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UNIFORM MONOTONICITY OF MARKOV PROCESSES AND ITS RELATED PROPERTIES

Masaaki Kijima
The University of Tsukuba, Tokyo

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Abstract A real valued discrete time Markov process $\{X_n\}$ is defined to be uniformly monotone in the negative (positive) direction if $P(x,y) = Pr\{X_{n+1} \le y \mid X_n = x\}(\overline{P}(x,y) = Pr\{X_{n+1} \ge y \mid X_n = x\}$, respectively) is totally positive of order 2 in $-\infty < x, y < \infty$ (see Shaked and Shanthikumar (1988) and Shanthikumar (1988)). The uniform monotonicity is a stronger notion than the ordinary stochastic monotonicity. A monotonicity theorem of the same form as in Daley (1968) is first established. Based on this theorem, uniform monotonicity as well as IFR and DFR properties in the Lindley waiting time processes, Markov jump processes and the associated counting processes is discussed. Uniform comparison of (random) sums of random variables is also made. Finally, some applications are given to demonstrate a potential use of this study.

1 Introduction

In the two decades, a series of papers was published in which monotonicity and comparability of random processes with respect to some partial orderings were studied, see e.g. [3, 7, 15, 20]. The results obtained there have been successfully used in the study of applied probability such as queueing theory [4, 5, 20] and reliability theory [20]. Partial orderings being considered in the literature are mostly the stochastic ordering $\prec_d$, the convex ordering $\prec_c$ and the concave ordering $\prec_{cv}$ (see e.g. [20] for the definitions and properties). Other important orderings are the uniform orderings $\prec_{(-)}$ and $\prec_{(+)}$ which are introduced in Whitt [21] and further studied in Keilson and Sumita [11]. These orderings are potentially useful notions, and indeed they are now quite frequently used in applications, see e.g. [13, 16, 21, 22].

When the process of interest is a temporally homogeneous Markov process, the monotonicity is stated in terms of the Markov operator T governing the process. In [3], Daley introduced the notion of stochastically monotone Markov operators for the ordering $\prec_d$. Then, as the monotonicity theorem, he showed that $TP_1 \prec_d TP_2$ for any pair of probability measures P_1 and P_2 with $P_1 \prec_d P_2$ if and only if (iff) T is stochastically monotone. For the orderings $\prec_c$ and $\prec_{cv}$, theorems of similar type have been built. For the uniform ordering, Shaked and Shanthikumar [18] and Shanthikumar [19] have introduced Markov processes that are increasing with respect to the ordering $\prec_{(+)}$. Our first task is therefore to give a definition of uniformly monotone Markov operators and to reformulate their result as a monotonicity theorem of the uniform orderings. These together with some related topics are discussed in Section 3. Section 2 is devoted to some preliminaries.

Based on the results in Sections 2 and 3, we then consider particular Markov processes. The Lindley waiting time process is a spatially homogeneous Markov process modified by a retaining boundary at zero. A sufficient condition under which the Lindley process is uniformly monotone is derived in Section 4. The DFR property of the number served during a busy period is also given there as a byproduct. In Section 5, Markov jump processes and the associated counting processes are considered. Especially, we are interested in uniform monotonicity and the IFR property of the counting processes. In Section 6, uniform comparison of two counting processes associated with sums of independent random variables (rv's) is established. The final section is concerned with uniform comparison of two random sums of identically and independently distributed (iid) rv's. Examples are given to show a potential use of our results.

2 Some Preliminaries

Let X be an rv with cumulative distribution function (cdf) $F_X(x) = \Pr[X \le x]$, $x \in R = (-\infty, \infty)$. The survival probability function (spf) is defined as $\overline{F}_X(x) = \Pr[X \ge x]$, $x \in R$. $\overline{F}_X(x)$ is then left continuous. Note the difference from the ordinary definition of spf's.

The uniform orderings are first considered in Whitt [21] and the definition below is given in Keilson and Sumita [11].

Definition 2.1. Let X and Y be rv's having cdf's $F_X(x)$ and $F_Y(x)$ respectively. X is said to be uniformly smaller than Y in the negative (positive respectively) direction if $F_X(x)F_Y(y) \ge F_X(y)F_Y(x)$ $(\overline{F}_X(x)\overline{F}_Y(y) \ge \overline{F}_X(y)\overline{F}_Y(x))$ for all $x < y$. In this case, we write either $X \prec_{(-)} Y$ or $F_X \prec_{(-)} F_Y$ $(X \prec_{(+)} Y$ or $F_X \prec_{(+)} F_Y)$, according to convenience.

Remark 2.1. (i) The definition of $X \prec_{(+)} Y$ above is slightly different from the one given in [11], since we use an unusual definition of spf's. If F_X and F_Y are absolutely continuous then, of course, they agree with each other.

(ii) Since, by the definition of spf's, $\overline{F}_{-X}(-x) = F_X(x)$, it follows that $X \prec_{(-)} Y$ iff $-Y \prec_{(+)} -X$. Hence $\prec_{(-)}$ and $\prec_{(+)}$ have a dual relation. In what follows, we study one of either $\prec_{(-)}$ or $\prec_{(+)}$ only. The results for $\prec_{(-)}$ are easily converted to the ordering $\prec_{(+)}$ via the duality and vice versa.

(iii) Let $\prec_d$ denote the ordinary stochastic ordering, i.e. $X \prec_d Y$ iff $F_X(x) \ge F_Y(x)$ for all $x \in R$. $\prec_d$ is weaker than both $\prec_{(-)}$ and $\prec_{(+)}$, see Keilson and Sumita [11].

Suppose $F_X(x)$ is absolutely continuous having a probability density function (pdf) $f_X(x)$.

Define

$$r_X(x) = \frac{f_X(x)}{\overline{F}_X(x)}; \quad h_X(x) = \frac{f_X(x)}{F_X(x)}, \quad -\infty < x < \infty, \tag{2.1}$$

whenever $\overline{F}_X(x) > 0$ and $F_X(x) > 0$ respectively. As we shall see below, $r_X(x)$ and $h_X(x)$ clarify the meaning of the uniform orderings. We note that if $\Pr[X \geq 0] = 1$ then $r_X(x)$ is the ordinary failure rate of X, see e.g. [1]. $h_X(x)$ is not commonly used in probability theory. It appears naturally, however, in considering e.g. the maximum of independent rv's. To see this, let $Z = \max\{X, Y\}$, where X and Y are mutually independent. It is easy to see that $h_Z(x) = h_X(x) + h_Y(x)$. Similarly, one has $r_Z(x) = r_X(x) + r_Y(x)$ where in turn $Z = \min\{X, Y\}$. The ordering $\prec_{(+)}$ is the ordering of $r_X(x)$ as pointed out by Pinedo and Ross [16]. That is, $X \prec_{(+)} Y$ iff $r_X(x) \geq r_Y(x)$ for all $x \in R$. For, $X \prec_{(+)} Y$ iff $\overline{F}_X(x+t)/\overline{F}_X(x) \leq \overline{F}_Y(x+t)/\overline{F}_Y(x)$ for any $t > 0$ and $x \in R$. It can be similarly shown that $X \prec_{(-)} Y$ iff $h_X(x) \leq h_Y(x)$ for all $x \in R$.

We next define two classes of cdf's of interest (see Kijima [13]).

Definition 2.2. A function $f(x,y)$ of two variables defined on $R \times R$ is totally positive of order 2 in $-\infty < x, y < \infty$ (denote $f \in TP_2$) if $f(x_1,y_1)f(x_2,y_2) \geq f(x_1,y_2)f(x_2,y_1)$ for all $x_1 < x_2$ and $y_1 < y_2$ (see Karlin [8]). An rv X having cdf $F_X(x)$ is said to be Pólya frequency of order 2 (write $X \in PF_2$ or $F_X \in PF_2$) if $F_X(x-y) \in TP_2$ in $-\infty < x, y < \infty$. X is called IFR (write $X \in IFR$ or $F_X \in IFR$) if $\overline{F}_X(x-y) \in TP_2$.

When $\Pr[X \geq 0] = 1$ and it has a pdf, the above definition for IFR agrees with the ordinary definition. The class of IFR distributions is widely used in applied probability because of the probabilistic meaning of its own. The class of PF_2 distributions seems not common. However, it plays an important role in the subsequence.

Remark 2.2. (i) As in Remark 2.1(ii), one easily sees that $X \in IFR$ iff $-X \in PF_2$. Thus, the classes of PF_2 and IFR distributions are in a dual relation.

(ii) Let $F \in IFR$. As is well known (see e.g. p26 of [1]), $F(x)$ is logconcave in $\{x : \overline{F}(x) > 0\}$ so that $F(x)$ is absolutely continuous except possibly at the right hand end point of the interval of support of $F(x)$. Thus, if $F(x)$ has a full support on R then it has a pdf. If $F \in PF_2$, the dual relation stated above leads to the conclusion that $F(x)$ is absolutely continuous except possibly at the left hand end point of its support.

(iii) Let $F_i(y) = F(y - x_i)$ $(i = 1,2)$ with $x_1 < x_2$ where $F(x)$ is a cdf. It is then easy to see that $F_1 \prec_{(-)} F_2$ iff $F \in PF_2$. Similarly, $F_1 \prec_{(+)} F_2$ iff $F \in IFR$. Thus the uniform orderings $\prec_{(-)}$ and $\prec_{(+)}$ characterize the classes of PF_2 and IFR distributions, respectively.

(iv) If X has a pdf, then the term "IFR" is understood through the failure rate function $r_X(x)$ in (2.1). It is readily seen that $X \in IFR$ iff $r_X(x)$ is increasing in x (in this paper, the terms "increasing" and "decreasing" are used in the weak sense). Similarly, $X \in PF_2$ iff $h_X(x)$ is decreasing in x.

3 Uniformly Monotone Markov Processes

We consider a temporally homogeneous discrete time Markov process $\{X_n;\ n \geq 1\}$ on the state space R. The one-step transition function $q(x, A)$ governing the Markov process is assumed to be $\boldsymbol{B}$-measurable in $x \in R$ and a probability measure on R in $A \in \boldsymbol{B}$, where $\boldsymbol{B}$ is the ordinary Borel field. For each fixed $x \in R$, the transition function defines transition cdf $P(x, y) = q(x, (-\infty, y]) = \Pr[X_{n+1} \leq y \mid X_n = x]$. As for spf's, the transition spf is similarly defined and is denoted by $\overline{P}(x, y) = q(x, [y, \infty)) = \Pr[X_{n+1} \geq y \mid X_n = x]$.

The transition cdf $P(x, y)$ defines a Markov operator T in a usual way. Let $F(x)$ be a cdf on R. Define

$$(TF)(y) = \int_{-\infty}^{\infty} P(x, y) F(dx). \tag{3.1}$$

The n-folded operator T^n is defined by $T^n F = T(T^{n-1}F)$ with $T^0 F = F$. The spf of TF is given by

$$\overline{(TF)}(y) = \int_{-\infty}^{\infty} \overline{P}(x, y) F(dx). \tag{3.2}$$

Here and hereafter, an integral is considered as a Riemann-Stieltjes integral. If we write $(TF)(y)$, this means that the transition cdf $P(x, y)$ is assumed to be Riemann-Stieltjes integrable with respect to $F(x)$ for fixed y. The integral exists, for example, if $P(x, y)$ is continuous with respect to x. The Riemann-Stieltjes integrability is the key in our analyses since it makes integration by parts possible under certain circumstances.

Definition 3.1. A Markov operator T is called uniformly monotone in the negative (positive respectively) direction, which we denote by $T \in UM(-)$ ($T \in UM(+)$), if $P(x, y) \in TP_2$, ($\overline{P}(x, y) \in TP_2$). If either $T \in UM(-)$ or $T \in UM(+)$, we say that T is uniformly monotone.

Remark 3.1. (i) For a Markov process $\{X_n;\ n \geq 0\}$ governed by T, consider the process $\{Y_n\}$ such that $Y_n \stackrel{\mathrm{d}}{=} -X_n$ ($\stackrel{\mathrm{d}}{=}$ stands for equality in law). This process is discussed also in [18, 19]. If T has transition cdf $P(x, y)$, $\{Y_n\}$ is governed by $\overline{P}(-x, -y)$ since $\Pr[Y_{n+1} \leq y | Y_n = x] = \Pr[X_{n+1} \geq -y | X_n = -x]$ by definition. Let $(-T)$ be the Markov operator governing $\{Y_n\}$. It is then easy to see that $T \in UM(+)$ iff $(-T) \in UM(-)$.

(ii) In [3], Daley introduced a stochastically monotone operator T. T is said to be stochastically monotone (write $T \in SM$) if $P(x_1, y) \geq P(x_2, y)$ for all $x_1 < x_2$ and for all $y \in R$. Suppose $T \in UM(-)$, i.e., $P(x_1, y)P(x_2, y') \geq P(x_1, y')P(x_2, y)$ for $x_1 < x_2$ and $y < y'$. By letting $y' \to \infty$, one sees that $P(x_1, y) \geq P(x_2, y)$. Hence, uniform monotonicity is a stronger notion than stochastic monotonicity.

As stated in the introductory section, the monotonicity theorem for stochastically monotone Markov operators of Daley [3] is that $TF_1 \prec_d TF_2$ for every pair of cdf's F_1 and F_2 with $F_1 \prec_d F_2$ iff $T \in SM$. The next theorem states that the monotonicity theorem also holds for uniformly monotone Markov operators.

Theorem 3.1. $TF_1 \prec_{(-)} TF_2$ for any cdf's F_1 and F_2 such that $F_1 \prec_{(-)} F_2$ iff $T \in UM(-)$.

Proof. To prove necessity, let $x_1 < x_2$ and let $F_i(x) = U(x - x_i)$ $(i = 1,2)$ where $U(x) = 1$ for $x \geq 0$ and $U(x) = 0$ for $x < 0$. It is then evident that $F_1 \prec_{(-)} F_2$. For these cdf's, one has $TF_1 = P(x_1, \cdot) \prec_{(-)} P(x_2, \cdot) = TF_2$. Thus, when $y_1 < y_2$, $P(x_1,y_1)P(x_2,y_2) \geq P(x_1,y_2)P(x_2,y_1)$, which is desired. The sufficiency follows by mimicking the proof of Corollary 2.3 in Shaked and Shanthikumar [18], where the Riemann-Stieltjes integrability ensures the integration by parts (see Kijima [12] for details). □

Applications of Theorem 3.1 yield the following corollaries.

Corollary 3.1. If $T \in UM(-)$ then $T^n \in UM(-)$, $n \geq 1$.

Proof. We note that $P^{(n)}(x,y) = \int_{-\infty}^{\infty} P^{(n-1)}(z,y)P(x,dz)$ with $P^{(1)}(x,y) = P(x,y)$ where $P^{(n)}(\cdot,\cdot)$ denotes the n-step transition cdf. For $x_1 < x_2$, write $P_i^{(n)}(y) = P^{(n)}(x_i,y)$ $(i = 1,2)$. Then $P_i^{(2)}(y) = (TP_i)(y)$. Since $T \in UM(-)$ so that $P_1 \prec_{(-)} P_2$, one has $P_1^{(2)} \prec_{(-)} P_2^{(2)}$ from Theorem 3.1. An inductive argument then proves the corollary. □

Corollary 3.2. Let F and G be cdf's on R and define

$$H(y) = \int_{-\infty}^{\infty} F(y-u)G(du), \quad -\infty < y < \infty. \tag{3.3}$$

Then, $H \in PF_2$ (IFR respectively) for any $G \in PF_2$ (IFR) iff $F \in PF_2$ (IFR).

Proof. Let $x_1 < x_2$ and denote $A_i(y) = A(y - x_i)$ $(i = 1,2)$ for a cdf $A(x)$. Note that $A \in PF_2$ iff $A_1 \prec_{(-)} A_2$. It is easy to see from (3.3) that

$$H_i(y) = \int_{-\infty}^{\infty} F(y - x_i - u)G(du) = \int_{-\infty}^{\infty} F(y-v)G_i(dv) \stackrel{\text{def}}{=} (TG_i)(y).$$

Further note that $F \in PF_2$ iff $T \in UM(-)$. The desired result then follows from Theorem 3.1. □

Consider a Markov process $\{X_n;\ n \geq 0\}$ on R governed by operator T having transition cdf $P(x,y)$. Suppose one restricts the state space to $R_+ = [0,\infty)$ by imposing an absorbing boundary at 0. Let

$$P_a(x,y) = \begin{cases} 0, & y < 0, \\ 1, & x \leq 0 \text{ and } y \geq 0, \\ P(x,y), & \text{otherwise.} \end{cases} \tag{3.4}$$

Denoting by ${}_aT$ the operator corresponding to $P_a(x,y)$, it is easily verified from (3.4) that if $T \in UMj$ $(j = (-),(+))$ then so is ${}_aT$. An absorbing Markov process whose state space is $R_- = (-\infty, 0]$ where state 0 is absorbing is similarly obtained from $\{X_n\}$. The preservation of uniform monotonicity in the both directions for this case is also readily checked. Hence uniform monotonicity is preserved under a modification by imposing absorbing boundaries. Suppose in turn that the state space is restricted to R_+ by imposing a retaining boundary at 0. Define now

$$P_r(x,y) = \begin{cases} 0, & y < 0, \\ 1, & x < 0 \text{ and } y \geq 0, \\ P(x,y), & \text{otherwise.} \end{cases} \tag{3.5}$$

Note the difference between (3.4) and (3.5). In (3.5), $P_r(0,y) = P(0,y)$ for $y \geq 0$ so that state "0" is not absorbing. Noting this difference, it is obvious that uniform monotonicity is also preserved by placing retaining boundaries.

We next consider conditional rv's X_n^+ and X_n^- of X_n given $\{X_m \in R_+,\ 0 \leq m \leq n\}$ and $\{X_m \in R_-,\ 0 \leq m \leq n\}$, respectively, i.e.

$$X_n^+ \stackrel{\mathrm{d}}{=} \{X_n \mid X_m \in R_+,\ 0 \leq m \leq n\}; \quad X_n^- \stackrel{\mathrm{d}}{=} \{X_n \mid X_m \in R_-,\ 0 \leq m \leq n\}. \tag{3.6}$$

These rv's are considered only when the conditional probabilities are not zero. Let $P^+(x,y) = P(x,y) - P(x,0-)$ for $x, y \geq 0$ and let $P^-(x,y) = P(x,y)$ for $x, y \leq 0$. The corresponding spf's are given by $\overline{P}^+(x,y) = P^+(x,\infty) - P^+(x,y-) = \overline{P}(x,y)$ for $x,y \geq 0$ and $\overline{P}^-(x,y) = P(x,0) - P(x,y-)$ for $x, y \leq 0$ respectively. We write $(T^+F)(y) = \int_0^\infty P^+(x,y)F(dx)$, $y \geq 0$, for F defined on R_+. Similarly, $(T^-F)(y) = \int_{-\infty}^0 P^-(x,y)F(dx)$, $y \leq 0$, for F defined on R_-. Note that T^+ and T^- may not be proper operators so as to define lossy processes since $(T^+F)(\infty)$ and $(T^-F)(0)$ can be strictly less than unity. Let F_n^+ be the cdf of X_n^+. The operator T^+ governs $\{X_n^+\}$ in the following manner (in what follows, we call $\{X_n^+\}$ the conditional process of $\{X_n\}$ governed by T^+). For a given F_n^+,

$$F_{n+1}^+(y) = \frac{(TF_n^+)(y) - (TF_n^+)(0-)}{1 - (TF_n^+)(0-)} = \frac{(T^+F_n^+)(y)}{(T^+F_n^+)(\infty)}, \quad y \geq 0. \tag{3.7}$$

It is then easy to see from (3.7) that $F_n^+ \prec_{(-)} F_{n+1}^+$ iff $T^+F_{n-1}^+ \prec_{(-)} T^+F_n^+$. Also, although $P^+(x,\infty)$ may be strictly less than unity, one can easily show that the content in Theorem 3.1 is still valid (see Kijima [12]). If we use the same notation even for improper operators, this means that $T^+ \in UM(-)$ iff $T^+F_0^+ \prec_{(-)} T^+F_1^+$ for any cdf's F_0^+ and F_1^+ with $F_0^+ \prec_{(-)} F_1^+$. The same results hold for T^-. It is hence of interest to know if $T \in UMj$ $(j = (-),(+))$ implies $T^+ \in UMj$. The answer is negative. Since $\overline{P}^+(x,y) = \overline{P}(x,y)$ for $x,y \geq 0$, $T \in UM(+)$ implies only $T^+ \in UM(+)$. The fact that $T \in UM(+)$ does not necessarily imply $T^+ \in UM(-)$ is readily verified (cf. Theorem 2.3 of [13]). Also, if $T \in UM(-)$ then only so is T^-. The next theorem summarizes these results.

Theorem 3.2. If $T \in UMj$ $(j = (-),(+))$ then both ${}_aT$ and ${}_rT$ are UMj. If $T \in UM(+)$ then $T^+ \in UM(+)$ and if $T \in UM(-)$ then $T^- \in UM(-)$.

Finally, we define the notion of uniformly monotone Markov processes. Recall that a Markov operator T together with an initial cdf $F_0(x)$ completely determines the stochastic behavior of a Markov process $\{X_n; n \geq 0\}$. Let $F_n(x)$ be the cdf of X_n, i.e. $F_n(x) = \Pr[X_n \leq x]$. Given $F_0(x)$, one has $F_1 = TF_0$. Suppose now $T \in UM(-)$. It is readily seen from Theorem 3.1 that if $X_0 \prec_{(-)} X_1$ then $X_n \prec_{(-)} X_{n+1}$ for all $n \geq 0$. Similarly, if $X_1 \prec_{(-)} X_0$ then $X_{n+1} \prec_{(-)} X_n$, $n \geq 0$.

Definition 3.2. A real valued Markov process $\{X_n;\ n \geq 0\}$ is said to be uniformly increasing in the negative (positive) direction, which we write as $X_n \uparrow (-)$ $(X_n \uparrow (+))$, if

$X_n \prec_{(-)} X_{n+1}$ ($X_n \prec_{(+)} X_{n+1}$) for all $n \geq 0$. If $X_n \succ_{(-)} X_{n+1}$ ($X_n \succ_{(+)} X_{n+1}$), it is called uniformly decreasing in the negative (positive) direction and is denoted by $X_n \downarrow (-)$ ($X_n \downarrow (+)$).

Of interest is uniform monotonicity of the conditional processes X_n^+ and X_n^-. Suppose $X_0 = 0$ *a.s.* Then, from Theorem 3.2, one sees that $X_n^+ \uparrow (+)$ if $T \in UM(+)$. If $T \in UM(-)$, in turn, $X_n^- \downarrow (-)$. The stochastic monotonicities of X_n^+ and X_n^- are then guaranteed, respectively. We note that, even if $T \in SM$, stochastic monotonicity of X_n^+ and that of X_n^- are not assured in general. As we shall see later, stochastic monotonicity of the conditional processes plays a key role in the study of distribution properties of the number served during a busy period in queues.

4 The Waiting Time Processes

Consider a standard GI/G/1 queueing system having inter-arrival time cdf $A(x)$ and service time cdf $B(x)$. The generic inter-arrival and service time rv's are denoted by A and B respectively. As the stability condition, we require that $E[A] > E[B]$. Let W_n denote the waiting time of the n-th customer with $W_0 = 0$ *a.s.* The cdf of W_n is designated by $W_n(x)$. Let $Q = B - A$ so that its cdf is given by

$$Q(y) = \int_{-\infty}^{\infty} B(y+x)A(dx), \quad -\infty < y < \infty. \tag{4.1}$$

The spf of $Q(y)$ is given by

$$\overline{Q}(y) = \int_{-\infty}^{\infty} \overline{B}(y+x)A(dx), \quad -\infty < y < \infty, \tag{4.2}$$

with $\overline{B}(x) = 1$ for $x < 0$. Consider the integral transformation

$$(T^H F)(y) = \int_{-\infty}^{\infty} Q(y-x)F(dx), \quad -\infty < y < \infty, \tag{4.3}$$

for F defined on R. The waiting time process $\{W_n;\ n \geq 0\}$ is the spatially homogeneous process except for a retaining boundary at 0. As in (3.5), let

$$Q_r(x,y) = \begin{cases} 0, & y < 0, \\ 1, & x < 0 \text{ and } y \geq 0, \\ Q(y-x), & \text{otherwise,} \end{cases} \tag{4.4}$$

and define

$$(TF)(y) = \int_{-\infty}^{\infty} Q_r(x,y)F(dx), \quad -\infty < y < \infty. \tag{4.5}$$

It is well known that $W_{n+1}(y) = (TW_n)(y)$, $n \geq 0$, starting with $W_0(y) = U(y)$.

In order to use the results in Section 3, one needs to have a uniformly monotone operator. The property of T^H in (4.3) should be described in terms of distribution properties of the non-negative rv's A and B.

Lemma 4.1. (i) If $A \in PF_2$ and $B \in IFR$ then $Q \in IFR$. (ii) If $A \in IFR$ and $B \in PF_2$ then $Q \in PF_2$.

Proof. To prove (i), we note that $-A \in IFR$ from Remark 2.2(ii). Corollary 2.2 then ensures $Q = B - A \in IFR$. Statement (ii) follows similarly. □

As we have seen in Section 3, if $T^H \in UMj$ $(j = (-), (+))$ then so is T in (4.5). Hence $W_n \uparrow j$ when $W_0 = 0$ *a.s.*

Theorem 4.1. Let $W_0 = 0$ *a.s.* Then, (i) if the inter-arrival time A is PF_2 and the service time B is IFR then $W_n \uparrow (+)$, and (ii) if $A \in IFR$ and $B \in PF_2$ then $W_n \uparrow (-)$.

Let N_B be the number served during a busy period in a GI/G/1 queue. If we denote by $\{W_n^H;\ n \geq 0\}$ the spatially homogeneous process governed by T^H, it is easy to see that

$$N_B = \inf\{n \geq 1 : W_n^H \in (-\infty, 0) \mid W_0^H = 0\}. \tag{4.6}$$

Suppose $A \in PF_2$ and $B \in IFR$ so that $T^H \in UM(+)$. Let $f_n = \Pr[N_B = n]$, $n \geq 1$. Denote by $r(n)$ the failure rate of N_B, i.e. $r(n) = f_n / \sum_{k \geq n} f_k$. Since the queue is stable, the distribution $(f_n)_0^\infty$ is proper. By definition, it is not hard to see that

$$r(n) = \int_0^\infty P(x, 0-) F_{n-1}^+(dx) = 1 - \int_0^\infty \overline{P}(x, 0) F_{n-1}^+(dx), \quad n \geq 1, \tag{4.7}$$

where $P(x, y) = Q(y - x)$ and $F_n^+(x)$ is the cdf of the conditional rv W_n^+ of W_n^H (see Section 3). Note that $\overline{P}(x, 0)$ is increasing with respect to x so that the integral $\int_0^\infty f(x) \overline{P}(dx, 0)$ is well defined for a bounded and monotone function $f(x)$. It then follows from integration by parts in (4.7) that

$$r(n) = 1 - \overline{P}(0,0) - \int_0^\infty \overline{F}_{n-1}^+(x) \overline{P}(dx, 0), \quad n \geq 1. \tag{4.8}$$

Hence

$$r(n+1) - r(n) = \int_0^\infty [\overline{F}_{n-1}^+(x) - \overline{F}_n^+(x)] \overline{P}(dx, 0), \quad n \geq 1. \tag{4.9}$$

In order for $r(n)$ to be decreasing, $X_{n-1}^+ \prec_d X_n^+$ is enough from (4.9). This is so, since $T^H \in UM(+)$ implies that $X_n^+ \uparrow (+)$. Thus the next theorem holds.

Theorem 4.2. Let $A \in PF_2$ and $B \in IFR$. Then $N_B \in DFR$.

Remark 4.1. For an exponential distribution $E(x) = 1 - e^{-\mu x}$, $\mu > 0$, one has $h_E(x) = \mu/(e^{\mu x} - 1)$ and $r_E(x) = \mu$. Since $E \in PF_2$ iff $h_E(x)$ is decreasing in x and $E \in IFR$ iff $r_E(x)$ is increasing in x, exponential variates are both PF$_2$ and IFR. Hence Theorem 4.2 of Kijima [13] is a special case of Theorem 4.2.

5 Pure Jump Processes and Associated Counting Processes

Let $\{X_n : n \geq 0\}$ be a Markov process defined on R_+. $\{X_n\}$ is said to be a pure jump process if $X_0 = 0$ *a.s.* and $\Pr[X_{n+1} \geq X_n] = 1$ for all $n \geq 1$. Pure jump processes often appear in shock models to describe the cumulative damages incurred by a system. Many authors have studied the first passage times of such processes into the upper set of a pre-specified level since it represents the time to failure of the system (see e.g. [2, 17, 18] and references thereof). Of interest in this section is the counting process associated with the pure jump process $\{X_n\}$ defined by

$$N(t) = \sup\{n : X_n \leq t\}, \quad t \geq 0. \tag{5.1}$$

Some special cases of such counting processes have been of great importance in the literature. For example, when X_n is a sum of iid rv's, $N(t)$ is the ordinary renewal process. If $X_n = \sum_{k=1}^{n} Y_k$ and Y_{n+1} depends only on the value of X_n, then $N(t)$ is the g-renewal process introduced in Kijima and Sumita [14]. In this section, we study IFR properties of the counting process associated with some Markov jump process. Recall that IFR distributions are characterized by uniform ordering $\prec_{(+)}$ (see Remark 2.2(iii)).

Let $\{X_n\}$ be governed by transition cdf $P(x,y)$, i.e. $P(x,y) = \Pr[X_{n+1} \leq y | X_n = x]$, $x, y \in R_+$, independent of n. Suppose $P(x,y) = 0$ for $x > y$. Then $\{X_n\}$ is a pure jump process, provided that $X_0 = 0$. When $X_n = x$, the increment $Y_{n+1} = X_{n+1} - X_n$ is distributed by $P(x, y+x)$, $y \geq 0$. Thus, the associated counting process $N(t)$ becomes the g-renewal process [14]. Suppose now $P(x,y) \in TP_2$ in $x, y \geq 0$. When $X_0 = 0$, it is obvious that $X_0 \prec_{(-)} X_1$. Thus, from Theorem 3.1, one has $X_n \prec_{(-)} X_{n+1}$ for $n \geq 0$. Let $F_n(x)$ be the cdf of X_n, i.e. $F_n(x) = \Pr[X_n \leq x]$. Then, $X_n \prec_{(-)} X_{n+1}$ for all $n \geq 0$ iff $F_n(x) \in TP_2$ in $n \geq 0$ and $x \geq 0$. It should be noted that $\Pr[N(t) \geq n] = F_n(t)$ since no negative drifts are allowed for $\{X_n\}$. It follows that $N(t) \prec_{(+)} N(t')$ for $t < t'$. Furthermore, since

$$F_{n+1}(t) = \int_0^\infty P(y,t) F_n(dy) = \int_0^\infty F_n(y) P(-dy, t) - F_n(t) P(t,t) \tag{5.2}$$

(note that $P(x,y)$ is decreasing in x for each y if $P(x,y) \in TP_2$), one has

$$\frac{F_{n+1}(t)}{F_n(t)} \leq \int_0^\infty \frac{F_{n-1}(y)}{F_{n-1}(t)} P(-dy, t) - P(t,t) = \frac{F_n(t)}{F_{n-1}(t)}. \tag{5.3}$$

Hence $N(t) \in IFR$ for any $t \geq 0$ or, equivalently, $\{X_n\}$ is an IFR process. This fact has been proved in Shaked and Shanthikumar [18] in a different content. The next theorem summarizes the above results.

Theorem 5.1. Let $P(x,y) \in TP_2$ in $x, y \geq 0$. Then $N(t) \prec_{(+)} N(t')$ for $t < t'$ and $N(t) \in IFR$ for any $t > 0$.

Remark 5.1. (i) Let $\{X_n\}$ be governed by $P(x,y)$, not necessarily $P(x,y) = 0$ for $x > y$. Let T_u be the first passage time of X_n into (u, ∞), $u > 0$. Define a modified Markov process

$\{\tilde{X}_n\}$ of $\{X_n\}$ by making (u,∞) absorbing. If we define $\tilde{F}_n(x) = \Pr[\tilde{X}_n \le x]$, one has $\Pr[T_u \ge n+1] = \tilde{F}_n(u)$. Let $\tilde{P}(x,y)$ denote the transition cdf governing $\{\tilde{X}_n\}$. It has been seen in Section 3 that $P(x,y) \in TP_2$ implies $\tilde{P}(x,y) \in TP_2$. It can then be proved by a similar manner to the above that $T_u \in IFR$ for any $u > 0$. Thus, $\{X_n\}$ is an IFR process, provided that the governing transition cdf $P(x,y) \in TP_2$. For the case of discrete state space, see Kijima [13].

(ii) As pointed out in Remark 2.1(ii), the orderings $\prec_{(+)}$ and $\prec_{(-)}$ are in a dual relation. The results in Theorem 5.1 are transferred into the context of $\prec_{(-)}$ as follows. Suppose that $P(x,y)$ is absolutely continuous in y for each x and that $\overline{P}(x,y) = \Pr[X_{n+1} \ge |X_n = x] \in TP_2$ in $x, y \ge 0$. Then $N(t) \prec_{(-)} N(t')$ for $t < t'$ and $N(t) \in PF_2$ for each t. These facts readily follow from noting $\Pr[N(t) \le n-1] = \Pr[X_n > t]$.

Another pure jump process of interest is a sum of independent non-negative rv's (not necessarily identically distributed). Let $X_n = \sum_{k=1}^{n} Y_k$ and let Y_k be distributed by $G_k(x)$, $x \ge 0$. The cdf of X_{n+1} is then given by $F_{n+1}(x) = F_n(x) * G_{n+1}(x)$, $n \ge 0$, where $*$ denotes convolution. Here $F_0(x) = U(x)$. The next lemma is related to [9].

Lemma 5.1. Suppose $G_n \in PF_2$ for $n \ge 1$. Then, (i) $F_n \in PF_2$, and (ii) $F_n(x) \in TP_2$ in $n \ge 0$ and $x \ge 0$.

Proof. Part (i) follows from Corollary 3.2. For (ii), we note that

$$\frac{F_{n+1}(y)}{F_n(y)} \ge \int_0^x \frac{F_n(y-z)}{F_n(y)} G_{n+1}(dz), \quad x < y.$$

Since $F_n(x) \in PF_2$ in x, one has $F_n(y-z)/F_n(y) \ge F_n(x-z)/F_n(x)$ for $x < y$. Thus $F_{n+1}(y)/F_n(y) \ge F_{n+1}(x)/F_n(x)$ for $x < y$, meaning $F_n(x) \in TP_2$ in $n \ge 0$ and $x \ge 0$. □

Let $N(t)$ be the associated counting process with $\{X_n\}$. As before, $N(t) \prec_{(+)} N(t')$ for $t < t'$ iff $F_n(x) \in TP_2$ in $n \ge 0$ and $x \ge 0$. Thus the assumption that $G_n \in PF_2$ for $n \ge 1$ guarantees the uniform monotonicity of $N(t)$ with respect to $t > 0$. The IFR property of $N(t)$ for each t, however, requires an additional condition.

Theorem 5.2. Suppose $G_n \in PF_2$ for $n \ge 1$. Then $N(t) \prec_{(+)} N(t')$ for $t < t'$. If in addition $G_n \prec_d G_{n+1}$ for $n \ge 1$ then $N(t) \in IFR$ for any $t > 0$.

Proof. Let $r_t(n)$ be the failure rate of $N(t)$, i.e. $r_t(n) = \Pr[N(t) = n]/\Pr[N(t) \ge n]$, $n \ge 0$. It then follows that

$$r_t(n) = 1 - \frac{F_{n+1}(t)}{F_n(t)}, \quad n \ge 0. \tag{5.4}$$

Since $F_n(x) \in TP_2$ in $n \ge 0$ and $x \ge 0$, one sees that

$$\frac{F_{n+1}(t)}{F_n(t)} \le \int_0^t \frac{F_{n-1}(t-x)}{F_{n-1}(t)} G_{n+1}(dx) \le \int_0^t \frac{F_{n-1}(t-x)}{F_{n-1}(t)} G_n(dx). \tag{5.5}$$

The second inequality in (5.5) follows from $G_n \prec_d G_{n+1}$ (see e.g. Stoyan [20] for the properties of $\prec_d$). Hence, from (5.4), $r_t(n)$ increases in n. □

Finally, we consider the ordinary renewal process. Let $X_n = \sum_{k=1}^{n} Y_k$ where Y_k are now iid. Let $G(x)$ be the cdf of Y_k and define for $n \geq 0$

$$h(n,t) = \begin{cases} \Pr[N(t) = n] = G^{(n)}(t) - G^{(n+1)}(t), & t \geq 0, \\ 0 & t < 0. \end{cases} \tag{5.6}$$

Here $G^{(n)}(t)$ denotes the n-fold convolution of $G(t)$ with itself. Because of the temporal homogeneity, more about $N(t)$ can be stated. For two discrete rv's X and Y having the probability distributions $p_n^X = \Pr[X = n]$ and $p_n^Y = \Pr[Y = n]$ respectively, X and Y are ordered in the sense of likelihood ratio ordering (write $X \prec_\ell Y$) iff p_n^Y/p_n^X is increasing in n. We note that the ordering $\prec_\ell$ is stronger than $\prec_{(-)}$ and $\prec_{(+)}$. An rv X is said to be strongly unimodal (denote $X \in SU$) if $\Pr[X = n]$ is PF_2 in n. The class of SU distributions is deeply related to the ordering $\prec_\ell$ and $SU \subset IFR \cap PF_2$ [13].

Theorem 5.3. Suppose $G \in IFR \cap PF_2$. Then, (i) $h(n,t) \in PF_2$ in t for any $n \geq 0$, (ii) $N(t) \prec_\ell N(t')$ for $0 \leq t < t'$, and (iii) $N(t) \in SU$ for each $t \geq 0$.

Proof. Note that if $G \in IFR$ then $\hat{G}(x) \in PF_2$ in $-\infty < x < \infty$ where $\hat{G}(x) = 1 - G(x)$ for $x \geq 0$ and $\hat{G}(x) = 0$ for $x < 0$ [13]. Thus, $h(0,t) \in PF_2$ in t. Next consider $h(n+1,t) = \int_0^\infty h(n,t-y)G(dy)$. Suppose that it has been shown that $h(n,t) \in PF_2$ in t. Note that, in the proof of Theorem 3.1, the fact $P(x,y)$ being increasing in y is not used but the key is its TP_2 property (see [12]). Hence an application of Corollary 3.2 proves that $h(n+1,t) \in PF_2$ in t since $G \in PF_2$. Hence Part (i) follows. This fact in turn implies that

$$h(n,x_1)h(n,x_2-y) \geq h(n,x_1-y)h(n,x_2), \quad y \geq 0 \tag{5.7}$$

for $x_1 < x_2$. Integrating the both sides of (5.7) over R_+ with respect to $G(x)$, it follows that

$$h(n,x_1)h(n+1,x_2) \geq h(n+1,x_1)h(n,x_2). \tag{5.8}$$

Hence, $N(t) \prec_\ell N(t')$ for $0 \leq t < t'$ by definition. To prove Part (iii), it suffices to show that $h(n,t)$ in (5.6) is PF_2 in n for each $t \geq 0$. This follows from (5.8) and similar arguments to the proof of Theorem 5.2. □

Remark 5.2. (i) If $f(x) = \frac{d}{dx}F(x) \in PF_2$ (i.e. $F \in SU$ [4]) then the results of Theorem 5.3 are immediately derived from Theorem 3 of Karlin and Proschan [9]. However, we require only $F \in IFR \cap PF_2$ which is implied by $f(x) \in PF_2$.

6 Uniform Comparison of Sums of Independent Random Variables

Let (X_i, Y_i), $i \geq 1$, be independent pairs of rv's such that $X_i \prec_{(-)} Y_i$. Applying Theorem 5.2 in p.124 of Karlin [8] repeatedly, one sees that if X_i and Y_i are PF_2 for all $i \geq 1$, then the sums $\sum_{i=1}^n X_i$ and $\sum_{i=1}^n Y_i$ are also ordered in the same ordering. For $\prec_{(+)}$, the additional assumptions that X_i and Y_i are IFR for all $i \geq 1$ are needed to guarantee the preservation of the ordering. That is, one has the following lemma.

Lemma 6.1. Let $X_i \prec_{(-)} Y_i$, $i \geq 1$, and suppose that both X_i and Y_i are PF_2 for all $i \geq 1$. If $\{X_i\}$ are mutually independent and so are $\{Y_i\}$ then $\sum_{i=1}^n X_i \prec_{(-)} \sum_{i=1}^n Y_i$.

For a sequence of rv's $\{X_i\}$, let $N_X(t)$ be the associated counting process, i.e. $N_X(t) = \sup\{n : \sum_{i=1}^n X_i \leq t\}$. For $\{Y_i\}$, counting process $N_Y(t)$ is defined similarly. Of interest is then to know if there is an ordering between $N_X(t)$ and $N_Y(t)$ when X_i and Y_i are ordered. In the next theorem, we prove it in the affirmative under the same conditions as in Lemma 6.1.

Theorem 6.1. Under the same assumptions as in Lemma 6.1, one has $N_Y(t) \prec_{(+)} N_X(t)$.

Proof. Let $G_n^X(x)$ ($G_n^Y(x)$) be the cdf of X_n (Y_n) and let $F_n^X(x)$ ($F_n^Y(x)$) be the cdf of $\sum_{i=1}^n X_i$ ($\sum_{i=1}^n Y_i$ respectively). From Lemma 6.1, $F_n^X(x)F_n^Y(y) \geq F_n^X(y)F_n^Y(x)$ for $x < y$. Then

$$\frac{F_{n+1}^X(t)}{F_n^X(t)} = \int_0^t \frac{F_n^X(t-x)}{F_n^X(t)} G_{n+1}^X(dx) \geq \int_0^t \frac{F_n^Y(t-x)}{F_n^Y(t)} G_{n+1}^X(dx).$$

Now, since $G_{n+1}^X \prec_{(-)} G_{n+1}^Y$ implies $G_{n+1}^X \prec_d G_{n+1}^Y$, one has the desired result. □

Example 6.1. In this example, we consider the following scheduling problem. A single machine processes a set of n jobs available at time zero. Each job i requires a random processing time X_i. $\{X_i\}$ are mutually independent and they are ordered as $X_1 \prec_{(-)} \cdots \prec_{(-)} X_n$. All jobs have the common due time d. The objective is to obtain a scheduling list $(i_1, i_2, \cdots, i_n)$, a permutation of $\{1, 2, \cdots, n\}$, to maximize, in the sense of a stochastic ordering, the number of completion of jobs. This problem is the so-called single machine scheduling problem considered by many authors (see e.g. [6]), except that the processing times are stochastically ordered. Suppose $X_i \in PF_2$ for all i. Under the assumption $X_1 \prec_{(-)} \cdots \prec_{(-)} X_n$, it can be shown that the list $L = \{1, 2, \cdots, n\}$ minimizes $\sum_{i=1}^k X_i$ for all k in the sense of $\prec_{(-)}$. From Theorem 6.1, it then follows that the associated counting process $N_L(t)$ is maximized for any t among all possible counting processes in the sense of $\prec_{(+)}$. Thus the number of completion of jobs before the due time, $N_L(d)$, is maximized in the sense of $\prec_{(+)}$ if the list L is used.

7 Random Sum of iid Random Variables

Of other importance is the comparison of random sums of iid rv's. Namely, let (X_i, Y_i), $i \geq 1$, be iid pairs of rv's and let M and N be independent integer-valued rv's which are independent of (X_i, Y_i). Then we are interested in when the preservation that $M \prec_j N$ and

$X_i \prec_j Y_i$ ($j = (-), (+)$) imply

$$\sum_{i=1}^{M} X_i \prec_j \sum_{i=1}^{N} Y_i \tag{7.1}$$

is guaranteed. Here the empty sum equals zero. Note that (7.1) is true for stochastic ordering and convex and concave orderings (see [20]). In this section, we seek a sufficient condition under which the above statement is true.

Theorem 7.1. Suppose $X_i \in PF_2$ ($\in IFR$ respectively). If $M \prec_{(-)}$ $(\prec_{(+)})$ N then $\sum_{i=1}^{M} X_i \prec_{(-)} (\prec_{(+)}) \sum_{i=1}^{N} X_i$.

Proof. Let $P_n^M = \Pr[M \le n]$, $n \ge 0$, and $P_{-1}^M = 0$. Further, let $F_n(x) = \Pr[\sum_{i=1}^{n} X_i \le x]$, $n \ge 0$, $x \ge 0$. The cdf of $\sum_{i=1}^{M} X_i$ is then given by

$$\Pr[\sum_{i=1}^{M} X_i \le x] = \sum_{n=0}^{\infty} F_n(x) \Delta[P_n^M] \stackrel{\text{def}}{=} (TM)(x), \tag{7.2}$$

where $\Delta[P_n^M]$ denotes the first difference of (P_n^M). Note that (7.2) can be understood as the discrete counterpart of (3.1). Thus, a similar proof to Theorem 3.1 provides the desired result, since $F_n(x) \in TP_2$ in $n \ge 0$ and $x \ge 0$, from Lemma 5.1(ii), if X_i are PF$_2$. □

Theorem 7.2. Let (X_i, Y_i), $i \ge 1$, be iid pairs of rv's and suppose that either X_i or Y_i for all $i \ge 1$ are exponentially distributed. Further let M be a geometric rv. Then $X_i \prec_{(+)} Y_i$ imply that $\sum_{i=1}^{M} X_i \prec_{(+)} \sum_{i=1}^{M} Y_i$.

Proof. We prove the theorem only for the case that X_i and Y_i are discrete rv's. For general rv's, a routine limiting argument leads to the desired result. Suppose first that X_i are geometric rv's with $\Pr[X_i = k] = (1-r)r^{k-1}$, $k \ge 1$. Let $r(k)$, $k \ge 1$, be the failure rate of Y_i. By assumption, one has $1 - r \ge r(k)$, $k \ge 1$. For geometric rv M, let $\Pr[M = k] = (1-\rho)\rho^k$, $k \ge 0$. Let $\{Z(n); n = 0, 1, \cdots\}$ be a Markov chain on $\{0, 1, \cdots\}$ with 0 absorbing, governed by the transition probability matrix

$$Q = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ (1-\rho)r(1) & \rho r(1) & 1-r(1) & 0 & \cdots \\ (1-\rho)r(2) & \rho r(2) & 0 & 1-r(2) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

It is easy to see that the first passage time T of $Z(n)$ from state 1 to state 0 is equal in law to $\sum_{i=1}^{M} Y_i$ given $M \ge 1$. Denote by $r_T(k)$, $k \ge 1$, the failure rate of T and let $\hat{Z}(n) \stackrel{\text{d}}{=} \{Z(n) \mid Z(n) \ge 1\}$, $n \ge 1$. One then has

$$r_T(k) = (1-\rho) \sum_{n=1}^{\infty} r(n) \Pr[\hat{Z}(k-1) = n], \quad k \ge 1.$$

Since $r(n) \le 1 - r$ and $\sum_{n=1}^{\infty} \Pr[\hat{Z}(k) = n] = 1$, it follows that $r_T(k) \le (1-\rho)(1-r)$, the right hand side of which is the failure rate of $\sum_{i=1}^{M} Y_i$ given $M \ge 1$. If Y_i are geometric rv's, one gets the reversed inequality. Since $\sum_{i=1}^{M} Y_i = 0$ iff $M = 0$, one obtains the theorem. □

The next corollary can be readily proved from the above theorems.

Corollary 7.1. Let (X_i, Y_i), $i \geq 1$, be iid pairs of rv's and let M and N be independent integer-valued rv's. Suppose X_i and Y_i are all IFR. Further suppose that there exist exponential iid rv's E_i and a geometric rv G such that $X_i \prec_{(+)} E_i \prec_{(+)} Y_i$ and $M \prec_{(+)} G \prec_{(+)} N$. Then $\sum_{i=1}^{M} X_i \prec_{(+)} \sum_{i=1}^{G} E_i \prec_{(+)} \sum_{i=1}^{N} Y_i$.

The conditions in the above theorems seem too restrictive. However, it is useful in developing bounds for performance measures of stochastic systems, since an explicit evaluation is often possible only for exponential distributions. The following examples show a potential usefulness of the theorem.

Example 7.1. (Shock Model) Consider a device subject to a sequence of shocks occurring randomly in time as events in a renewal process $N(t)$ with the inter-occurrence time cdf $F(x)$. Suppose that each shock causes an IFR random damage and that damages $\{X_i\}$ on successive shocks are iid. Then, the cumulative damage by the time is given by $\sum_{i=1}^{N(t)} X_i$. If $F \in PF_2$ then, from Theorem 5.2, $N(t) \in IFR$. Let $\rho(t) = \Pr[N(t) \geq 1] = F(t)$ and let $M(t)$ be geometrically distributed with mean $\rho(t)/(1-\rho(t))$. Note that the failure rate of $N(t)$ is bounded below by $1-\rho(t)$. For any sequence of exponential rv's $\{E_i\}$ such that $X_i \prec_{(+)} E_i$, it follows from Corollary 7.1 that $\sum_{i=1}^{N(t)} X_i \prec_{(+)} \sum_{i=1}^{M(t)} E_i$. If the failure rate of X_i is bounded below by r, the failure rate of $\sum_{i=1}^{N(t)} X_i$ is bounded below by $r(1-F(t))$, $t > 0$.

Example 7.2. (Comparison of M/G/1 Queues) Consider a single server $M/G/1$ queue having a Poisson stream of customers at rate ρ and service time distribution function $F(x)$ with finite mean. Denote by $F_R(x)$ the residual lifetime distribution of $F(x)$, i.e. $F_R(x) = \int_0^x (1-F(u))du / \int_0^\infty (1-F(u))du$. Let $\{X_n\}$ be a sequence of iid rv's having cdf $F_R(x)$ and let N be geometrically distributed with mean $\frac{\rho}{1-\rho}$, which is independent of $\{X_n\}$. It is well-known that the generic waiting time W for the $M/G/1$ queue is given by $W = \sum_{i=1}^{N} X_i$. Let $r_R(x)$ be the failure rate of $F_R(x)$. It then follows that

$$r_R(x) = \frac{1-F(x)}{\int_x^\infty (1-F(u))du} = \frac{1}{E[Y \mid Y > x]}$$

where Y is the generic service time of the $M/G/1$ queue. Hence, if $r_2^{-1} \leq E[Y \mid Y > x] \leq r_1^{-1}$ for any $x \geq 0$, the failure rate of W is bounded as $r_1(1-\rho) \leq r_W(x) \leq r_2(1-\rho)$, $x > 0$.

In applying the above results, it is the key that the failure rate of the rv of interest is bounded from below or above. The requirement is satisfied for classes of cdf's that are often used in application. Consider, for example, a mixture of exponential rv's. Let X be an rv having distribution function $F(x) = \sum_{i=1}^{\infty} p_i(1-e^{-r_i x})$, i.e. X has a completely monotone density. The failure rate of X is given by

$$r_X(x) = \frac{\sum_{i=1}^{\infty} p_i r_i e^{-r_i x}}{\sum_{i=1}^{\infty} p_i e^{-r_i x}}, \quad x \geq 0.$$

It is easy to see that $\underline{r} = \inf\{r_i\} \le r_X(x) \le \sup\{r_i\} = \overline{r}$, regardless of the mixing distribution $\{p_i\}$. Next consider a sum of exponential rv's. Suppose Y has a density function $f(x) = \int_0^x r_1 e^{-r_1(x-y)} r_2 e^{-r_2 y} dy$ with $r_1 < r_2$. The failure rate of Y is given by

$$r_Y(x) = \frac{r_1 r_2 (e^{-r_1 x} - e^{-r_2 x})}{r_2 e^{-r_1 x} - r_1 e^{-r_2 x}}, \quad x \ge 0.$$

Since $r_Y(x)$ increases in x and $\lim_{x\to\infty} r_Y(x) = r_1$, $r_Y(x) \le r_1$ for $x \ge 0$. Hence a lower bound with respect to the uniform ordering $\prec_{(+)}$ is readily obtained for a random sum of such rv's based on Corollary 7.1.

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Masaaki KIJIMA
Graduate School of Systems Management
University of Tsukuba, Tokyo
3-29-1 Otsuka, Bunkyo-ku, 112 Tokyo, Japan