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OPTIMAL SELECTION PROBLEM WITH THREE STOPS

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Abstract Optimal selection problem with multiple choices, like secretary, dowry or marriage problems have attracted attention of many applied mathematicians and are also of great significance for those who are looking for "the best partner". In this paper we consider a variation of the optimal selection problem with three stops allowed which is often referred to as the secretary problem with multiple choices where the objective is to find an optimal stopping rule so as to maximize the probability of selecting three absolute bests.

We also present a set of dynamic programming equations for the problems both of selecting the best object with three stops and of selecting two absolute bests with three stops. For those problems the optimal stopping rules are numerically calculated in aid of computer programming.

1. Introduction

Optimal selection problems with multiple choices, like secretary, dowry or marriage problems, have attracted attention not only of many applied mathematicians but also of economists for more than 30 years and are also of great significance for those who are looking for "the best partner". Starting from the paper by Gilbert and Mosteller [2], various models of the optimal selection problem have been proposed. For example see [3], [4], [5], [8] and [9].

In this paper we consider a variation of the optimal selection problem with three stops, often referred to as the secretary problem which possesses the following framework: The N objects which can be totally ordered according to some criterion and appear in a sequential order at random, which implies that all permutations are equally likely. At each time an object appears the decision maker is able to rank the objects appeared so far according to the above criterion. Whenever an object appears, after observing its relative rank the decision maker must either accept it and then stop or reject it and continue to observe further incoming objects. He is asked to find a stopping rule which maximizes the probability of selecting the best object within three stops.

Even though many versions of the original secretary problem have been studied by several authors like Gilbert and Mosteller [2], Tamaki [4], Stadje [5] and Sakaguchi [8], the "No-Information" secretary problem with three stops has yet been unsolved. The "No-Information" problem as treated later in §2, §3 and §4 is described as follows: The decision maker is allowed to have three stops (choices) and his objective is to choose the three absolute best objects among N objects which randomly appear in a sequential order.

We derive an optimal stopping rule and the maximum probability of selecting the three absolute best objects by using OLA policy which is proposed by Ross [6]. This policy can be defined for a class of Markov decision processes whose optimality equation is given by

$$V(i) = \max\{R(i), -C(i) + \sum_{j=i+1}^{N} p_{ij} V(j)\},\ i = 1, 2, \cdots, N-1,\ V(N) = R(N).$$

where the state space is finite, $\{1,2,\cdots,N\}$ with the particular transition probabilities $\{p_{ij} \mid p_{ij} \geq 0 \text{ for } i<j,\ p_{ij}=0 \text{ for } i \geq j\}$. The decision maker facing state i must decide whether to accept terminal reward $R(i)$ and stop, or by paying the cost $C(i)$, to move to the next state j according to transition probability p_{ij}. Let $V(i)$ be the maximum of the expected reward when starting from an initial state i. Define

$$B \equiv \{N\} \cup \{i \mid R(i) \geq -C(i) + \sum_{j=i+1}^{N} p_{ij}R(j),\ 1 \leq i \leq N-1\},$$

which represents the set of states in which immediate stopping is at least as good as continuing for exactly one more period and then stopping. The policy that indicates stopping at i if and only if $i \in B$ is called the OLA policy. B is called closed if there exists an integer i^* such that B can be written as $B=\{i \mid i^* \leq i \leq N\}$. Ross [6] proves that if B is closed, then such OLA policy is optimal.

In §5 we present a set of dynamic programming (abbreviated by DP) equations for the problem of selecting the best object with three stops and derive the optimal stopping policy and the probability of achieving this object through numerical calculations. Furthermore, we derive a set of DP equations for the two absolute best objects with three stops by analogous numerical approach. Such a numerical approach in aid of computer programming is quite useful in deriving an optimal stopping rule for any N whenever a set of DP equations is available.

Gilbert & Mosteller [2] provides some numerical results for the problem of selecting the best object with multiple choice, and our numerical results are inspired by their work. It is important to note that for such a type of multi-choice problems, if we derive a set of DP equations, then the computer programming approach gives the optimal stopping policy for any N.

2. Selecting three absolute bests—Model

The i-th best object among the objects observed so far is called "i-candidate" and denoted by C_i. We need only C_1, C_2 and C_3 through out this paper, since other candidates cannot be chosen. Let N be the number of objects. We are allowed to make three stops and "win" if the accepted are the three absolute best ones. We are asked to find a stopping policy that maximizes the probability of win.

Three kinds of states are defined as follows: Suppose the decision-maker is facing the i-th object. The state $(i, C_k, 1\ell; 1)$, $k=1,2,3$; $\ell=1,2$, means that (1) the i-th object is C_k, (2) the first two stops were made at C_1 and C_ℓ in this order, and (3) still one stop left to be made. The state $(i, C_k; 2)$, $k=1,2$, means that (1) the i-th object is C_k and (2) still two stops left to be made. The state $(i;3)$ means that (1) the i-th object is C_1, and (2) three stops left to be made.

An optimal policy satisfies the following condition: if an object is better (worse) than some previously accepted (rejected) one, then accept (reject) it. Hence the optimal first stop should be made at C_1 and the optimal second stop should be made either at C_1 or at C_2.

We denote by $V_1(i, C_k, 1\ell), V_2(i, C_k)$, and $V_3(i)$, the probability of win under an optimal policy when starting from the state $(i, C_k, 1\ell; 1), (i, C_k; 2)$ and $(i;3)$, respectively. Given the i-th object is C_1, the conditional probability that the j-th object $(i<j \leq N)$ is the earliest C_1 is $i/j(j-1)$. Given the i-th object is C_1, the conditional probability that the j-th $(i<j\leq N)$ is the earliest C_1 or the earliest C_2 is $i(i-1)/j(j-1)(j-2)$. And given the i-th is C_1, the conditional probability that the j-th $(i<j\leq N)$ is the earliest C_1 or the earliest C_2 or the earliest C_3 is $i(i-1)(i-2)/j(j-1)(j-2)(j-3)$. Thus we obtain the

following set of DP equations.

$$V_1(i, C_k, 1\ell) = \frac{i(i-1)(i-2)}{N(N-1)(N-2)}, \qquad (3 \le i \le N-1,\ k=1,2,\ \ell=1,2), \tag{2.1}$$

$$V_1(N, C_k, 1\ell) = 1. \qquad \{\text{stop is optimal}\}$$

$$V_1(i, C_3, 1\ell) = \max\{\frac{i(i-1)(i-1)}{N(N-1)(N-2)},$$

$$\sum_{j=i+1}^{N} \frac{i(i-1)(i-2)}{j(j-1)(j-2)(j-3)}[\sum_{k=1}^{3} V_1(j, C_k, 1\ell)]\}, \tag{2.2}$$

$$V_1(N, C_3, 1\ell) = 1.$$

$$V_2(i, C_1) = \sum_{j=i+1}^{N} \frac{i(i-1)(i-2)}{j(j-1)(j-2)(j-3)}[\sum_{k=1}^{3} V_1(j, C_k, 1\ell)],\ 3 \le i \le N-1, \tag{2.3}$$

$$V_2(N, C_1) = 0,$$

$$V_2(2, C_1) = \frac{1}{3}\{\frac{12}{N(N-1)(N-2)} + V_1(3, C_3, 1\ell)\}. \qquad \{\text{stop is optimal}\}$$

$$V_2(i, C_2) = \max\{\sum_{j=i+1}^{N} \frac{i(i-1)(i-2)}{j(j-1)(j-2)(j-3)}[\sum_{k=1}^{3} V_1(j, C_k, 12)],$$

$$\sum_{j=i+1}^{N} \frac{i(i-1)}{j(j-1)(j-2)}[\sum_{k=1}^{2} V_2(j, C_k)]\}, \qquad 3 \le i \le N-1, \tag{2.4}$$

$$V_2(N, C_2) = 0,$$

$$V_2(2, C_2) = \max\{\frac{1}{3}[\frac{12}{N(N-1)(N-2)} + V_1(3, C_3, 12)],$$

$$\sum_{j=3}^{N} \frac{2}{j(j-1)(j-2)}[\sum_{k=1}^{2} V_2(j, C_k)]\}.$$

$$V_3(i) = \max\{\sum_{j=i+1}^{N} \frac{i(i-1)}{j(j-1)(j-2)}[\sum_{k=1}^{2} V_2(j, C_k)],\ \sum_{j=i+1}^{N} \frac{i}{j(j-1)} V_3(j)\},$$

$$2 \le i \le N-1,\ V_3(N) = 0, \tag{2.5}$$

$$V_3(1) = \max\{\frac{1}{6}[\frac{12}{N(N-1)(N-2)} + V_1(3, C_3, 11)] + \frac{1}{2}V_2(2, C_2),$$

$$\sum_{j=2}^{N} \frac{1}{j(j-1)} V_3(i)\}.$$

The first expression inside the max $\{\cdots\}$ is the probability of win due to immediate stopping in current state and the second one inside the max $\{\cdots\}$ is the probability of win due to continuing and behaving optimally hereafter. For computational convenience, we introduce the following notation:

$F_1(i) \equiv V_1(i, C_k, 1\ell)/i(i-1)(i-2),\ k=1,2,\ \ell=1,2,\ 3 \le i \le N-1,$

$F_2(i) \equiv V_1(i, C_3, 1\ell)/i(i-1)(i-2),\ \ell=1,2,\ 3 \le i \le N-1,$

$F_3(i) \equiv V_2(i, C_1)/i(i-1)(i-2),\ 3 \le i \le N-1,$

$F_4(i) \equiv V_2(i, C_2)/i(i-1)(i-2),\ 3 \le i \le N-1,$

$F_5(i) \equiv V_3(i)/i(i-1),\ 2 \le i \le N-1,$

Then we have the following set of "reduced" DP equations.

$$F_1(i)=\frac{1}{N(N-1)(N-2)}, \qquad 3\le i\le N. \tag{R1}$$

$$F_2(i)=\max\{\frac{1}{N(N-1)(N-2)}, \sum_{j=i+1}^{N}\frac{1}{j-3}[\frac{2}{N(N-1)(N-2)}+F_2(j)]\}, \qquad 3\le i\le N-1, \tag{R2}$$

$$F_2(N)=\frac{1}{N(N-1)(N-2)}.$$

$$F_3(i)=\sum_{j=i+1}^{N}\frac{1}{j-3}[\frac{2}{N(N-1)(N-2)}+F_2(j)], \qquad 3\le i\le N-1, \tag{R3}$$

$$F_3(N)=0, V_2(2,C_1)=\frac{4}{N(N-1)(N-2)}+2F_2(3).$$

$$F_4(i)=\max\{F_3(i),\frac{1}{i-2}\sum_{j=i+1}^{N}[F_3(j)+F_4(j)]\}, \qquad 3\le i\le N-1, \tag{R4}$$

$$F_4(i)=0, V_2(2,C_2)=\max\{\frac{4}{N(N-1)(N-2)}+2F_2(3), 2\sum_{j=3}^{N}[F_3(j)+F_4(j)]\}.$$

$$F_5(i)=\max\{\sum_{j=i+1}^{N}[F_3(j)+F_4(j)],\frac{1}{i-1}\sum_{j=i+1}^{N}F_5(j)\}, \qquad 2\le i\le N-1, \tag{R5}$$

$$F_5(N)=0, V_3(1)=\max\{\frac{2}{N(N-1)(N-2)}+F_2(3)+\frac{1}{2}V_2(2,C_2), \sum_{j=2}^{N}F_5(j)\}.$$

3. Selecting three absolute bests—Optimal stopping times

The optimal first stopping time is determined by $(R5)$, the optimal second by $(R3)$ & $(R4)$, and the optimal third by $(R1)$ & $(R2)$.

Theorem 1.

The optimal third stop is made at the earliest C_1 after the second stop, at the earliest C_2 after the second stop, or at the earliest C_3 after r_1, where r_1 is given by

$$r_1=\min\{i\mid \sum_{j=1}^{N-1}\frac{1}{j-2}\le\frac{1}{3},\ 3\le i\le N-1\}. \tag{3.1}$$

Proof. For $(R2)$, we have the OLA stopping region

$$B_1\equiv(N,C_3,1\ell;1)\cup\{(i,C_3,1\ell;1)\mid\frac{1}{N(N-1)(N-2)}\ge\sum_{j=i+1}^{N}\frac{1}{j-3}[\frac{3}{N(N-1)(N-2)}]\}, \tag{3.2}$$

for $\ell=1,2$. Then $B_1=\{(i,C_3,1\ell;1)\mid r_1\le i\le N\}$ and it turned out to be closed. Thus the theorem is proved. □

For this optimal third stopping policy, we obtain by $(R2)$,

$$F_2(i) = \begin{cases} \dfrac{1}{N(N-1)(N-2)(i-2)}\{(r_1-3)F_2(r_1-1)+2(r_1-i-1)\}, 3 \le i \le r_1-1 \\ \dfrac{1}{N(N-1)(N-2)}, \quad r_1 \le i \le N, \end{cases} \tag{3.3}$$

where

$$F_2(r_1-1) = 3 \cdot \sum_{j=r_1-1}^{N-1} \frac{1}{j-2}. \tag{3.4}$$

Let

$$K(i) = \frac{1}{i-3}\{F_2(i) + \frac{2}{N(N-1)(N-2)}\} \tag{3.5}$$

for $i \ge 4$. Then we have by (3.3),

$$K(i) = \begin{cases} \dfrac{r_1-3}{N(N-1)(N-2)(i-2)(i-3)}\{F_2(r_1-1)+2\}, \ 4 \le i \le r_1-1 \\ \dfrac{3}{N(N-1)(N-2)(i-3)}, \qquad r_1 \le i \le N, \end{cases} \tag{3.6}$$

and

$$F_3(i) = \begin{cases} \dfrac{1}{N(N-1)(N-2)(i-2)}\{(r_1-3)F_2(r_1-1)+2(r_1-i-1)\}, 3 \le i \le r_1-1 \\ \dfrac{3}{N(N-1)(N-2)} \displaystyle\sum_{j=i+1}^{N} \frac{1}{j-3}, \qquad r_1 \le i \le N-1. \end{cases} \tag{3.7}$$

For $(R3)$, we have the OLA stopping region,
$B_2 \equiv (N, C_2; 2) \cup (N-1, C_2; 2)$

$$\cup \left\{ (i, C_2; 2) \left| \begin{array}{l} \dfrac{4}{N(N-1)(N-2)} + 2F_2(3) \ge 2 \displaystyle\sum_{j=3}^{N-1} 2 \sum_{k=j+1}^{N} K(k), \quad i=2 \\ \displaystyle\sum_{j=i+1}^{N} K(j) \ge \frac{1}{i-2} \sum_{j=i+1}^{N-1} 2 \sum_{k=j+1}^{N} K(k), \qquad 3 \le i \le N-2 \end{array} \right. \right\}. \tag{3.8}$$

B_2 can be written as

$$B_2 = (N, C_2; 2) \cup \{(i, C_2; 2) \mid \Phi_1(i) \ge 0, \ 1 \le i \le N-1\} \tag{3.9}$$

where

$$\Phi_1(i) = \begin{cases} 2\displaystyle\sum_{j=4}^{N}(3-j)K(j) + F_2(3) + \frac{2}{N(N-1)(N-2)}, & i=2, \\ \dfrac{1}{N(N-1)(N-2)}\{3(r_1-3)F_2(r_1-1) + 6(2r_1-N-i-2) & \\ \qquad -2(r_1-3)(F_2(r_1-1)+2)\displaystyle\sum_{j=i}^{r_1-2}\frac{1}{j-1}\}, & 3 \le i \le r_1-2, \\ \dfrac{1}{N(N-1)(N-2)}\{9(i-2)\displaystyle\sum_{j=i}^{N-1}\frac{1}{j-2} - 6(N-i)\}, & r_1-1 \le i \le N-1. \end{cases} \tag{3.10}$$

Theorem 2.
The optimal second stop is made either at the earliest C_1 after the first stop, or at the earliest C_2 after r_2, where r_2 is given by,

$$r_2 = \min\{i \mid 3(r_1 - 3)F_2(r_1 - 1) + 6(2r_1 - N - i - 2) - 2(r_1 - 3)(F_2(r_1 - 1) + 2)\sum_{j=i}^{r_1-2} \frac{1}{j - i} \geq 0,\ 2 \leq i \leq N - 1\}. \tag{3.11}$$

Proof. It is sufficient to show that B_2 is closed. This statement is equivalent to showing that, if there exists an integer r_2 such that $\Phi_1(r_2) \geq 0$, then $\Phi_1(i)$ is also non-negative for any $i \geq r_2$. Let $\Phi_1'(i) \equiv N(N-1)(N-2)\Phi_1(i)$. We have for $i \geq r_1 - 1$,

$$\Phi_1'(i) - \Phi_1'(i+1) = 9(\frac{5}{3} - \sum_{j=i+1}^{N-1} \frac{1}{j-2}) \geq 0, \tag{3.12}$$

by the definition of r_1. Therefore $\Phi_1(i)$ is a non-increasing function for $r_1 - 1 \leq i \leq N - 1$ and $\Phi_1'(N-1) = 3 > 0$. On the other hand, we have for $3 \leq i \leq r_1 - 3$,

$$\begin{aligned}\Phi_1'(i) - \Phi_1'(i+1) &= -4\{\frac{r_1 - 3}{i - 1} - \frac{3}{2} + \frac{r_1 - 3}{2(i-1)}F_2(r_1 - 1)\} \\ &< -4\{\frac{3}{2}(\frac{r_1 - 3}{i - 1} - 1)\} < 0,\end{aligned} \tag{3.13}$$

by using the fact $F_2(r_1 - 1) > 1$. Therefore $\Phi_1(i)$ is a non-decreasing function for $3 \leq i \leq r_1 - 2$. Since $\Phi_1(2)$ can be rewritten as

$$\Phi_1(2) = \Phi_1(3) - \sum_{j=4}^{N} K(j) - 2\sum_{j=5}^{N} K(j) \tag{3.14}$$

and $K(i) > 0$ for all i, $\Phi_1(2) < 0$ if $\Phi_1(3) < 0$. Thus $\Phi_1(i)$ is proved to be unimodal. Therefore B_2 is closed and the states $(r_1, C_3, 1\ell; 1)$, $\ell = 1, 2$, belong to B_2. Thus the proof is completed. □

For this optimal second stopping policy, we obtain

$$F_4(i) = \begin{cases} \frac{1}{N(N-1)(N-2)(i-1)(i-2)}\{(r_1 - 3)(r_2 - 1 - i)(F_2(r_1 - 1) + 2) \\ \quad - (r_2 - 1 - i)(r_2 + i - 4) + (r_2 - 2)F_4(r_1, r_2)\}, & 3 \leq i \leq r_2 - 1, \\ \frac{1}{N(N-1)(N-2)(i-2)}\{(r_1 - 3)F_2(r_1 - 1) + 2(r_1 - 1 - i)\}, & r_2 \leq i \leq r_1 - 1, \\ \frac{3}{N(N-1)(N-2)} \sum_{j=i+1}^{N} \frac{1}{j-3}, & r_1 \leq i \leq N - 1, \end{cases} \tag{3.15}$$

where

$$\begin{aligned}F_4(r_1, r_2) \equiv\ & 2(r_1 - 3)(F_2(r_1 - 1) + 2)\sum_{j=r_2}^{N} \frac{1}{j-2} + 2(3N - 5r_1 + 2r_2 + 3) \\ & - 2(r_1 - 3)F_2(r_1 - 1).\end{aligned} \tag{3.16}$$

Let us consider the optimal first stopping time. We have the OLA stopping region for $(R5)$,

$B_3 \equiv (N;3)$

$$\cup \left\{ (i;3) \left| \begin{array}{ll} \dfrac{2}{N(N-1)(N-2)} + F_2(3) + \dfrac{1}{2}V_2(2,C_2) \geq \displaystyle\sum_{j=2}^{N-1} \sum_{k=j+1}^{N} [F_3(k)+F_4(k)], i=1 \\ \displaystyle\sum_{j=i+1}^{N} [F_3(j)+F_4(j)] \geq \frac{1}{i-1} \sum_{j=i+1}^{N-1} \sum_{k=j+1}^{N} [F_3(k)+F_4(k)], \quad 2 \leq i \leq N-1 \end{array} \right. \right\} \tag{3.17}$$

and B_3 can be rewritten as

$$B_3 = (N;3) \cup \{(i;3) \mid \Phi_2(i) \geq 0, 1 \leq i \leq N-1\}, \tag{3.18}$$

where

$$\Phi_2(i) = \begin{cases} \dfrac{2}{N(N-1)(N-2)} + F_2(3) + V_2(2,C_2) + \displaystyle\sum_{j=2}^{N} (2-j)[F_3(j)+F_4(j)], i=1 \\ \displaystyle\sum_{j=i+1}^{N} (2i-j)[F_3(j)+F_4(j))], \quad 2 \leq i \leq N-1. \end{cases} \tag{3.19}$$

Theorem 3.
The optimal first stop is made at the earliest C_1 after r_3, where

$$r_3 = \min\{i \mid \sum_{j=i+1}^{r_2-1} (2i-j)[F_3(j)+F_4(j)] + 2\sum_{j=r_2}^{n} (2i-j)F_3(j) \geq 0, 1 \leq i \leq N-1\}. \tag{3.20}$$

Proof. Let $\Phi_2'(i) \equiv N(N-1)(N-2)\Phi_2(i)$. By (3.7) and (3.17), we have

$$\Phi_2'(i) = \begin{cases} 6\displaystyle\sum_{j=i+1}^{N} (2i-j) \sum_{k=j+1}^{N} \frac{1}{k-3}, & r_1 - 1 \leq i \leq N-1, \\ 2\displaystyle\sum_{j=i+1}^{r_1-1} \frac{(2i-j)}{j-2}\{(r_1-3)F_2(r_1-1) + 2(r_1-1-j)\} & \\ \quad + 6\displaystyle\sum_{j=r_1}^{N-1} (2i-j) \sum_{k=j+1}^{N} \frac{1}{k-3}, & r_2 - 1 \leq i \leq r_1 - 2, \\ \displaystyle\sum_{j=i+1}^{r_2-1} \frac{(2i-j)}{j-2}\{(r_1-3)(r_2-2)(F_2(r_1-1)+2) - (r_2-1-j)(r_2+i-4) & \\ \quad + (r_2-2)F_4(r_1,r_2) - 2(j-1)(j-2)\} & \\ \quad\quad + 2\displaystyle\sum_{j=r_2}^{r_1-1} \frac{(2i-j)}{j-2}\{(r_1-3)F_2(r_1-1) + 2(r_1-1-j)\} & \\ \quad\quad + 6\displaystyle\sum_{j=r_1}^{N-1} (2i-j) \sum_{k=j+1}^{N} \frac{1}{k-3}, & 2 \leq i \leq r_2 - 2. \end{cases} \tag{3.21}$$

It is sufficient to prove that B_3 is closed. For $r_1 - 1 \le i \le N-2$, we have

$$\Phi_2'(i) - \Phi_2'(i+1) = 3\{9(i-1)\sum_{j=i+1}^{N-1}\frac{1}{j-2} - 6(N-i-1)\} \ge 0, \tag{3.22}$$

by using the fact $\Phi_1(i) \ge 0$ for $r_1 - 1 \le i \le N-1$. Therefore $\Phi_2(i)$ is a non-increasing function for $r_1 - 1 \le i \le N-1$ and $\Phi_2(N-1) = 0$.

For $r_2 - 1 \le i \le r_1 - 3$, we have

$$\begin{aligned}&\Phi_2'(i) - \Phi_2'(i+1)\\ &= -4(r_1-3)(F_2(r_1-1)+2)\sum_{j=i+1}^{r_1-2}\frac{1}{j-1} + 6(r_1-3)F_2(r_1-1) + 12(2r_1-N-3-i) \ge 0,\end{aligned} \tag{3.23}$$

by the definition of r_2. And we have

$$\Phi_2'(r_1-2) - \Phi_2'(r_1-1) = 2\{9(r_1-3)\sum_{j=r_1-1}^{N-1}\frac{1}{j-2} - 6(N-r_1+1)\} \ge 0, \tag{3.24}$$

by using the fact $\Phi_1(r_1-1) \ge 0$. Therefore $\Phi_2(i)$ is a non-increasing function for $r_2 - 1 \le i \le N-1$.

We show that $\Phi_2(i)$ is a non-decreasing function for $2 \le i \le r_2 - 2$. We have for $3 \le i \le r_2 - 2$

$$\Phi_2(i-1) - \Phi_2(i) = (i-2)[F_3(i) + F_4(i)] - 2\sum_{j=i+1}^{N}[F_3(j) + F_4(j)]. \tag{3.25}$$

Since in state $(i, C_2; 2)$ the decision of continuing is optimal for $3 \le i \le r_2 - 2$ by Theorem 2, we have

$$(i-2)F_4(i) = \sum_{j=i+1}^{N}[F_3(j) + F_4(j)], \tag{3.26}$$

and

$$(i-2)F_3(i) \le \sum_{j=i+1}^{N}[F_3(j) + F_4(j)]. \tag{3.27}$$

Then by (3.25), (3.26) and (3.27) we have

$$\Phi_2(i-1) - \Phi_2(i) \le 0,$$

Therefore, $\Phi_2(i)$ is a non-decreasing function for $2 \le i \le r_2 - 2$. We need to show that if $\Phi_2(2) < 0$ then $\Phi_2(1) < 0$. We show this in the case $r_2 \ge 3$. By using the fact $V_2(2, C_2) = 2\sum_{j=3}^{N}[F_3(j) + F_4(j)] \ge \frac{4}{N(N-1)(N-2)} + 2F_2(3)$, we obtain

$$\Phi_2(1) = \Phi_2(2) - \sum_{j=3}^{N}[F_3(j) + F_4(j)] + \frac{2}{N(N-1)(N-2)} + F_2(3) < 0.$$

Thus the proof is completed. □

The proof for the case $r_2 = 2$ is similar and will be omitted.

We are able to calculate the value, $V_3(1)$, which is the probability of win under the optimal stopping policy, by using (3.3), (3.7) and (3.15).

4. Selecting three absolute bests—Asymptotic behaviors

In this section several asymptotic behaviors are considered.

Theorem 4.

Let $d_i = \lim_{N\to\infty} r_i/N,\ i = 1,2,3$, then we obtain

$$d_1 = e^{-\frac{1}{3}}, \tag{4.1}$$

and $d_2 \cong 0.437$, which is the unique root x of the equation

$$d_1^{-1}(1+x) - \log x = \frac{17}{6}, \tag{4.2}$$

and $d_3 \cong 0.1668$, which is the unique root x of the equation

$$-\log x = (\frac{3}{2}x^2 - 6d_1 x + 6d_1 d_2 - 3d_2^2 - c)/(6d_1 d_2 - 3d_2^2), \tag{4.3}$$

where $c = \frac{3}{2}d_1^2 - 29d_1 d_2 + \frac{21}{2}d_2^2 + 6d_2 + \frac{3}{2} - 3d_2^2 \log d_2$.

The asymptotic value of the maximum probability of win is given by

$$\lim_{N\to\infty} V_3(1) = d_3^3 - 3d_1 d_3^2 + (6d_1 d_2 - 3d_2^2)d_3 \cong 0.163. \tag{4.4}$$

Proof. The asymptotic values d_1, d_2 and d_3 are straightforward by (3.1), (3.11) and (3.26), respectively. $F_5(i)$ is obtained by Theorem 3, but the complete form of $F_5(i)$ need not to be obtained,

$$F_5(i) = \begin{cases} \dfrac{1}{i-1}\displaystyle\sum_{j=i+1}^{N} F_5(j), & 2 \le i \le r_3 - 1 \\ \displaystyle\sum_{j=i+1}^{N} [F_3(j) + F_4(j)], & r_3 \le i \le N-1. \end{cases} \tag{4.6}$$

Therefore we obtain for $2 \le i \le r_3 - 1$,

$$F_5(i) = \frac{(r_3-2)(r_3-1)}{(i-1)i} F_5(r_3 - 1), \tag{4.7}$$

and for large N, by $(R5)$

$$V_3(1) = 2F_5(2) = (r_3 - 1)(r_3 - 2)F_5(r_3 - 1). \tag{4.8}$$

Noting that

$$(r_3 - 2)F_5(r_3 - 1) = \sum_{j=r_3}^{N-1} \sum_{k=j+1}^{N} [F_3(k) + F_4(k)], \tag{4.9}$$

we can obtain by (3.7), (3.15), (4.8) and (4.9)

$$
\begin{aligned}
V_3(1) = \frac{3}{N(N-1)(N-2)}\{ & \sum_{j=r_3}^{r_1-1} \frac{j-r_3}{j-2}[(r_1-3)F_2(r_1-1)+2(r_1-1-j)] \\
& + \sum_{j=r_3}^{r_2-1} \frac{j-r_3}{(j-1)(j-2)}[(r_1-3)(r_2-1-j)(F_2(r_1-1)+2) \\
& \qquad - (r_2-1-j)(r_2-4+j)+(r_2-2)F_4(r_1,r_2)] \\
& + \sum_{j=r_2}^{r_1-1} \frac{j-r_3}{j-2}[(r_1-3)F_2(r_1-1)+2(r_1-1-j)] \\
& + 6\sum_{j=r_1}^{N-1}(j-r_3)\sum_{k=j+1}^{N}\frac{1}{k-3}\}.
\end{aligned}
\tag{4.10}
$$

Therefore (4.4) is obtained by (4.10). □

The following Table 1 gives r_1, r_2, r_3 and $V_3(1)$ for some values of N.

Table 1

N	r_1	r_2	r_3	$V_3(1)$	N	r_1	r_2	r_3	$V_3(1)$
5	5	3	2	0.4333	100	73	45	17	0.1667
10	8	5	2	0.2343	200	145	88	34	0.1646
15	12	7	3	0.1957	300	216	132	51	0.1639
20	16	10	4	0.1859	400	288	176	67	0.1635
25	19	12	5	0.1806	500	359	219	84	0.1633
30	23	14	6	0.1770	1000	718	438	167	0.1630
35	26	16	6	0.1749					
40	30	18	7	0.1734					
45	33	21	8	0.1721					
50	37	23	9	0.1712					

5. The other two problems with three stops

5.1 Selecting the best with three stops

We define that the process is in state $(i;s)$, $s=1,2,3$, if the decision maker still has s stops left to be made and the i-th object is C_1 and he is facing this C_1. Let $Vs(i)$, $s=1,2,3$, be the maximum probability of "win" under an optimal policy when starting from the state $(i;s)$. When the objective is fulfilled, we call this event "win".

We have the following set of DP equations.

$$
Vs(i) = \max\{\sum_{j=i+1}^{N}\frac{i}{j(j-1)}Vs-1(j)+\frac{i}{N}, \sum_{j=i+1}^{N}\frac{i}{j(j-1)}Vs-1(j)\},
$$

$$
s \le i \le N-1,\ Vs(N)=1,\ s=1,2,3,\ V_0(j)\equiv 0. \tag{5.1}
$$

These equations can be solved numerically and give the optimal stopping policy and the probability of "win" for any N.

We have the following optimal policy for any N:

1st stop should be made at the earliest C_1 after r_3.
2nd stop should be made either at the earliest C_1 after the 1st stop or at the earliest C_2 after r_2.
3rd stop should be made either at the earliest C_1 after the 2nd stop or at the earliest C_3 after r_1.
Table 2 gives the values r_1, r_2, r_3 and $Pr(\text{win}) = V_3(1)$ for some N.

Table 2

N	r_1	r_2	r_3	$V_3(1)$	N	r_1	r_2	r_3	$V_3(1)$
5	3	2	1	0.9083	100	38	23	15	0.7385
10	4	3	2	0.8055	200	74	45	29	0.7353
15	6	4	3	0.7769	300	111	68	43	0.7342
20	8	5	3	0.7644	400	148	90	57	0.7340
25	10	6	4	0.7588	500	185	112	71	0.7334
30	12	7	5	0.7540	1000	369	224	142	0.7327
35	14	8	6	0.7502					
40	16	10	6	0.7483					
45	17	11	7	0.7466					
50	20	12	8	0.7450					

These results show the following order relations

$$r_3^N \le r_2^N \le r_1^N,$$

and $Pr(\text{win}) = V_3^N(1)$ is a decreasing function of N, which hold for any N whenever a set of DP equations is available with the aid of a computer.

5.2 Selecting the two absolute bests with three stops

We define the state $(i, C_k, j,; s)$, $k = 1, 2$, $j = 0, 1, \cdots, 3-s$, $s = 1, 2, 3$, to mean that: (1) the decision maker is facing C_k at the i-th period, (2) he has already had j relatively best objects among the accepted ones up to this period i, and (3) he has still s left to be made. We denote by $Vs(i, C_k, j)$ the maximum probability of "win" under an optimal policy when starting from the state $(i, C_k, j; s)$. Then we have the following set of DP equations.

$$V_1(i, C_1, 0) = 0, \qquad 3 \le i \le N. \tag{5.2}$$

$$V_1(i, C_2, 0) = 0, \qquad 3 \le i \le N. \tag{5.3}$$

$$\begin{aligned} V_1(i, C_1, 1) &= \max\{\frac{i(i-1)}{N(N-1)}, \sum_{j=i+1}^{N} \frac{i(i-1)}{j(j-1)(j-2)}[\sum_{k=1}^{2} V_1(j, C_k, 0)]\} \\ &= \frac{i(i-1)}{N(N-1)}, \qquad 3 \le i \le N. \end{aligned} \tag{5.4}$$

$$\begin{aligned} V_1(i, C_2, 1) = \max\{\frac{i(i-1)}{N(N-1)}, \sum_{j=i+1}^{N} \frac{i(i-1)}{j(j-1)(j-2)}[\sum_{k=1}^{2} V_1(j, C_k, 1)]\}, \\ 3 \le i \le N-1, \ V_1(N, C_2, 1) = 1. \end{aligned} \tag{5.5}$$

$$V_1(i,C_1,2)=\max\{\frac{i(i-1)}{N(N-1)},\sum_{j=i+1}^{N}\frac{i(i-1)}{j(j-1)(j-2)}[\sum_{k=1}^{N}V_1(j,C_k,0)]\}$$
$$=\frac{i(i-1)}{N(N-1)},\qquad 3\le i\le N\ . \tag{5.6}$$

$$V_1(i,C_2,2)=\max\{\frac{i(i-1)}{N(N-1)},\sum_{j=i+1}^{N}\frac{i(i-1)}{j(j-1)(j-2)}[\sum_{k=1}^{2}V_1(j,C_k,1)]\},$$
$$3\le i\le N-1,\ V_1(N,C_2,2)=1\ . \tag{5.7}$$

$$V_2(i,C_1,0)=\max\{\sum_{j=i+1}^{N}\frac{i(i-1)}{j(j-1)(j-2)}[\sum_{k=1}^{2}V_1(j,C_k,1)],$$
$$\sum_{j=i+1}^{N}\frac{i(i-1)}{j(j-1)(j-2)}[\sum_{k=1}^{2}V_2(j,C_k,0)]\}\ ,$$
$$2\le i\le N-1,\ V_2(N,C_1,0)=0\ . \tag{5.8}$$

$$V_2(i,C_2,0)=\max\{\sum_{j=i+1}^{N}\frac{i(i-1)}{j(j-1)(j-2)}[\sum_{k=1}^{2}V_1(j,C_k,0)],$$
$$\sum_{j=i+1}^{N}\frac{i(i-1)}{j(j-1)(j-2)}[\sum_{k=2}^{2}V_2(j,C_k,0)]\}\ ,$$
$$2\le i\le N-1,\ V_2(N,C_2,0)=0\ . \tag{5.9}$$

$$V_2(i,C_1,1)=\max\{\sum_{j=i+1}^{N}\frac{i(i-1)}{j(j-1)(j-2)}[\sum_{k=1}^{2}V_1(j,C_k,2)]+\frac{i(i-1)}{N(N-1)},$$
$$\sum_{j=i+1}^{N}\frac{i(i-1)}{j(j-1)(j-2)}[\sum_{k=1}^{2}V_2(j,C_k,0)]\}\ ,$$
$$2\le i\le N-1,\ V_2(N,C_1,1)=1\ . \tag{5.10}$$

$$V_2(i,C_2,1)=\max\{\sum_{j=i+1}^{N}\frac{i(i-1)}{j(j-1)(j-2)}[\sum_{k=1}^{2}V_1(j,C_k,2)]+\frac{i(i-1)}{N(N-1)},$$
$$\sum_{j=i+1}^{N}\frac{i(i-1)}{j(j-1)(j-2)}[\sum_{k=1}^{2}V_2(j,C_k,1)]\}\ ,$$
$$2\le i\le N-1,\ V_2(N,C_2,1)=1\ . \tag{5.11}$$

$$V_3(i,C_1,0)=\max\{\sum_{j=i+1}^{N}\frac{i(i-1)}{j(j-1)(j-2)}[\sum_{k=1}^{2}V_2(j,C_k,1)],$$
$$\sum_{j=i+1}^{N}\frac{i(i-1)}{j(j-1)(j-2)}[\sum_{k=1}^{2}V_3(j,C_k,0)]\}\ ,$$
$$2\le i\le N-1,\ V_3(N,C_1,0)=0\ . \tag{5.12}$$

$$V_3(1,C_1,0)=\max\{\frac{1}{2}[V_2(2,C_1,1)+V_2(2,C_2,1)],\sum_{j=2}^{N}\frac{1}{j(j-1)}V_3(j,C_1,0)\}.$$

$$V_3(i,C_2,0)=\max\{\sum_{j=i+1}^{N}\frac{i(i-1)}{j(j-1)(j-2)}[\sum_{k=1}^{2}V_2(j,C_k,0)],$$

$$(\sum_{j=i+1}^{N} \frac{i(i-1)}{j(j-1)(j-2)}[\sum_{k=1}^{2} V_3(j,C_k,0)]\}),$$
$$(2 \le i \le N-1,\ V_3(N,C_2,0)=0). \quad (5.13)$$

These equations give both the optimal stopping policy and the probability of win for any N with the aid of a computer.

We have the following optimal stopping policy for any N:

1st stop should be made at the earliest C_1 after r_5.

2nd stop should be made either at the earliest C_1 after r_3, or at the earliest C_2 after r_4 if $j=1$, and should be made at the earliest C_2 after r_2 if $j=0$.

3rd stop should be made either at the earliest C_1 after the 2nd stop, or at the earliest C_2 after r_1.

Table 3 gives the values, r_1, r_2, r_3, r_4, r_5 and $Pr(\text{win}) = V_3(1, C_1, 0)$ for some N.

Table 3

N	r_5	r_3	r_2	r_4	r_1	$V_3(1)$
5	2	2	2	3	4	0.6500
10	2	3	3	6	7	0.5044
15	3	4	4	8	10	0.4683
20	4	4	5	11	13	0.4514
25	5	5	6	13	16	0.4418
30	5	6	8	15	19	0.4369
35	6	7	9	18	22	0.4335
40	7	8	10	20	25	0.4306
45	8	9	11	23	28	0.4282
50	9	10	12	25	31	0.4262
100	17	19	24	49	62	0.4185
200	32	38	46	98	122	0.4148
300	48	57	69	146	183	0.4135
400	64	75	92	195	244	0.4129
500	80	94	115	244	304	0.4126
100	160	187	230	486	608	0.4118

After computational working, we have the following order relations

$$r_5^N \le r_3^N \le r_2^N \le r_4^N \le r_1^N,$$

and $Pr(\text{win}) = V_3^N(1, C_1, 0)$ is a decreasing function of N, which hold for any N whenever a set of DP equations is available with the aid of a computer.

7. Remarks

This paper indicates the possibility that the problem of recognizing absolutely ℓ best objects with m stops $(1 \le \ell \le m \le N)$ is solved by either analytical or computational approach. If we derive a set of DP equations for such problem (each ℓ and m), then by using OLA policy we will obtain the solution, or numerical solutions of the DP equations will give both the optimal stopping policy and the probability of win for any N.

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