They proved the following lemma.

Lemma 3.1 *Let x^* be optimal to Q and let $u_i^* = q_i(x^*), i = 1, 2$. Define λ^* by*

$$(17) \quad \lambda^* = \left(\frac{\partial r(u_1^*, u_2^*)}{\partial u_2} \right) / \left(\frac{\partial r(u_1^*, u_2^*)}{\partial u_1} \right).$$

Then an optimal solution of the following problem $Q(\lambda)$ with $\lambda = \lambda^$ is optimal to Q .*

$$(18) \quad Q(\lambda) : \text{minimize } \{q_1(x) + \lambda q_2(x) \mid x \in \mathcal{S}\}. \quad \square$$

The following lemma is obtained by specializing Lemma 3.1 to problem P . Let x^* and x^λ be optimal to P and $P(\lambda)$ respectively.

Lemma 3.2 *Let λ^* be defined by*

$$(19) \quad \lambda^* = 2 \sum_{e \in E} h_e(x^*(e)) / |E|.$$

Then x^{λ^} is optimal to P .*

Proof. First note that for any vector $x \in S$,

$$(20) \quad \begin{aligned} \text{var}(x) &= \sum_{e \in E} [h_e(x(e)) - \frac{1}{|E|} \sum_{e \in E} h_e(x(e))]^2 \\ &= \sum_{e \in E} \{h_e(x(e))\}^2 - \frac{1}{|E|} \left\{ \sum_{e \in E} h_e(x(e)) \right\}^2 \end{aligned}$$

Let $\mathcal{S} = S$ and let

$$q_1(x) = \sum_{e \in E} \{h_e(x(e))\}^2, \quad q_2(x) = \sum_{e \in E} h_e(x(e))$$

and

$$r(u_1, u_2) = u_1 - \frac{1}{|E|} (u_2)^2.$$

Then it is easy to see that for each x

$$\text{var}(x) = q_1(x) - \frac{1}{|E|} \{q_2(x)\}^2.$$

Therefore P can be rewritten into

$$\text{minimize } \{q_1(x) - \frac{1}{|E|} \{q_2(x)\}^2 \mid x \in S\}.$$

Since $r(u_1, u_2)$ is quasiconcave and differentiable on R^2 , it turns out that P is a special case of Q . As a result, by $\partial r(u_1, u_2)/\partial u_1 = 1$ and $\partial r(u_1, u_2)/\partial u_2 = -2u_2/|E|$, the lemma follows from Lemma 3.1. \square

Although this lemma states that $P(\lambda)$ for an appropriate λ can solve P , such λ is not known unless P is solved. A straightforward approach to resolve this dilemma is to solve $P(\lambda)$ for all λ ; the one with the minimum $\text{var}(x)$ is an optimal solution. For certain special cases, this idea leads to a pseudopolynomial algorithm for P , which will be discussed in the last section. Though an optimal solution x^λ of $P(\lambda)$ for $\lambda = \lambda^*$ satisfies

$$\lambda = 2 \sum_{e \in E} h_e(x^\lambda(e))/|E|,$$

this condition is not, in general, sufficient for x^λ to be optimal to P (see [18]). Therefore, a binary search technique cannot be applied to search for λ^* .

4 The Outline of an Approximation Scheme for P

In this section, we shall develop an ϵ -approximation scheme for P that requires to solve $P(\lambda)$ a number of times polynomial in the input size and $1/\epsilon$.

Lemma 4.1 *For an n -dimensional vector $y = (y_1, y_2, \dots, y_n)$ with $y_1 \leq y_2 \leq \dots \leq y_n$, let $\bar{y} = \sum_{i=1}^n y_i/n$. Then we have*

$$(21) \quad (y_n - y_1)^2/2 \leq \sum_{i=1}^n (y_i - \bar{y})^2 \leq \frac{n-1}{2} \cdot (y_n - y_1)^2.$$

Proof. First it is easy to see that

$$(22) \quad \sum_{i=1}^n (y_i - \bar{y})^2 = \frac{1}{n} \sum_{i=1}^{n-1} \sum_{j=i+1}^n (y_j - y_i)^2$$

holds. By $|y_j - y_i| \leq y_n - y_1$ for all i, j with $1 \leq i, j \leq n$, the second inequality of (21) immediately follows. Since

$$(23) \quad \begin{aligned} \sum_{i=1}^n (y_i - \bar{y})^2 &\geq (y_1 - \bar{y})^2 + (y_n - \bar{y})^2 \\ &\geq (y_1 - \frac{y_1 + y_n}{2})^2 + (y_n - \frac{y_1 + y_n}{2})^2 = \frac{1}{2}(y_n - y_1)^2, \end{aligned}$$

the first inequality of (21) follows (the inequality of (23) follows since $z = (y_1 + y_n)/2$ minimizes $(y_1 - z)^2 + (y_n - z)^2$). \square

Now let us consider problem P_{FAIR} . Let $d(x)$ denote the objective value of this problem for $x \in S$, and let x° denote its optimal solution.

Lemma 4.2 For any $x \in S$, we have

$$(24) \quad \{d(x)\}^2/2 \leq \text{var}(x) \leq \frac{|E|-1}{2} \cdot \{d(x)\}^2$$

Proof. Let $n = |E|$. For the sake of simplicity, let $h_{e_1}(x(e_1)), h_{e_2}(x(e_2)), \dots, h_{e_n}(x(e_n))$ with $h_{e_1}(x(e_1)) \leq h_{e_2}(x(e_2)) \leq \dots \leq h_{e_n}(x(e_n))$ be the sorted list of $\{h_e(x(e)) \mid e \in E\}$. Then $d(x) = h_{e_n}(x(e_n)) - h_{e_1}(x(e_1))$ follows. Applying Lemma 4.1 after letting $y_i = h_{e_i}(x(e_i))$ for each i with $1 \leq i \leq n$, (24) is immediately obtained. \square

Lemma 4.3 For any optimal solution x^* of P and any optimal solution x° of P_{FAIR} , we have

$$(25) \quad \{d(x^\circ)\}^2/2 \leq \text{var}(x^*) \leq \frac{|E|-1}{2} \{d(x^\circ)\}^2.$$

Proof. Since $d(x^\circ) \leq d(x^*)$ holds by the optimality of x° , the first inequality of (25) follows from the first inequality of (24). Since $\text{var}(x^*) \leq \text{var}(x^\circ)$ holds by the optimality of x^* , the second inequality of (25) follows from the second inequality of (24). \square

Lemma 4.4 For any optimal solution x^* of P , we have

$$(26) \quad \max_{e \in E} h_e(x^*(e)) \leq v_{MAXIMIN} + \sqrt{|E|-1} \cdot d(x^\circ),$$

$$(27) \quad \min_{e \in E} h_e(x^*(e)) \geq v_{MINIMAX} - \sqrt{|E|-1} \cdot d(x^\circ).$$

Proof. Let

$$(28) \quad v^* = \max_{e \in E} h_e(x^*(e)), \quad v_* = \min_{e \in E} h_e(x^*(e)).$$

By the minimality of $v_{MINIMAX}$ and the maximality of $v_{MAXIMIN}$,

$$(29) \quad v^* \geq v_{MINIMAX}$$

and

$$(30) \quad v_* \leq v_{MAXIMIN}$$

follow. If either (26) or (27) does not hold,

$$(31) \quad d(x^*) = v^* - v_* > \sqrt{|E|-1} \cdot d(x^\circ)$$

follows from (30) or (29) respectively. By the first inequality of (24),

$$(32) \quad \frac{1}{2} \cdot \{d(x^*)\}^2 \leq \text{var}(x^*)$$

holds. Then it follows that

$$\begin{aligned}
 \text{var}(x^*) &\leq \frac{|E|-1}{2} \cdot \{d(x^\circ)\}^2 \text{ (by the second inequality of (25))} \\
 &< \frac{|E|-1}{2} \cdot \frac{1}{|E|-1} \cdot \{d(x^*)\}^2 \text{ (by (31))} \\
 (33) \quad &= \{d(x^*)\}^2/2 \leq \text{var}(x^*). \text{ (by (32))}
 \end{aligned}$$

This is a contradiction. \square

Lemma 4.5 For λ^* defined in (19),

$$(34) \quad 2v_{MINIMAX} - 2\sqrt{|E|-1} \cdot d(x^\circ) \leq \lambda^* \leq 2v_{MAXIMIN} + 2\sqrt{|E|-1} \cdot d(x^\circ)$$

holds.

Proof. Immediate from (19), (26) and (27). \square

Now we shall describe the outline of an ϵ -approximation scheme for P . First note that if $d(x^\circ) = 0$, it is obvious that $\text{var}(x^\circ) = 0$ and thus x° is optimal to P . As a result, if $d(x^\circ) = 0$, an exact optimal solution of P can be obtained by solving P_{FAIR} . Therefore assume $d(x^\circ) > 0$ in the following discussion.

Define

$$(35) \quad \delta \equiv d(x^\circ)\sqrt{8\epsilon/|E|}$$

$$(36) \quad K \equiv \lceil (2v_{MAXIMIN} - 2v_{MINIMAX} + 4\sqrt{|E|-1} \cdot d(x^\circ))/\delta \rceil,$$

$$(37) \quad \lambda_0 \equiv 2v_{MINIMAX} - 2\sqrt{|E|-1} \cdot d(x^\circ),$$

$$(38) \quad \lambda_K \equiv 2v_{MAXIMIN} + 2\sqrt{|E|-1} \cdot d(x^\circ),$$

$$(39) \quad \lambda_k \equiv \lambda_0 + \frac{k(\lambda_K - \lambda_0)}{K}, \quad k = 1, \dots, K-1,$$

where $\lceil a \rceil$ denotes the smallest integer not less than a . Then solve $P(\lambda)$ for $\lambda = \lambda_0, \lambda_1, \dots, \lambda_K$. Among $K+1$ solutions obtained, the one with minimum $\text{var}(x^{\lambda_k})$ is output as an ϵ -approximate solution of P . This is proved as follows.

Lemma 4.6 Let $\lambda_0, \lambda_1, \dots, \lambda_K$ be as defined above, and let λ_{k^*} satisfy

$$(40) \quad \text{var}(x^{\lambda_{k^*}}) = \min_{0 \leq k \leq K} \text{var}(x^{\lambda_k}).$$

Then $x^{\lambda_{k^*}}$ is an ϵ -approximate solution of P .

Proof. By Lemma 4.5 and (35)-(39), there exists l with $0 \leq l \leq K$ such that

$$(41) \quad |\lambda_l - \lambda^*| \leq \delta/2$$

holds. Since $\text{var}(x^{\lambda_l}) \geq \text{var}(x^{\lambda^*})$ holds by (40), it is sufficient to show that x^{λ_l} is an ϵ -approximate solution. Define δ' by

$$(42) \quad \delta' \equiv \lambda_l - \lambda^*.$$

For the sake of simplicity, let

$$(43) \quad \tilde{z}_1 = \sum_{e \in E} \{h_e(x^{\lambda_l}(e))\}^2, \tilde{z}_2 = \sum_{e \in E} h_e(x^{\lambda_l}(e)),$$

$$(44) \quad z_1^* = \sum_{e \in E} \{h_e(x^*(e))\}^2, z_2^* = \sum_{e \in E} h_e(x^*(e)).$$

Since x^{λ_l} is optimal to $P(\lambda_l)$, we have

$$(45) \quad z(x^{\lambda_l}) = \tilde{z}_1 - \lambda_l \tilde{z}_2 \leq z_1^* - \lambda_l z_2^*.$$

It then follows that

$$\begin{aligned} \text{var}(x^{\lambda_l}) &= \tilde{z}_1 - \frac{1}{|E|}(\tilde{z}_2)^2 \quad (\text{by (20)}) \\ &\leq z_1^* - (\lambda^* + \delta')z_2^* + (\lambda^* + \delta')\tilde{z}_2 - \frac{1}{|E|}(\tilde{z}_2)^2 \quad (\text{by (42) and (45)}) \\ &= z_1^* - (\lambda^* + \delta')z_2^* - \frac{1}{|E|}(\tilde{z}_2 - \frac{|E|}{2}(\lambda^* + \delta'))^2 + \frac{|E|}{4}(\lambda^* + \delta')^2 \\ &\leq z_1^* - (\lambda^* + \delta')z_2^* + \frac{|E|}{4}(\lambda^* + \delta')^2 \\ &= z_1^* - \frac{2}{|E|} \cdot (z_2^*)^2 - \delta' \cdot z_2^* \\ &\quad + \frac{1}{|E|} \cdot (z_2^*)^2 + \delta' \cdot z_2^* + \frac{|E|}{4}(\delta')^2 \quad (\text{by substituting } \lambda^* = \frac{2z_2^*}{|E|} \text{ from (19)}) \\ &= z_1^* - \frac{1}{|E|} \cdot (z_2^*)^2 + \frac{|E|}{4}(\delta')^2 \\ &= \text{var}(x^*) + \frac{|E|}{4}(\delta')^2 \quad (\text{by (20)}) \\ (46) \quad &\leq \text{var}(x^*) + \frac{|E|}{16}\delta^2. \quad (\text{by (41) and (42)}) \end{aligned}$$

Therefore

$$\begin{aligned} \frac{\text{var}(x^{\lambda_l}) - \text{var}(x^*)}{\text{var}(x^*)} &\leq \frac{|E| \delta^2}{16 \cdot \text{var}(x^*)} \quad (\text{by (46)}) \\ &\leq \frac{2|E| \delta^2}{16\{d((x^*))\}^2} \quad (\text{by the first inequality of (25)}) \\ (47) \quad &= \epsilon. \quad (\text{by (35)}) \end{aligned}$$

This implies that x^{λ^t} is an ϵ -approximate solution. \square

5 Description of an ϵ -approximation Scheme for P

Based on the results given in the previous section, we shall describe an ϵ -approximation scheme for P .

Procedure APPROX

Input: The minimum variance problem P with $h_e, e \in E$.

Output: An ϵ -approximate solution of P .

Step 1: Solve $P_{MINIMAX}$ and $P_{MAXIMIN}$ and let $v_{MINIMAX}$ and $v_{MAXIMIN}$ be their optimum values, respectively. Solve P_{FAIR} and let x° and $d(x^\circ)$ be its optimal solution and optimum value, respectively.

Step 2: If $d(x^\circ) = 0$, then output x° as an optimal solution of P and halt. Else go to Step 3.

Step 3: Compute $\delta, \lambda_0, \lambda_1, \dots, \lambda_K$ and K by (35)-(39).

Step 4: For each $k = 0, 1, \dots, K$, compute x^{λ^k} .

Step 5: Compute $x^{\lambda^{k^*}}$ determined by

$$(48) \quad \text{var}(x^{\lambda^{k^*}}) = \min_{0 \leq k \leq K} \text{var}(x^{\lambda^k}).$$

and output $x^{\lambda^{k^*}}$ as an ϵ -approximate solution of P . Halt. \square

Theorem 5.1 *Procedure APPROX correctly computes an ϵ -approximate solution of P in*

$$(49) \quad O(T |E| / \sqrt{\epsilon} + T_{MINIMAX} + T_{MAXIMIN} + T_{FAIR})$$

time, where T is the time required to compute an optimal solution x^λ of $P(\lambda)$ and $T_{MINIMAX}$, $T_{MAXIMIN}$ and T_{FAIR} are the times required to solve $P_{MINIMAX}$, $P_{MAXIMIN}$ and P_{FAIR} respectively.

Proof. The correctness follows from Lemma 4.6. The running time is analyzed as follows. Problems $P_{MINIMAX}$, $P_{MAXIMIN}$ and P_{FAIR} can be solved in $T_{MINIMAX}$, $T_{MAXIMIN}$ and T_{FAIR} time respectively. Thus Step 1 requires $O(T_{MINIMAX} + T_{MAXIMIN} + T_{FAIR})$ time. Step 2 requires $O(|E|)$ time to output an $|E|$ -dimensional vector x° . Since

$$(50) \quad 2v_{MAXIMIN} - 2v_{MINIMAX} + 4\sqrt{|E| - 1} \cdot d(x^\circ) \leq 4\sqrt{|E| - 1} \cdot d(x^\circ), \quad (\text{by (10)})$$

$$(51) \quad K \leq \sqrt{2|E|(|E| - 1)/\epsilon} = O(|E| / \sqrt{\epsilon})$$

follows. Thus, K is determined in $O(\log(|E|/\sqrt{\epsilon})) = O(\log |E| - \log \epsilon)$ time by applying the binary search. After K is obtained, λ_k is computed in constant time for each k from (39). Hence, Step 3 requires $O(K) = O(|E|/\sqrt{\epsilon})$ time. By (51), $O(T|E|/\sqrt{\epsilon})$ time is required in Step 4. Step 5 clearly requires $O(K) = O(|E|/\sqrt{\epsilon})$ time. The total time required by APPROX is therefore given by (49). \square

Corollary 5.1 *If T , $T_{MINIMAX}$, $T_{MAXIMIN}$ and T_{FAIR} are all polynomial in the input size, procedure APPROX is a fully polynomial time approximation scheme. \square*

6 Special Cases Satisfying (A1) and (A2)

We shall present three special cases satisfying assumptions (A1) and (A2) for which procedure APPROX given in the previous section provides an FPAS (fully polynomial time approximation scheme). They are defined as follows.

(a) $h_e(x(e))$ defined on R is linear and increasing for each $e \in E$, i.e.,

$$(52) \quad h_e(x(e)) = a_e x(e) + b_e,$$

where $a_e > 0$ and b_e is a real constant. In addition, a feasible set S is a closed subset of R^E which is described by the set of linear equalities and/or inequalities containing the constraint $\sum_{e \in E} x(e) = c$, where $c \in R$ is a fixed constant.

(b) $h_e(x(e))$ defined on R is the same as the one in (a). A feasible set S is equal to the set B (or IB) defined in (12) (or (13) respectively).

(c) $h_e(x(e))$ is defined by

$$(53) \quad h_e(x(e)) = a_e \{x(e)\}^{b_e},$$

where $a_e > 0$ and $1/2 \leq b_e < 1$. S is the set of real or integer bases of a polymatroid, i.e., $S = B_0$ or IB_0 defined in Section 2.

Lemma 6.1 *In case (a), assumptions (A1) and (A2) are satisfied, and $P(\lambda)$ can be solved in polynomial time for any λ .*

Proof. Since all of problems $P_{MINIMAX}$, $P_{MAXIMIN}$ and P_{FAIR} can be formulated as linear programs as easily verified, these problems can be solved in polynomial time if Khachian's or Karmarkar's algorithm is applied [22], [16]. To show that $P_{MINIMAX}$ has a finite optimal value, we show that $\{\max_{e \in E} h_e(x(e)) | x \in S\}$ is bounded below by a certain constant. Suppose otherwise. Then there exists $x_M \in S$ such that $\max_{e \in E} h_e(x_M(e)) < -M$ for any $M > 0$. By $a_e > 0$ for all e , this implies $\max_{e \in E} x_M(e) < c$ for a sufficiently large $M > 0$, which is a contradiction to $\sum_{e \in E} x_M(e) = c$ assumed in (a). Since S is closed as assumed in (a), there exists a finite optimal value for $P_{MINIMAX}$. The case of $P_{MAXIMIN}$ or P_{FAIR} can be similarly treated. Thus, assumption (A1) is satisfied.

In order to prove (10) of (A2), let $x_{MINIMAX}$ and $x_{MAXIMIN}$ be optimal to $P_{MINIMAX}$ and $P_{MAXIMIN}$ respectively. If $v_{MAXIMIN} > v_{MINIMAX}$,

$$\begin{aligned} h_e(x_{MAXIMIN}(e)) &\geq v_{MAXIMIN} \quad (\text{from definition of } v_{MAXIMIN}) \\ &> v_{MINIMAX} \\ &\geq h_e(x_{MINIMAX}(e)) \quad (\text{from definition of } v_{MINIMAX}) \end{aligned}$$

holds for each $e \in E$. Since h_e is increasing, $x_{MAXIMIN}(e) > x_{MINIMAX}(e)$ follows. This implies $\sum_{e \in E} x_{MAXIMIN}(e) (= c) > \sum_{e \in E} x_{MINIMAX}(e) (= c)$, which is a contradiction to the assumption that $\sum_{e \in E} x(e)$ for $x \in S$ is a constant. Finally, since

$$\begin{aligned} \{h_e(x(e))\}^2 - \lambda h_e(x(e)) &= (a_e x(e) + b_e)^2 - \lambda(a_e x(e) + b_e) \\ (54) \qquad \qquad \qquad &= a_e^2 \{x(e)\}^2 + (2a_e b_e - \lambda a_e) x(e) + b_e^2 - \lambda b_e, \end{aligned}$$

$\{h_e(x(e))\}^2 - \lambda h_e(x(e))$ is convex for any λ . Thus, $P(\lambda)$ is a convex program, and can be solved in polynomial time by applying the ellipsoid method by Kozlov et. al [23]. \square

It should be remarked that for practical purposes the simplex method may be more efficient than Khachian's and Karmarkar's algorithms in order to solve $P_{MINIMAX}$, $P_{MAXIMIN}$ and P_{FAIR} , though polynomial time solvability is not guaranteed. For a convex program, practically more efficient algorithms than the one by [23] are available. Thus we suggest that for practical purposes such algorithms should be used in order to solve $P(\lambda)$ though polynomial time solvability is not guaranteed. The following theorem is an immediate consequence of Corollary 5.1 and Lemma 6.1.

Theorem 6.1 *Problem P for the case (a) has a fully polynomial time approximation scheme. \square .*

In order to treat case (b), we need some preliminary results. First we assume the following.

(B1) An upper bound M for $|f(X)|$ is known.

(B2) There exists a polynomial time algorithm that minimizes $\{f(X) \mid X \in \mathcal{D}'\}$, where \mathcal{D}' is an arbitrarily given distributive lattice defined over $E' (\subseteq E)$.

Under assumption (B1), Grötschel, Lovász, Schrijver [12] proved that minimizing $\{f(X) \mid X \in \mathcal{D}'\}$ in (B2) can be solved in polynomial time by using the ellipsoid method of Khachian [22]. In most of applications, however, there exists a more efficient algorithm to do this job.

First we consider the case of $S = B$. Let us consider the following problem.

$$(55) \quad P_{sum} : \text{minimize } \left\{ \sum_{e \in E} w_e(x(e)) \mid x \in B \right\}.$$

Here w_e for each e is a convex function from R to $R \cup \{-\infty, +\infty\}$ satisfying

$$(56) \quad \lim_{y \rightarrow +\infty} w_e^+(y) = +\infty, \quad \lim_{y \rightarrow -\infty} w_e^-(y) = -\infty,$$

where w_e^+ and w_e^- denote the right and left derivatives of w_e respectively. Then [6] showed the following lemma (see also [11]).

Lemma 6.2 *If (56) is satisfied, there exists a finite optimal solution for P_{sum} . In addition, P_{sum} can be solved by calling the algorithm assumed in (B2) a polynomial number of times if $w_e^+(x(e))$ and $w_e^-(x(e))$ can be evaluated in constant time for each $x(e) \in R$. \square*

Now consider the case of $S = IB$. For each $e \in E$, let $\bar{w}_e : Z \rightarrow R$ be a real-valued function such that the piecewise linear extension, denoted by w_e , of \bar{w}_e on R is a convex function, where $w_e(y) = \bar{w}_e(y)$ for $y \in Z$ and w_e restricted on each unit interval $[y, y + 1]$, $y \in Z$, is a linear function. We also assume (56). Now consider the integer version (denoted by \bar{P}_{sum}) of problem P_{sum} . Then [6] proved the following lemma (see also [11]).

Lemma 6.3 *If (56) is satisfied, there exists a finite optimal solution for \bar{P}_{sum} . In addition, \bar{P}_{sum} (i.e., problem P_{sum} with w_e and B replaced by \bar{w}_e and IB respectively) can be solved by calling the algorithm assumed in (B2) a polynomial number of times if $w_e^+(x(e))$ and $w_e^-(x(e))$ can be evaluated in constant time for each $x(e) \in Z$. \square*

The following lemma has been shown also by [6] (see also chapters 8 and 9 of [15]).

Lemma 6.4 *Suppose that S is equal to either the set B or IB , and h_e is increasing for each $e \in E$. If*

$$(57) \quad \lim_{y \rightarrow +\infty} h_e(y) = +\infty, \quad \lim_{y \rightarrow -\infty} h_e(y) = -\infty,$$

all of problems $P_{MINIMAX}$, $P_{MAXIMIN}$ and P_{FAIR} have finite optimal values and can be solved by calling the algorithm assumed in (B2) a polynomial number of times. \square

Lemma 6.5 *In case (b), assumptions (A1) and (A2) are satisfied, and $P(\lambda)$ for any λ can be solved by calling the algorithm assumed in (B2) a polynomial number of times.*

Proof. From Lemma 6.4, $P_{MINIMAX}$, $P_{MAXIMIN}$ and P_{FAIR} can be solved in polynomial time since h_e is increasing by $a_e > 0$ as assumed in case (b). Thus assumption (A1) is satisfied. Since $S = B$ or IB contains the constraint of $\sum_{e \in E} x(e) = f(E)$ as seen from (12) or (13), (A2) is satisfied as proved in the same manner as in the case (a). Since h_e is linear, $\{h_e(x(e))\}^2 - \lambda h_e(x(e))$ is convex for any λ as was shown in the proof of Lemma 6.1. The right and left derivatives of $\{h_e(x(e))\}^2 - \lambda h_e(x(e))$ are equivalent to $2a_e h_e(x(e)) - \lambda a_e$. Thus $P(\lambda)$ can be solved in polynomial time for either $S = B$ or IB from Lemmas 6.2 and 6.3. \square

[6] (see also Chapter 5 of [15]) proved that an algorithm for P_{sum} (or \bar{P}_{sum}) can be modified to solve $P_{MINIMAX}$ and $P_{MAXIMIN}$ for $S = B$ or IB without any significant change, and that $P_{MINIMAX}$ and $P_{MAXIMIN}$ for $S = B$ or IB can be assumed to be solved in the same time complexity as P_{sum} or \bar{P}_{sum} respectively. Therefore we shall use the notation $\tau(|E|, f)$ (resp. $\bar{\tau}(|E|, f)$) to denote the time complexity required to solve either $P(\lambda)$, $P_{MINIMAX}$ or $P_{MAXIMIN}$ with $S = B$ (resp. $S = IB$). [7] showed the following lemma.

Lemma 6.6 (i) *Problem P_{FAIR} with $S = B$ has an efficient algorithm whose time complexity is the same as that for P_{sum} .*

(ii) *Problem P_{FAIR} with $S = IB$ has an efficient algorithm that requires a polynomial number of calls of the algorithm assumed in (B2). Its running time is $O(\bar{\tau}(|E|, f) + |E|(\log M + \log |E|))$.* \square

From (B2), Lemmas 6.5 and 6.6, we have the following theorem.

Theorem 6.2 *Problem P for the case (b) has a fully polynomial time approximation scheme. Its running time for $S = B$ (or IB) is $O(\tau(|E|, f)|E|/\sqrt{\epsilon} + \tau(|E|, f))$ or $O(\bar{\tau}(|E|, f)|E|/\sqrt{\epsilon} + \bar{\tau}(|E|, f) + |E|(\log M + \log |E|))$ respectively.* \square

Finally let us consider the case (c). In this case

$$(58) \quad \{h_e(x(e))\}^2 - \lambda h_e(x(e)) = a_e^2 \{x(e)\}^{2b_e} - \lambda a_e \{x(e)\}^{b_e}$$

holds. Since $\lambda^* \geq 0$ by (19) and $h_e(x(e)) \geq 0$, $\lambda \geq 0$ can be assumed without loss of generality. Thus, it is easy to see from $1/2 \leq b_e < 1$ that $\{h_e(x(e))\}^2 - \lambda h_e(x(e))$ is convex for any $\lambda \geq 0$ and satisfies (57) if we let $w_e(x(e)) = \{h_e(x(e))\}^2 - \lambda h_e(x(e))$ for $x(e) \geq 0$ and $w_e(x(e)) = -\infty$ for $x(e) < 0$. Thus, from Lemmas 6.2 and 6.3, $P(\lambda)$ can be solved in polynomial time by calling the algorithm given in (B2) a polynomial number of times since a polymatroid is a special class of a submodular system. In addition, from Lemma 6.4, problems $P_{MINIMAX}$, $P_{MAXIMIN}$ and P_{FAIR} with such h_e can also be solved in polynomial time by calling the algorithm given in (B2) a polynomial number

of times since each $h_e(x(e))$ is nondecreasing by $a_e > 0$. For $S = B_0$, the existence of optimal solutions of $P_{MINIMAX}$, $P_{MAXIMIN}$ and P_{FAIR} is obvious since S is bounded and closed. For $S = IB_0$, the existence of optimal solutions for such three problems is also obvious since S is bounded and hence is a finite set. Thus assumption (A1) is satisfied. Assumption (A2) is also satisfied as proved in the same manner as cases (a) and (b). From Theorem 5.1 and the above discussion, we have the following theorem.

Theorem 6.3 *Problem P for the case (c) has a fully polynomial time approximation scheme. Its running time for $S = B$ (or IB) is $O(\tau'(|E|, f)|E|/\sqrt{\epsilon} + \tau'(|E|, f))$ or $O(\tau'(|E|, f)|E|/\sqrt{\epsilon} + \bar{\tau}'(|E|, f) + |E|(\log M + \log |E|))$ respectively. Here $\tau'(|E|, f)$ (resp. $\bar{\tau}'(|E|, f)$) denote the time required to solve problem P_{sum} (resp. \bar{P}_{sum}) with h_e of (53) for $S = B_0$ (resp. $S = IB_0$). \square*

7 A Pseudopolynomial Time Algorithm for a Simple Case

In this section, we shall consider the case in which S is expressed by (6) while we do not impose any restriction on the form of h_e . Notice that this is a special case of $S = IB$. We shall first give basic properties. It is well known in the theory for parametric programming (see for example [8], [13]) that $z(\lambda)$ (the optimal objective value of $P(\lambda)$) is a piecewise linear concave function as illustrated in Fig. 1, with a finite number of joint points $\lambda_{(1)}, \lambda_{(2)}, \dots, \lambda_{(N)}$ with $\lambda_{(1)} < \lambda_{(2)} < \dots < \lambda_{(N)}$. Here N denotes the number of total joint points, and let $\lambda_{(0)} = -\infty$ and $\lambda_{(N+1)} = \infty$ by convention. In what follows, for two real numbers a, b with $a \leq b$, (a, b) and $[a, b]$ stand for the open interval $\{x \mid a < x < b\}$ and the closed interval $\{x \mid a \leq x \leq b\}$ respectively. The following two lemmas are known in the parametric combinatorial programming.

Lemma 7.1 [13] *For any $\lambda' \in (\lambda_{(k-1)}, \lambda_{(k)})$, $k = 1, \dots, N + 1$, $x^{\lambda'}$ is optimal to $P(\lambda)$ for all $\lambda \in [\lambda_{(k-1)}, \lambda_{(k)}]$. \square*

Let for $k = 1, \dots, N + 1$

$$(59) \quad S_k^* \equiv \{x \in S \mid x \text{ is optimal to } P(\lambda) \text{ for all } \lambda \in [\lambda_{(k-1)}, \lambda_{(k)}]\} .$$

Lemma 7.2 [13] *For any two $x, x' \in S_k^*$ with $1 \leq k \leq N + 1$,*

$$\sum_{e \in E} \{h_e(x(e))\}^2 = \sum_{e \in E} \{h_e(x'(e))\}^2 \text{ and } \sum_{e \in E} h_e(x(e)) = \sum_{e \in E} h_e(x'(e))$$

hold. \square

Lemmas 7.1 and 7.2 imply that in order to determine $z(\lambda)$ for all λ , it is sufficient to compute $x^{\lambda'}$ for an arbitrary $\lambda' \in (\lambda_{(k-1)}, \lambda_{(k)})$ for each $k = 1, 2, \dots, N + 1$. Let x_k^* stand for any $x \in S_k^*$.

Eisner and Severance [2] proposed an algorithm that determines $z(\lambda)$ for all λ and x_k^* , $k = 1, \dots, N + 1$ for a large class of combinatorial parametric problems including $P(\lambda)$ as a special case. They showed that the running time of their algorithm is proportional to (the number of joint points) \times (the time required to solve $P(\lambda)$ for a given λ). Since $P(\lambda)$ for a fixed λ can be viewed as the resource allocation problem with a separable objective function, it can be solved in $O(|E| \cdot L^2)$ time by applying the dynamic programming technique (see Chapter 3 of [15] for the details). Here $L = c - l(E)$.

Lemma 7.3 $z(\lambda)$ for all λ and x_k^* , $k = 1, \dots, N + 1$, can be determined in $O(N |E| L^2)$ time. \square

Lemma 7.4 (Chapter 10 of [15])

$$(60) \quad N \leq |E|^2 L^2. \quad \square$$

Thus, by Lemmas 7.3 and 7.4, we have the following theorem.

Theorem 7.1 If S is expressed by (6), problem P can be solved in $O(|E|^3 L^4)$ time. \square

Notice that this running time is not polynomial in the input size but pseudopolynomial. Now let us consider the case in which h_e is expressed by (52) or (53).

Theorem 7.2 If S is expressed by (6) with $l \geq 0$, and each $h_e, e \in E$, is given by (52) or (53), problem P has a fully polynomial time approximation scheme. Its running time is $O(|E| \cdot \max\{|E|, |E| \log(L/|E|)\} / \sqrt{\epsilon} + \max\{|E|, |E| \log(L/|E|)\} + |E|(\log L + \log |E|))$.

Proof. Since in this case $P(\lambda)$ is a special case of \bar{P}_{sum} with IB equal to S expressed in (6), such problem can be solved in polynomial time (the best known algorithm is given by [3] which runs in $O(\max\{|E|, |E| \log(L/|E|)\})$ time. See also [9], [15]). Problems $P_{MINIMAX}$ and $P_{MAXIMIN}$ can be solved in the same time complexity. P_{FAIR} can be solved in $O(\max\{|E|, |E| \log(L/|E|)\} + |E|(\log L + \log |E|))$ time as shown by [7] and [19]. Thus the theorem follows from Theorems 6.2 and 6.3. \square

Acknowledgements

The author would like to thank Professor S. Fujishige of Tsukuba University for his helpful comments, and Professor T. Ibaraki of Kyoto University for his support and encouragement. He also would like to express his gratitude to anonymous referees for valuable comments, which helped much to improve the original version of the paper. The suggestion provided by one of the referees was greatly helpful in order to improve the running time of the proposed algorithm and to generalize the results of the original version of the paper. It is deeply acknowledged.

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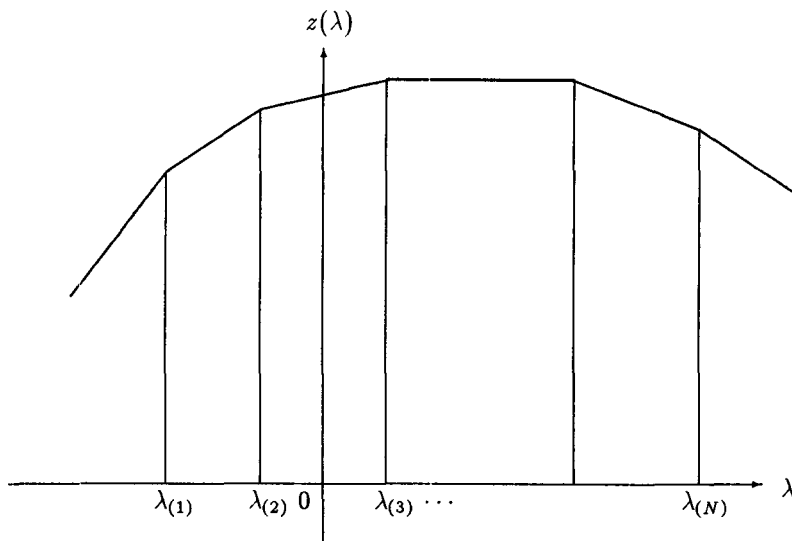


Fig. 1 Illustration of $z(\lambda)$