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COMPARISON OF THE LOSS PROBABILITY OF THE $GI^X/GI/1/k$ QUEUES WITH A COMMON TRAFFIC INTENSITY

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Abstract The loss probabilities of the $GI^X/GI/1/k$ queues, which means the batch arrival $GI/GI/1/k$ queues, are compared when the queues have a common waiting capacity k, a common batch size distributions and a common traffic intensity. We prove that, if the interarrival time and service time distributions are NBUE (NWUE), then the loss probabilities are not greater (less) than those of the corresponding $M^X/M/1/k$ queues. The stronger results are obtained for the $M/G/1/k$ queues with single arrivals.

1. Introduction

In analyses of queueing models, the exponential assumptions, which means that the exponential distributions are assumed for the interarrival times, the service times and so on, have been used widely. It seems to have been believed that, if these distribution are less random than the exponential ones, the exponential assumptions give safety bounds for a large class of queues with respect to stationary characteristics, for example, the queue length. However, it has been verified only for a small class of queues. The problem is a special case of stochastic comparison of queues, but we need to compare queues with a common traffic intensity.

Stoyan and Stoyan [11] introduced the new stochastic order and studied the waiting times of the $GI/GI/1$ queues. Miyazawa [6] took a different approach and gave the classes of the $GI/GI/1$ queues for which the exponential assumptions provide safety bounds with respect to the system queue length distributions, where the system queue length means the number of customers in the system. Related problems of stochastic comparison have been studied widely for the $GI/GI/1$ queues (for example see Stoyan [10]). However, we have only a few results for queues other than $GI/GI/1$. Especially, for queues with finite waiting capacity, only special cases such as pure loss systems have been studied (see Section 7.5 of Stoyan [10]).

In this paper, we discuss comparison of the $GI^X/GI/1/k$ queues, which means the batch arrival $GI/GI/1/k$ queues, where k denotes the number of the waiting positions not including a service position. The detailed description of our $GI^X/GI/1/k$ is given in Section 2. There are two typical models on the batch acceptance policy. One is called a partially rejected model, in which all free waiting positions are occupied by customers in an arriving batch and the remained customers in the batch are lost. The other is called a totally rejected model, in which all customers in an arriving batch are lost if the number of free positions are less than the batch size (see Baba [1]). We are only concerned with a partially rejected model. Unfortunately, the similar argument is not enough for applying to the totally rejected model (see Section 5). We compare loss probabilities in the steady state, *i.e.*, the stationary probability of the event that arriving customers can not enter the system. It will be noticed that this is equivalent to consider the time-stationary probability of the system being empty.

The approach taken in this paper is a similar to Miyazawa [6], which is based on the relationships between time and customer stationary characteristics. In this paper, we derive them by using the so called basic equations, which are recently developed by Miyazawa [7, 8, 9]. To handle multiple arrivals of customers in a formal way, we define conditional probabilities at arrival and departure epochs of batches and customers in terms of point processes. We discuss them in Section 2. In Section 3, we consider the case of $M/GI/1/k$, *i.e.*, the single and Poisson arrivals case. The main result is obtained in Section 4. It is proved that, if the interarrival time and service time distributions are NBUE (NWUE), then the loss probabilities are not greater (less) than that of $M^X/M/1/k$ for a fixed batch size distribution and a fixed traffic intensity. Here NBUE (NWUE) denoted a distribution F satisfying:

$$\int_t^{+\infty}(1-F(u))du \leq (\geq) m_F(1-F(t)) \quad (\forall t \geq 0),$$

where m_F is the mean of F. Refer to Stoyan [10] for the details of NBUE and NWUE distributions. In Section 5, we give remarks on a totally rejected model and on the effect of a batch size distribution.

2. The basic equations

For the $GI^X/GI/1/k$ queue, we assume that the batch size of arriving customers, the interarrival times and the service times are *i.i.d.* sequences of random variables, respectively, and those sequences are independent of one another. Accepted customers by the system are served by a single server. FCFS (First Come First Served) is a typical service discipline for this queue, but it is not necessary because the system queue length is only concerned with. We assume that the service discipline is non-preemptive and the server is busy as far as there are customers in the system. Since we assume a partially rejected model, $k+1-n$ customers in an arriving batch can enter the system when the batch finds n customers in the system.

In this section, we consider some of relationships between time and customer stationary characteristics in the $GI^X/GI/1/k$ queue, which are called the basic equations. In Section 4, more detailed relationships will be discussed.

Let F, G and H be the interarrival time, service time and batch size distributions, respectively. T, S, and X denote random variables subjected to F, G, and H, respectively. We assume that $X \geq 1$ *a.s* and that $E(T), E(S)$ and $E(X)$ are positive and finite, where E denotes the expectation, which will be used in the restrictive sense later. Define the arrival rate of customers $\lambda_c = \frac{E(X)}{E(T)}$, the arrival rate of batches $\lambda_b = \frac{1}{E(T)}$ and the traffic intensity $\rho = \lambda_c E(S)$. The key assumptions of this paper are that there exists a steady state for a given $\rho > 0$ and finite k and that arrival of a batch and completion of service do not occur at once *w.p.*1. These are satisfied at least when F has density.

Denote the stationary departure rate of customers by λ_d, where lost customers are not counted as departure from the queue on the contrary to Franken et al. [3] and Miyazawa [9]. This definition of λ_d leads somewhat different expressions from their ones. Let $l(t), u(t)$ and $r(t)$ be the system queue length, the residual arrival time and the residual service time at time t, respectively. All of those characteristic are defined in the steady state. Hence, $Y(t) = (u(t), r(t), l(t))$ is a stationary processes, and it is Markov by our assumptions. Without loss of generality, we can assume that $Y(t)$ is right continuous *w.p.*1.

To get the basic equations on $\{Y(t)\}$, we need the distributions at the arriving epochs of customers and batches, and at the service completion epochs denoted by P_c, P_b and P_d, respectively. Since the distributions P_c and P_b play important roles in our discussion, we give their formal definitions. Let t_n be the arrival time of the nth batch. We number customers in the nth batch as (n, i) $(i = 1, 2, \cdots, X_n)$, where X_n is the size of the nth batch. Let

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$t_{n,i}$ be the arrival time of the customer (n,i). Of course, $t_n = t_{n,i}$ for all i but we assume that the (n,i)th customer sees the system in which the customers from $(n,1)$ to $(n,i-1)$ have arrived. This convention is useful to treat the partial rejected model in a formal way. From our assumptions, we can assume that all random quantities are defined in a suitable probability space $(\Omega, \mathcal{F}, P)$ with a time-shift operator $\{T_s\}$ on Ω, and that P is stationary with respect to $\{T_s\}$. We now define P_c and P_b by:

$$P_c(A) = \frac{1}{\lambda_c} E\left(\sum_{0<t_{n,i}\leq 1} T_{t_{n,i}} I_A\right) \quad (A \in \mathcal{F}),$$

$$P_b(A) = \frac{1}{\lambda_b} E\left(\sum_{0<t_n\leq 1} T_{t_n} I_A\right) \quad (A \in \mathcal{F}),$$

where E denotes the expectation by P, I_A is an indicator function of a set A and $T_s f(\omega) = f(T_s\omega)$ for a function f defined on Ω. P_d can be defined in a similar way using the departure epochs of customers. The meaning of these definitions are clear. For example, if we let $A = \Omega$, then the expectations of the right-hand terms just counts customers and batches, respectively, arrived in the time interval (0,1]. Hence they should be λ_c and λ_b, respectively. We remark that P_c, P_b and P_d express the conditional probability measures of P under the conditions that a customer arrives at the system at time 0, that the batch of customers arrive at time 0 and that the service of a customer is completed at time 0, respectively. Please refer Miyazawa [7, 8] or Franken et al. [3] for background of those definitions.

Let $A = \{l^- = j\}$ $(0 \leq j \leq k)$ in the definition P_c, where $l_n^- = l_n(0-)$. Then we have:

$$\begin{aligned}
\lambda_c P_c\{l^- = j\} &= E\left(\sum_{0<t_n\leq 1}\sum_{i=1}^{X_n} I_{\{l_n^-+i-1=j\}}\right) \\
&= E\left(\sum_{0<t_n\leq 1} I_{\{j+1-X_n\leq l_n^- < j+1\}}\right) \\
&= \lambda_b P_b\{j+1-X \leq l^- < j+1\} \quad (j = 0,1,\cdots,k).
\end{aligned} \tag{2.1}$$

The intuitive meaning of the equation (2.1) is clear since the event of the probability of the right-hand term is that there are customers in a batch who find l^- equal to j. However, we should be careful of normalizing constants λ_c and λ_b. By using the formal definitions, we can avoid those cumbersome things.

We are now in a position to give the basic equations. Let f be a bounded function from R^3 to R, and define $Z(t) = f(Y(t))$. Then the rate of state change of the real valued process $\{Z(t)\}$ is composed of two parts, the derivative of its continuous part and the difference at its jump epochs. Since $\{Z(t)\}$ is stationary, the expected rate of the total change must be zero. Hence we can get the following lemma, whose formal proof is given in Miyazawa [7, 8].

Lemma 2.1 If $Z(t)$ has the right-hand derivative $Z'(t)$ for any t, we have:

$$E(Z'(0)) = \lambda_b E_b(Z(0-) - Z(0+)) + \lambda_d E_d(Z(0-) - Z(0+)), \tag{2.2}$$

where E_b and E_d denote the expectations by P_b and P_d respectively.

The equation (2.2) is a general form of the basic equation. In the following, we choose suitable f's or equivalently $Z(t)$'s for our purpose. Similarly as l_n^-, we let $l^- = l(0-)$, $l = l(0)$ and $l^+ = l(0+)$. Define, for $j = 0,1,\cdots,k+1$,

$$Z(t) = I_{\{l(t)\geq j\}}.$$

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Since $Z'(t) = 0$, we have, from Lemma 2.1,

$$\lambda_b[P_b\{l^+ \geq j\} - P_b\{l^- \geq j\}] = \lambda_d[P_d\{l^- \geq j\} - P_d\{l^+ \geq j\}] \quad (j = 0, 1, \cdots, k+1) \quad (2.3)$$

Since $l^+ \leq l^- + X$ *a.s* P_b and $l^+ = l^- - 1$ *a.s* P_d, we have, from (2.3),

$$\lambda_b P_b\{j - X \leq l^- < j\} = \lambda_d P_d\{l^+ = j-1\} \quad (j = 1, 2, \cdots, k+1). \quad (2.4)$$

From (2.1) and (2.4), we have:

$$\lambda_c P_c\{l^- = j\} = \lambda_d P_d\{l^+ = j\} \quad (j = 0, 1, \cdots, k). \quad (2.5)$$

This is a finite version of Finch's formula. Let $p_{loss} = P_c\{l^- = k+1\}$, which is the loss probability of arriving customers. By summing up (2.5) for all j, we have:

$$\lambda_c(1 - p_{loss}) = \lambda_d, \quad (2.6)$$

or equivalently $p_{loss} = 1 - \frac{\lambda_d}{\lambda_c}$.

Next, we apply Lemma 2.1 for $Z(t) = r(t)I_{\{l(t)>0\}}$. Since $Z'(t) = -I_{\{l(t)>0\}}$, we have:

$$-(1 - p_0) = \lambda_b(-E_b(S; l^- = 0)) + \lambda_d(-E_d(S; l^+ > 0)),$$

where $p_0 = P\{l = 0\}$ and $E_d(S; l^+ > 0) = E_d(SI_{\{l^+>0\}})$. We will use the later notation for other expectations. Note that $E_b(S) = E_d(S) = E(S)$. Hence, using (2.4) for $j = 1$ and (2.6), we get:

$$\begin{aligned} 1 - p_0 &= E(S)[\lambda_d P_d\{l^+ = 0\} + \lambda_d P_d\{l^+ > 0\}] \\ &= \lambda_d E(S) \\ &= \rho(1 - p_{loss}). \end{aligned} \quad (2.7)$$

The equation (2.7) is a very simple relation but we will see that it plays a key role in the next section.

Remark 2.1 Since (2.7) is a version of Little's formula, more direct derivation is possible. We have not done it to show that our formulas are derived from (2.2) in a unified way. We also remark that (2.7) was obtained for the non-batch $GI/GI/1/k$ queue in Miyazawa [9].

Finally, we express p_{loss} in terms of P_b. This can be calculated by (2.4) and (2.5), but we directly derive it here. Let $A = \{l^- = k+1\}$ in the definition of P_c. Since the ith customer in an arriving batch find $\min(l^- + i - 1, k+1)$ customers in the system,

$$\begin{aligned} \lambda_c p_{loss} &= E(\sum_{0<t_n\leq 1} \sum_{i=1}^{X_n} I_{\{k+1\leq l^-+i-1\}}) \\ &= \lambda_b E_b(\sum_{i=1}^{X} I_{\{k+2-i\leq l^-\}}) \\ &= \lambda_b \sum_{n=1}^{+\infty} \sum_{i=1}^{n} P_b\{k+2-i \leq l^-\} P\{X = n\} \\ &= \lambda_b \sum_{i=1}^{+\infty} \sum_{n=i}^{+\infty} P_b\{k+2-i \leq l^-\} P\{X = n\} \\ &= \lambda_b \sum_{i=1}^{+\infty} P_b\{k+2-i \leq l^-\} P\{X \geq i\} \\ &= \lambda_b\{\sum_{i=k+2}^{+\infty} P\{X \geq i\} + \sum_{i=1}^{k+1} P_b\{k+2-i \leq l^-\} P\{X \geq i\}\}. \end{aligned} \quad (2.8)$$

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For $M^X/GI/1/k$, Baba [1] gave a equivalent expression to (2.8) (see (3.30) of his paper).

3. $M/GI/1/k$ with single arrivals

In this section, we briefly consider the $M/GI/1/k$ queue without batch arrivals. In this special case, results are more detailed than in a general case, which is discussed in the next section. Let $d_n := P_d\{l^+ = n\}$. We use the following well-known facts.

(i) If $\rho < 1$, then

$$d_n = \frac{q_n}{\sum_{i=0}^{k} q_i} \quad (n = 0, 1, \cdots, k),$$

where $\{q_n\}$ is the system queue length distribution at arbitrary time of the $M/GI/1$ queue with the same arrival rate λ and the same service time distribution G.

(ii) Denote $M/GI/1$ with a service time distribution G_i by $M/G_i/1$. For any two distributions G_1 and G_2, we define $G_1 \leq_{st} G_2$ if $1 - G_1(t) \leq 1 - G_2(t)$ for all $t > 0$, and $G_1 \leq_c G_2$ if $\int_t^\infty (1 - G_1(x))dx \leq \int_t^\infty (1 - G_2(x))dx$ for all $t > 0$. '$\leq_{st}$' and '$\leq_c$' are called stochastic and convex orders, respectively. Suppose that the traffic intensity ρ_i of $M/G_i/1$ is less than 1 for $i = 1, 2$ and $\rho_1 = \rho_2$. Then $G_1 \leq_c G_2$ and $m_{G_1} = m_{G_2}$ imply that $\{q_n^1\} \leq_{st} \{q_n^2\}$, where m_{G_i} and $\{q_n^i\}$ are the mean of G_i and the system queue length distribution of $M/G_i/1$, respectively ($i = 1, 2$).

(i) is given in Cohen [2] and Tijms [12]. (ii) can be obtained from (5.0.16) and Theorem 5.2.3 of Stoyan [10]. From (i) and $q_0 = 1 - \rho$, we have:

$$d_0 = \frac{1 - \rho}{\sum_{i=0}^{k} q_i}.$$

Then, from (2.5) for $j = 0$, (2.7) and the fact that $P_c\{l^- = 0\} = P\{l = 0\}$, which comes from Poisson property (see also (4.8) in the next section), we have:

$$p_{loss} = 1 - \frac{\sum_{i=0}^{k} q_i}{(1 - \rho) + \rho \sum_{i=0}^{k} q_i}. \tag{3.1}$$

Remark 3.1 (3.1) is obtained in a different way in Tijms [12]. For $M/M/1/k$, (3.1) agrees with the well-known loss formula $p_{loss} = q_{k+1}/(\sum_{i=0}^{k+1} q_i)$ since $q_i = (1-\rho)\rho^i$. However, this formula for p_{loss} does not hold for $M/GI/1/k$ in general because the system queue length distribution just before the arrival is different from the one just after the departure (see (2.5)).

From (3.1) and (ii), we have the next proposition.

Proposition 3.1 If $\rho_1 = \rho_2 < 1$, $G_1 \leq_c G_2$ and $m_{G_1} = m_{G_2}$, then p_{loss} of $M/G_1/1/k$ is less than p_{loss} of $M/G_2/1/k$.

Remark 3.2 If $G_2(G_1)$ is an exponential distribution, $G_1 \leq_c G_2$ is equivalent to that $G_1(G_2)$ is NBUE(NWUE) (see Proposition 1.6.1 of Stoyan [10]).

It is a natural question whether or not the condition $\rho_i < 1$ is essential in Proposition 3.1. From the analytic expression of $\{d_n\}$ (see (6.24) of Cohen [2]), Proposition 3.1 seems to be true for any $\rho_i > 0$ but we have not proved it yet. We here give this direction's result but for special cases.

Proposition 3.2 Let $p_{loss}(E_m)$ be the loss probability of customers in $M/E_m/1/k$ and suppose that $M/E_m/1/k$ has a traffic intensity ρ, where E_m denotes the mth order Erlang distribution. Then, for all $\rho > 0$, $p_{loss}(E_m)$ is non-increasing in $m = 1, 2, \cdots$.

The proof of this proposition is given in Appendix. For $M/GI/1/k$ with a common traffic intensity, it is interesting to see that (2.7) implies that we can not compare $\{p_n^1\}$ with

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$\{p_n^2\}$ in the sense of '$\le_{st}$', *i.e.*, their distribution functions cross at least one time, if their loss probabilities are different. This is the case of Proposition 3.1 if $G_1 \ne G_2$. In Table 3.1, we give numerical examples for such cases. From numerical experience, we conjecture that $\{p_n^1\} \le_c \{p_n^2\}$ if $G_1 \le_c G_2$, but it seems hard to prove it.

Table 3.1 The system queue length distributions of $M/E_m/1/k$

$---k = 5,\ \rho = 0.30---$

m	p_{loss}	$E(l)$	$p\{l \le 0\}$	$p\{l \le 1\}$	$p\{l \le 2\}$	$p\{l \le 3\}$	$p\{l \le\}$	$p\{l \le 5\}$
1	0.00051	0.427	0.70015	0.91020	0.97321	0.99212	0.99779	0.99949
2	0.00013	0.396	0.70004	0.92580	0.98286	0.99618	0.99919	0.99987
3	0.00007	0.386	0.70002	0.93173	0.98602	0.99728	0.99950	0.99993
4	0.00005	0.380	0.70001	0.93485	0.98757	0.99777	0.99962	0.99995
5	0.00004	0.377	0.70001	0.93677	0.98849	0.99804	0.99969	0.99996

$---k = 5,\ \rho = 0.70---$

m	p_{loss}	$E(l)$	$p\{l \le 0\}$	$p\{l \le 1\}$	$p\{l \le 2\}$	$p\{l \le 3\}$	$p\{l \le\}$	$p\{l \le 5\}$
1	0.03846	1.705	0.32692	0.55577	0.71596	0.82810	0.90659	0.96154
2	0.02245	1.586	0.31572	0.57539	0.75030	0.86236	0.93309	0.97755
3	0.01737	1.533	0.31216	0.58562	0.76627	0.87688	0.94315	0.98263
4	0.01494	1.504	0.31046	0.59177	0.77545	0.88487	0.94840	0.98506
5	0.01353	1.485	0.30947	0.59586	0.78140	0.88991	0.95162	0.98647

4. Main results

We back to a general case. We first derive two relationships from the basic equation (Lemma 2.1). Define, for any $x > 0$ and $j = 0, 1, \cdots, k+1$,

$$Z(t) = \min(x, u(t)) I_{\{l(t)=j\}}.$$

Since $Z'(t) = -1$ if $0 < u(t) \le x$, we have, from Lemma 2.1,

$$\begin{aligned} -P\{l = j, u \le x\} = &-\lambda_b E_b(\min(x,T); l^+ = j) \\ &+ \lambda_d[E_d(\min(x,u); l^- = j) - E_d(\min(x,u); l^+ = j)] \\ &(x > 0,\ j = 1, 2, \cdots, k+1), \end{aligned} \tag{4.1}$$

where $u = u(0)$. In the derivation of (4.1), we have used the fact that $u(0-) = u(0+) = u(0)$ *a.s* P_d. By summing up (4.1) from j to $k+1$, we have:

$$\begin{aligned} P\{l \ge j, u \le x\} = \lambda_b E_b(\min(x,T)) P_b\{l^+ \ge j\} - \lambda_d E_d(\min(x,u); l^+ = j-1) \\ (x > 0,\ j = 1, 2, \cdots, k+1). \end{aligned} \tag{4.2}$$

Note that the left-hand side and the first term of the right-hand side of (4.2) are bounded in x. Hence, the last term of the right-hand side is also bounded in x. Consequently, by letting x tend to infinity, we have:

$$P\{l \ge j\} = P_b\{l^+ \ge j\} - \lambda_d E_d(u; l^+ = j-1) \quad (j = 1, 2, \cdots). \tag{4.3}$$

We next define, for any $x > 0$ and $j = 0, 1, \cdots, k+1$,

$$Z(t) = \min(x, r(t)) I_{\{l(t)=j\}}.$$

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By the similar argument, we have:

$$-P\{l=j, r\le x\} = \lambda_b[E_b(\min(x,r); l^- = j) - E_b(\min(x,r); l^+ = j)] \\ - \lambda_d E_d(\min(x,S); l^+ = j) \quad (x>0,\ j=1,2,\cdots,k+1), \tag{4.4}$$

where $r = r(0)$, and it implies:

$$P\{l \ge j\} = \lambda_d E(S) P_d\{l^+ \ge j\} + \lambda_b E_b(r; l^+ \ge j > l^-) \quad (j=1,2,\cdots). \tag{4.5}$$

Remark 4.1 For the non-batch arrival case, (4.1) and (4.4) with $x = +\infty$ correspond to (4.3.19) and (4.3.44) of Franken et al. [3], for which they did not mind the finiteness of $E_d(u; l^- = j-1)$. Similar things can be said for the argument in Miyazawa [6]. The above discussion shows that those expectations are certainly finite.

Remark 4.2 In the derivation of (4.3) and (4.5), we have used the *i.i.d.* assumption of the interarrival and service times only for splitting $E_b(\min(x,T))$ and $E_d(\min(x,S))$. Hence, if we do not so, we can have a similar equation for the $G^X/G/1/k$ queue, *i.e.*, the batch arrival single server finite queue with a stationary input.

We now assume that the distributions F and G are NBUE. Then, the conditional expectations of the residual interarrival time and service time given any events determined by the past information are not greater than their unconditional ones, respectively. Hence, we have, for $j = 1,2,\cdots,k+1$

$$E_d(u; l^+ = j-1) \le E(T) P_d\{l^+ = j-1\} \tag{4.6}$$

$$E_b(r; j - X \le l^- < j) \le E(S) P_b\{j - X \le l^- < j\}. \tag{4.7}$$

We have, from (4.3) and (4.6),

$$P\{l \ge j\} \ge P_b\{l^+ \ge j\} - \lambda_d E(T) P_d\{l^+ = j-1\}$$

Hence, we have, from (2.4) and the fact that $l^+ \ge j$ if and only if $l^- \ge j - X$,

$$P\{l \ge j\} \ge P_b\{l^- \ge j - X\} - P_b\{j - X \le l^- < j\} \\ = P_b\{l^- \ge j\} \quad (j=1,2,\cdots,k+1), \tag{4.8}$$

Similarly, from (2.4), (4.5) and (4.7), we have:

$$P\{l \ge j\} \le \lambda_d E(S) P_d\{l^+ \ge j-1\} \\ = \lambda_b E(S) \sum_{i=j}^{k+1} P_b\{i - X \le l^- < i\} \quad (j=1,2,\cdots,k+1). \tag{4.9}$$

Define $\alpha = \lambda_b E(S)$. From (4.8) and (4.9), we get:

$$P_b\{l^- \ge j\} \le \alpha \sum_{i=j}^{k+1} P_b\{i - X \le l^- < i\} \quad (j=1,2,\cdots,k+1). \tag{4.10}$$

We note that the equalities hold in (4.8), (4.9) and (4.10) if F and G are exponential distributions.

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For convenience of calculation, let us introduce the following notations.

$$B_j = P_b\{l^- \geq j\},\ v_j = P\{X = j\},\ V_j = P\{X \geq j\}$$

Since

$$\begin{aligned} P_b\{i - X \leq l^- < i\} &= P\{X \geq i+1\}P_b\{l^- < i\} \\ &\quad + \sum_{n=1}^{i}(P\{l^- \geq i-n\} - P\{l^- \geq i\})P\{X = n\} \\ &= \sum_{n=1}^{i-1} B_{i-n}v_n + V_i - B_i, \end{aligned}$$

we get from (4.10) after some calculation:

$$B_j + \alpha\sum_{n=1}^{k+1} B_n V_{k+2-n} \leq \alpha\{\sum_{n=1}^{j-1} B_n V_{j-n} + \sum_{n=j}^{k+1} V_n\} \quad (j = 1, 2, \cdots, k+1). \tag{4.11}$$

On the other hand, from (2.8), we have:

$$\lambda_c p_{loss} = \lambda_b\{\sum_{n=1}^{k+1} B_n V_{k+2-n} + \sum_{n=k+2}^{+\infty} V_n\} \tag{4.12}$$

Hence, we have, from (4.11),

$$B_j + \rho p_{loss} \leq \alpha\{\sum_{n=1}^{j-1} B_n V_{j-n} + \sum_{n=j}^{+\infty} V_n\} \quad (j = 1, 2, \cdots, k+1). \tag{4.13}$$

Multiplying V_{k+2-j} to the both sides of (4.13) and summing up it from $j = 1$ to $k+1$, we have, by (4.12),

$$\begin{aligned} &[E(X)+\rho\sum_{n=1}^{k+1} V_n]p_{loss} \\ &\quad \leq \alpha\sum_{j=1}^{k+1} V_{k+2-j}[\sum_{n=1}^{j-1} B_n V_{j-n} + \sum_{n=j}^{+\infty} V_n] + \sum_{n=k+2}^{+\infty} V_n \\ &\quad = \alpha\{\sum_{n=1}^{k} B_n \sum_{j=n+1}^{k+1} V_{j-n}V_{k+2-j} + \sum_{j=1}^{k+1}\sum_{n=j}^{+\infty} V_n V_{k+2-j}\} + \sum_{n=k+2}^{+\infty} V_n \end{aligned} \tag{4.14}$$

For $k = 0$, we have, from (4.14),

$$p_{loss} \leq \frac{E(X) + \rho - 1}{E(X) + \rho} = 1 - \frac{1}{E(X) + \rho}. \tag{4.15}$$

The equality of this equation holds when F and G are exponentially distributed. Note that the last term of (4.15) agrees with the loss probability of $M^X/M/1/k$. For $k \geq 1$, we have, from (4.13) for $j = 1$,

$$B_1 \leq \rho(1 - p_{loss}).$$

This inequality and (4.13) for $j = 2$ follows:

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$$B_2 + \rho(1+\alpha)p_{loss} \leq \rho(1+\alpha) - \alpha.$$

In a similar way, we have $B_j + \beta_j p_{loss} \leq \gamma_j \quad j = 1, 2, \cdots, k$ repeatedly using (4.13), where β_j and γ_j are positive constants not depending B_i $(i = 1, 2, \cdots, k+1)$. Substituting those inequalities into (4.14), we finally obtain that $p_{loss} \leq$ (positive constant), and this positive constant must be p_{loss} of the corresponding $M^X/M/1/k$ queue because the equalities hold in the all inequalities when F and G are exponentially distributed. In case F and G are NWUE, all inequalities are reversed. Thus we have the next theorem.

Theorem 4.1 For the $GI^X/GI/1/k$ queue, if the interarrival and service time distributions, F and G, are NBUE (NWUE), then the loss probability p_{loss} is not greater (less) than the one of the batch arrival $M/M/1/k$ queue with the same traffic intensities ρ and α and the same batch size distribution H.

Remark 4.3 By rewriting (4.11) as:

$$B_j + \alpha \sum_{n=j}^{k+1} B_n V_{k+2-n} \leq \alpha \{ \sum_{n=1}^{j-1} B_n (V_{j-n} - V_{k+2-n}) + \sum_{n=j}^{k+1} V_n \} \quad (j = 1, 2, \cdots, k+1),$$

we can see that B_{k+1}, which is the probability that all customers in an arriving batch are lost, is also not greater than the one of the corresponding $M^X/M/1/k$.

Remark 4.4 By (2.7), the time stationary probability of the system empty is also not greater (less) than the one of the corresponding $M^X/M/1/k$ if F and G are NBUE (NWUE).

Remark 4.5 (4.13) and (4.14) give a procedure to calculate p_{loss} of $M^X/M/1/k$ in finite steps.

5. Concluding remarks

We give two remarks. First remark is on the effect of the batch size distribution H to the loss probability. One may conjecture that, if $H_1 \leq_c H_2$ and if all other characteristics are fixed, then the $p^1_{loss} \leq p^2_{loss}$ for the $GI^X/GI/1/k$ queue, where p^i_{loss} is the loss probability for the queue with the batch size distribution H_i $(i = 1, 2)$. This seems true at least for $M^X/M/1/k$, for which we can get explicit formula from (4.13) and (4.14). For example, we have for $k = 1, 2, 3$

$$p_{loss}(1) = 1 - \frac{1}{\rho} + \frac{E(X)}{\rho[E(X) + \rho(1 + V_2 + \alpha)]},$$
$$p_{loss}(2) = 1 - \frac{1}{\rho} + \frac{E(X)}{\rho[E(X) + \rho\{1 + V_2 + V_3 + (1 + 2V_2)\alpha + \alpha^2\}]},$$
$$p_{loss}(3) = 1 - \frac{1}{\rho}$$
$$+ \frac{E(X)}{\rho[E(X) + \rho\{1 + V_2 + V_3 + V_4 + (1 + 2(V_2 + V_3) + V_2^2)\alpha + (1 + 3V_2)\alpha^2 + \alpha^3\}]}.$$

where $p_{loss}(k)$ is the loss probability for $M^X/M/1/k$. Hence, $p_{loss}(k)$ for $k = 1, 2, 3$ are functions of $V_2, V_2 + V_3, V_2 + V_3 + V_4$, and not increasing with respect to them. We now suppose that two batch size distributions H_1 and H_2 satisfy $H_1 \leq_c H_2$ and have a common mean. We denote $\{V_n\}$ and $p_{loss}(k)$ of the queue with the batch size distribution H_i by $\{V^i_n\}$ and $p^i_{loss}(k)$, respectively. Since $\sum_{n=1}^{+\infty} V^1_n = \sum_{n=1}^{+\infty} V^2_n$, $H_1 \leq_c H_2$ is equivalent to:

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$$V_2^2 + V_3^2 + \cdots + V_j^2 \le V_2^1 + V_3^1 + \cdots + V_j^1 \quad (j = 2, 3, \cdots).$$

Hence we conclude that $p^1_{loss}(k) \le p^2_{loss}(k)$ for $k = 1, 2, 3$. For a general k, the expression of p_{loss} is very complicated and we need more detailed discussions. This will be given in a forthcoming paper.

Finally we briefly consider the totally rejected model. Let X^* be the number of customers accepted by the system when a batch arrives. If $l^- + X \le k+1$, $X^* = X$, and otherwise $X^* = 0$, where l^- and X are the same notations as used in Sections 2 and 4. The equations (2.3), (2.4), (2.7), (4.3) and (4.5) are true if we replace X by X^*. Hence, if F and G are NBUE, then we have (4.10) in which X is replaced by X^*. However, the equations (2.1) and hence (2.5) is not true in general. The loss probabilities of customers and batches are calculated in a similar way as (2.8). For example, let p^b_{loss} be the batch loss probability. Then we have:

$$\begin{aligned} \lambda_b p^b_{loss} &= E\Big(\sum_{0<t_n\le 1} I_{\{k+2\le l^- + X\}}\Big) \\ &= \lambda_b \sum_{n=1}^{+\infty} P_b\{l^- \ge k+2-n\} P\{X = n\} \\ &= \lambda_b \Big\{\sum_{n=1}^{k+1} B_n v_{k+2-n} + V_{k+2}\Big\}. \end{aligned} \tag{5.1}$$

On the other hand, we have:

$$P_b\{i - X^* \le l^- < i\} = \sum_{n=1}^{i-1} B_{i-n} v_n + V_i - B_i - (1 - B_i) V_{k+2}. \tag{5.2}$$

Thus, similarly as (4.11), we get, for $j = 1, 2, \cdots, k+1$,

$$B_j + \alpha \sum_{n=j}^{k+1} B_n (V_{k+2-n} - V_{k+2}) \le \alpha\Big\{\sum_{n=1}^{j-1} B_n (V_{j-n} - V_{k+2-n}) + \sum_{n=j}^{k+1} (V_n - V_{k+2})\Big\}. \tag{5.3}$$

Hence, from (5.1), we have, for $j = 1, 2, \cdots, k+1$,

$$\begin{aligned} &B_j + \alpha\Big\{p^b_{loss} + \sum_{n=j}^{k+1} B_n (V_{k+3-n} - V_{k+2})\Big\} \\ &\qquad \le \alpha\Big\{\sum_{n=1}^{j-1} B_n (V_{j-n} - V_{k+3-n}) + V_{k+2} + \sum_{n=j}^{k+1} (V_n - V_{k+2})\Big\} \end{aligned} \tag{5.4}$$

Consequently, we have Theorem 4.1 concerning B_{k+1}, which is the probability that an arriving batch finds the system full (see Remark 4.3). Once we have inequalities $B_j + \beta_j p_{loss} \le \gamma_j$ for each j as we did in Section 4, then we also have Theorem 4.1 concerning p^b_{loss}. However, it seems difficult to derive such inequalities from (5.4) since the left hand term of (5.4) contains $B_{j+1}, B_{j+2}, \cdots, B_{k+1}$. We conjecture that Theorem 4.1 holds also for the totally rejected model, but our argument in Section 4 is not applicable to prove it.

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Appendix

We prove Proposition 3.2. We can assume that an arriving customers brings m service phases each of which is exponentially distributed. Let $h(t)$ be the number of phases in the system at time t. Similarly as $l(t)$, we define h^-, h and h^+. Analogously as (2.5), (4.3) and (4.5), we have:

$$\lambda_c P_c\{h^- = j\} = \lambda_d P_d\{h^+ = j\} \quad (j = 0, 1, \cdots, km) \tag{A.1}$$
$$P\{h \geq j\} = P_c\{h^+ \geq j\} - \lambda_d E(T) P_d(h^+ = j-1) \quad (j = 1, 2, \cdots, (k+1)m), \tag{A.2}$$
$$P\{h \geq j\} = \lambda_d E(S) P_d\{h^+ \geq j-1\} + \frac{\lambda_c E(S)}{m} P_c(h^+ \geq j > h^-) \quad (j = 1, 2, \cdots, (k+1)m), \tag{A.3}$$

where we have used the fact that u and r are exponentially distributed. Let $a_j = P\{h \geq j\}$ for $j = 0, 1, \cdots, (k+1)m$. Note that $p_{loss} = a_{km+1}$. Then, after some calculations, $(A.1), (A.2)$ and $(A.3)$ imply:

$$a_j = \begin{cases} a_{j-m} - \dfrac{\rho}{m}(a_j - a_{j+1}) & (1 \leq j \leq km) \\ a_{j-m} + a_j - p_{loss} - \dfrac{\rho}{m}(a_j - a_{j+1}) & (km+1 \leq j \leq (k+1)m) \end{cases} \tag{A.4}$$

or, equivalently,

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$$\begin{aligned} a_j &= \frac{\rho}{m}(a_{j-m} + \cdots + a_{j-1}) - \rho p_{loss} \\ &= \frac{\rho}{m}(a_{j-m} + \cdots + a_{j-1}) - a_1 + \rho \quad (1 \le j \le (k+1)m), \end{aligned} \tag{A.5}$$

where $a_j = 1$ for $j \le 0$ and the last equality is obtained from (2.7). Note that $(A.5)$ agrees with (4.5) of Miyazawa (1976) when $k = +\infty$. Let $a'_j = P\{h \ge j\}$ $(j = 0, 1, \cdots, (m+1)(k+1))$ for $M/E_{m+1}/1/k$. Proposition 3.2 is obtained if we prove that $a_1 \le a'_1$. Suppose that $a_1 > a'_1$. Then, we can prove the following inequality by the similar argument as in Section 4 of Miyazawa (1976).

$$a_{im+j} - a_1 \ge \frac{m-j+1}{m} a'_{i(m+1)+j} + \frac{j-1}{m} a'_{i(m+1)+j+1} - a'_1 \ (i = 0, 1, \cdots, k,\ j = 1, \cdots, m). \tag{A.6}$$

Hence, we have:

$$a_{km+1} - a_1 \ge a'_{k(m+1)+1} - a'_1.$$

This inequality and (2.7) imply that $a_1 \le a'_1$, which contradict the supposition.

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