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A SIMPLIFIED CROSS DECOMPOSITION ALGORITHM FOR MULTIPLE RIGHT HAND CHOICE LINEAR PROGRAMMING

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Abstract Multiple right hand choice linear programming first studied by Johnson is considered in this paper. An algorithm is suggested for the general purpose multiple right hand choice linear programs. It is a simplified version of the cross decomposition method, one of the recent advances in mathematical programming. As a motivation of our modified dual step a significant relationship has been observed between the optimal solutions of primal subproblems and those of dual subproblems where there is no duality gap. Our computational results show a great improvement in computational efficiency over the simple enumeration method.

1. Introduction

In this paper we consider a Multiple Right Hand Choice Linear Programming (MRCLP), which was first studied by Johnson [4]. Some analytic results on the duality property and on the facets of feasible set were established succeedingly by Granot, Granot and Johnson [3] and by Johnson [5]. The purpose of this paper is to propose an efficient algorithm for MRCLP given as follows:

$$\text{(P)} \qquad \min\{cx \mid Ax = q, Dx = d \text{ for some } q \in Q,\ x \geq 0\}.$$

Here Q is a finite subset of R^m (or the corresponding matrix composed of columns of elements in Q); x denotes an ordered n-tuple of n decision variables; all matrices and vectors are of consistent structure with real coefficients. It is evident that (P) becomes an ordinary linear program when the cardinality of $Q, |Q|$, is equal to one.

The above problem can be justified in many real situations where the decision maker can consider many different desirable options of right hand side vectors and choose one of them arbitrarily. One example we may imagine is the problem for linear economic planning with a certain social decision criterion cx and the set Q of a number of a different bills for final products which can be regarded as alternative economic policies by economic planners. In this case, (P) is a problem for determining optimal economic policy q^* among candidate policies in Q and its corresponding activity level x^*. Another economic interpretation can be found in the study by Granot, Granot and Johnson [3]. In spite of such economic implications as above no effort has been made for systematic algorithms of MRCLP up to the present.

Our algorithmic interest starts from the decomposition structures that can be easily found in the mixed integer programming equivalence of the problem (P). The suggested algorithm follows the fundamental structure of the the cross decomposition method developed by Van Roy [6]. The Lagrangean relaxation method for mixed integer programming makes use of only the dual structure of the problem [2] and the Benders' procedure only the primal structure. But the cross decomposition exploits both of them in a single framework. Cross decomposition was first applied to capacitated facility location problems [7]. This method is appropriate for mathematical programs with a natural decomposition structure that can be identified in constraints and variables simultaneously. The problem (P), translated to an equivalent mixed integer program, well fits this decomposition structure.

Though we have adopted the fundamental structure of the cross decomposition method, we in fact have developed a simplified variant of the original one so that it contains minimal elements necessary for our purpose.

In the following sections we convert the problem (P) to an equivalent mixed integer program and construct basic framework for our method. A new optimality test is provided based upon an interesting observation on the optimal solutions of our subproblems. And we display our algorithm and discuss the validity of it. The algorithm proceeds mainly with solving a usual linear program iteratively. Computational results are given and support our attempt.

2. Related Problems and Assumptions

The problem (P) can be solved if we solve every linear program corresponding to each right hand side in Q. But such a trivial method may be very tedious as the number of considered right hand sides increases, and even the parametric linear programming technique will take much trouble unless the right hand sides share particular similarities easily dealt with parametrically.

To take advantage of the merits of the decomposition structure, the problem (P) can be expressed as a mixed integer program as follows:

$$\text{(MIP)} \qquad \min\{cx \mid Ax - Qy = 0, Dx = d, ey = 1,\ x \geq 0,$$
$$y_j \in \{0,1\},\ j = 1, \cdots, |Q|\},$$

where e is a $|Q|$-dimensional row vector with all components equal to 1. Observing that (P) and (MIP) have a common feasible solution set with respect to the variables x, we conclude that (P) is equivalent to (MIP) in the sense that both have identical optimal solutions and an optimal value. We can find a natural decomposition structure between the constraints $Ax - Qy = 0$ and the others, and between the variables x and y.

The problems defined in the following subsections, 'primal structure' and 'dual structure', are essential to develop the algorithm. The following assumptions and notations are given for convenience throughout the discussion.

$X = \{x \in R^n \mid Dx = d,\ x \geq 0\}$ is nonempty and bounded.
$X(q) = \{x \in R^n \mid Ax = q,\ Dx = d,\ x \geq 0\}$ is nonempty for every q in Q.
$F = \{(u, w) \mid uA + wD \leq c\}$ is nonempty.
Let Y denote the set $\{y \mid ey = 1,\ y_j \in \{0,1\},\ j = 1, \cdots, |Q|\}$.

Primal Structure

The algorithm starts first with fixing the integer variables in (MIP) to specified values, which corresponds to determining a specific right hand side for (P). That is, Qy equals a certain selection q in Q. This procedure leads to a so called primal subproblem with a selected right hand side vector q, appearing as follows:

$$(\mathrm{P}_q) \qquad \min\{cx \mid x \in X(q)\}. \qquad (1)$$

We call the above problem (P_q) the primal subproblem with right hand choice q. It is an ordinary linear program. We prefer the symbol (P_k) to (P_{q^k}) where q^k denotes the right hand choice selected for iteration k of the algorithm presented.

Let $v(\cdot)$ denote the optimal value of problem $(\cdot)$. From duality theorem,

$$\begin{aligned} v(\mathrm{P}_q) &= \min\{cx \mid x \in X(q)\} \\ &= \max\{uq + wd \mid (u,w) \in F\} \\ &= \max\{uq + wd \mid (u,w) \in E\}, \end{aligned} \tag{2}$$

where E is the set of extreme points of F. Now the primal master problem becomes:

$$(\mathrm{M}) \qquad \min\{z \mid uQy + wd \le z \text{ for all } (u,w) \in E,\ y \in Y\}.$$

Let (u^k, w^k) be a dual optimal solution for (P_k). Then we can consider the following relaxation for (M):

$$(\mathrm{M}_k) \qquad \min\{z \mid u^iQy + w^id \le z,\ i = 1, \cdots, k,\ y \in Y\}.$$

Each (M_k) is a trivial mixed integer program with one continuous variable z and $|Q|$ binary integer variables. Let $f_k(q) = \max\{u^iq + w^id \mid i = 1, \cdots, k\}$. Then the computation of $v(\mathrm{M}_k)$ can be mechanized as follows:

$$(i) \begin{cases} f_1(q) = u^1q + w^1d \\ f_k(q) = \max\{u^kq + w^kd,\ f_{k-1}(q)\} \quad \text{if } k \ge 2 \end{cases} \tag{3}$$

$$(ii)\ v(\mathrm{M}_k) = \min\{f_k(q) \mid q \in Q\} = f_k(\overline{q}) \quad \text{for some } \overline{q} \in Q. \tag{4}$$

If we store the value $f_k(q)$ for each q at iteration k, we need only $2|Q| - 1$ mechanical comparisons for the operations in (3) and (4). These computations are quite trivial and all the same independent of the iteration counter k. Moreover we do not need any storage requirement to reserve the extreme points generated during the algorithm.

Dual Structure

At each iteration we solve the Lagrangean relaxation problem, the dual subproblem, by relaxing constraints $Ax - Qy = 0$ in (MIP), which has a very simple decomposition structure. A candidate right hand choice for the next iteration is provided from this dual subproblem. Furthermore this dual structure may accelerate the convergence to optimality where there is no duality gap. The computational burden for performing this dual phase is almost negligible in our algorithm.

At iteration k, the dual subproblem with Lagrangean multiplier vector u^k with respect to constraints $Ax - Qy = 0$ is,

$$(\mathrm{D}_k) \qquad \min\{cx + u^k(Qy - Ax) \mid x \in X,\ y \in Y\}.$$

This problem is classified into two much simpler ones, one a linear programming problem and the other a 0-1 integer problem.

$$v(\mathrm{D}_k) = \min\{(c - u^kA)x \mid x \in X\} + \min\{u^kQy \mid y \in Y\}. \tag{5}$$

Lemma 1. Let (u^k, w^k) be a dual optimal solution to (P_k). Then

$$v(\mathrm{D}_k) = \min\{u^kq \mid q \in Q\} + w^kd. \tag{6}$$

Proof: Noting from Theorem 2 of Geoffrion [2], we get

$$\begin{aligned} v(\mathrm{P}_k) &= \min\{cx + u^k(q^k - Ax) \mid x \in X\} \\ &= \min\{(c - u^kA)x \mid x \in X\} + u^kq^k. \end{aligned}$$

Again from (5),

$$\begin{aligned} v(\mathrm{D}_k) &= v(\mathrm{P}_k) - u^k q^k + \min\{u^k Qy \mid y \in Y\} \\ &= u^k q^k + w^k d - u^k q^k + \min\{u^k q \mid q \in Q\} \\ &= \min\{u^k q \mid q \in Q\} + w^k d. \end{aligned}$$ □

From the above lemma, $v(\mathrm{D}_k)$ can be computed directly by the following simple steps.

$$(i)\text{Compute } \min\{u^k q \mid q \in Q\} = u^k \tilde{q} \quad \text{for some} \quad \tilde{q} \in Q. \tag{7}$$

$$(ii)\text{Set } v(\mathrm{D}_k) = u^k \tilde{q} + w^k d. \tag{8}$$

This procedure requires only simple $|Q|-1$ comparisons to compute $v(\mathrm{D}_k)$ and produces a candidate for the right hand side of the next iteration.

The Lagrangean dual problem is

$$(\mathrm{D}) \qquad \max\{v(\mathrm{D}_u) \mid u \in R^m\}.$$

It is well known that

$$v(\mathrm{D}_u) \le u(\mathrm{D}) \le v(\mathrm{P}) \le v(\mathrm{P}_q) \quad \text{for all } u \in R^m \text{ and } q \in Q.$$

3. Additional Optimality Test

The following interesting property may lead to a stronger optimality test in case there is no duality gap.

Theorem 1. (MIP) has no Lagrangean duality gap, with respect to the relaxed constraints $Ax - Qy = 0$, i.e.,

$$v(\mathrm{MIP}) = v(\mathrm{P}) = v(\mathrm{D}),$$

if and only if there exist u^* and y^* such that

(i) (u^*, w^*) for some w^* is a dual optimal solution of the linear program (P_{Qy^*}) and

(ii) (x^*, y^*) for some x^* is an optimal solution of (D_{u^*}).

Proof: Suppose first that $(i), (ii)$ hold. Let $q^* = Qy^*$. Then from Lemma 1,

$$v(\mathrm{D}_{u^*}) = u^* q^* + w^* d = v(\mathrm{P}_{q^*}).$$

This implies that (MIP) has no Lagrangean duality gap. Conversely suppose (MIP) has no Lagrangean duality gap. Let u^* be an optimal solution of the Lagrangean dual (D) and $q^* = Qy^*$ an optimal right hand choice for (P). Then

$$v(\mathrm{D}_{u^*}) = v(\mathrm{P}_{q^*}).$$

Choose any optimal solution x^* to (P_{q^*}). Since (x^*, y^*) is a feasible solution to (D_{u^*}), it should be also an optimal solution to (D_{u^*}). This directly implies (ii). Therefore, from relation (5),

$$v(\mathrm{D}_{u^*}) = \min\{(c - u^* A)x \mid x \in X\} + u^* q^*.$$

Consider the following linear programming problem.

$$\min\{(c - u^* A)x \mid x \in X\}.$$

Note that x^* is an optimal solution to the above problem. Let w^* be a dual optimal solution to the above problem. Then (u^*, w^*) is also an optimal dual solution to (P_{q^*}) from the complementarity theorem in linear programming [1, p. 134]. Hence we have proved (i). □

Let us consider the simple procedure of solving only the subproblems (P_q), (D_u) by turns; if (P_{q^i}) generates u^i as a dual optimal solution then we solve the problem (D_{u^i}) and if (D_{u^i}) yields y^{i+1} as an optimal solution then we solve $(\mathrm{P}_{q^{i+1}})$ where $q^{i+1} = Qy^{i+1}$. Then (MIP) has no duality gap if and only if we can find a sequence of optimal solutions with $u^i = u^{i+1}$ or $y^i = y^{i+1}$ for some i. This observation may motivate a new scheme for decomposition methods that traces the temporal behavior of optimal solutions of subproblems, in the variables u and y, in parallel with evaluating optimal values. Our optimality test 2 based upon the above algorithmic motivation tries to examine u^{i+1} in advance from solving (D_{u^i}). Van Roy [6] has established only the 'if' part of the above theorem [Theorem 3.4 and Lemma 5.1].

Corollary. Suppose (MIP) has no Lagrangean duality gap. Then y^* is an optimal integer choice and u^* is an optimal multiplier for (D) if and only if they satisfy the hypotheses (i) and (ii) of the above theorem.

Proof: Straightforward from the proof of the above theorem. □

4. Algorithm and Validation

Our algorithm for solving MRCLP is a simplified variant of the cross decomposition method and proceeds as follows.

Algorithm for MRCLP

Step 0. (Initialization).

Let $v_P = +\infty$, $v_D = -\infty$.

Step 1. (Primal subproblem).

(Increase the iteration counter by 1)

(i) Solve (P_k) for the given right hand side q^k($q^1$is chosed arbitrarily in Q) and find out an optimal solution x^k and a dual optimal solution (u^k, w^k). Let B be the current optimal basis that computes (u^k, w^k).

(ii) If $v_\mathrm{P} > v(\mathrm{P}_k)$, set $v_\mathrm{P} = v(\mathrm{P}_k)$ and $q' = q^k$, $x' = x^k$.

Step 2. (Master problem and optimaliry test 1)

Compute $v(\mathrm{M}_k) = f_k(\overline{q})$ as given in (3) and (4).

If $v_\mathrm{P} = v(\mathrm{M}_k)$, (9)

then stop. We have an optimal solution x' with the optimal value v_P, and its corresponding right hand choice q'.

Step 3. (Dual subproblem and optimality test 2)

(i) Find a $\tilde{q} \in Q$ that satisfies (7) and

$$B^{-1} \begin{bmatrix} \tilde{q} \\ d \end{bmatrix} \geq 0, \tag{10}$$

If such a $\tilde{q}$ exists, then set

$$\tilde{x} = \begin{bmatrix} B^{-1} \begin{bmatrix} \tilde{q} \\ d \end{bmatrix} \\ 0 \end{bmatrix} \text{ and stop.}$$

We have an optimal solution $\tilde{x}$ with the corresponding right hand choice $\tilde{q}$.

Otherwise, compute $v(\mathrm{D}_k)$ as in (8).

If $v(\mathrm{D}_k) > v_\mathrm{D}$, set $v_\mathrm{D} = v(\mathrm{D}_k)$.

(ii) If $v_\mathrm{D} = v_\mathrm{P}$, (11)

then stop, x' and q' are optimal. Otherwise go to step 4 with any $\tilde{q}$ that satisfies (7).

Step 4. (Determining q^{k+1})

If $v_P > f_k(\tilde{q})$ (12)
then set $q^{k+1} = \tilde{q}$, and, otherwise, set $q^{k+1} = \overline{q}$.
Go to step 1.

The main steps of the above algorithm are steps 1 and 3, each of which reflects a decomposition structure from primal and dual side respectively. Step 1 solves an ordinary linear program and requires major computational work. We may reach optimality the faster for step 3 when (MIP) happens to have no duality gap. If not, the computational burden of step 3 is almost negligible.

The step 4 reflects a natural choice for the next primal subproblem in the cross decomposition framework [6].

Lemma 2. If the above algorithm terminates, then we get an optimal solution of the problem (P).

Proof: It is possible for the algorithm to stop at step 2 or 3. The proof is trivial for step 2 and step 3 (ii). Let us prove for step 3 (i). Suppose there exists $\tilde{q}$ in Q that satisfies the condition in (7) and (10) at k-th iteration. Then,

$$u^k \tilde{q} + w^k d \leq v(\mathrm{P}_q) \tag{13}$$

for any q in Q which is clear from (2) and (7). From relation (10), (u^k, w^k) is a dual optimal solution for $(\mathrm{P}_{\tilde{q}})$. This implies that $v(\mathrm{P}_{\tilde{q}})$ is equal to the left hand side of (13). Hence $\tilde{q}$ is an optimal right hand choice. It is also clear that $\tilde{x}$ is an optimal solution for $(\mathrm{P}_{\tilde{q}})$ from (10). The proof is completed. □

Lemma 3. During the k-th iteration, unless relation (12) holds, then $v(\mathrm{P}_{\tilde{q}}) \geq v_P$. If $\tilde{q}$ is selected to be equal to one among $q^1, \cdots, q^k$, then relation (12) does not hold.

Proof: The first statement is a direct result of (2). Since

$$u^i q^i + w^i d = f_k(q^i) = v(\mathrm{P}_i) \geq v_P \quad \text{for any } i \leq k,$$

the second statement is also clear. □

The above lemma assures that our method eliminates the possibility of repetition in the scanning procedure of the right hand choice and helps to reduce the incumbent value.

Theorem 2. The above algorithm terminates within a finite number of iterations and solves the problem (P).

Proof: Unless the conditions in the optimality test 2 are satisfied during the algorithm, the condition (9) in the optimality test 1 should hold within $|Q|$ iterations from Lemma 3. This directly leads to the stated convergence from Lemma 2. □

Theorem 3. If the conditions in step 3 (i) are satisfied at an iteration, then the problem (MIP) has no Lagrangean duality gap with respect to the relaxed constraints $Ax - Qy = 0$.

Proof: Suppose the optimality condition in step 3 (i) holds at k-th iteration. Let $u^* = u^k$ and y^* be such that $y^* \in Y$ and $Qy^* = \tilde{q}$. It is clear from (7) and (10) that u^* and y^* satisfies the hypotheses $(i), (ii)$ in Theorem 1. Hence the proof is evident. □

5. Computational Results and Conclusions

The algorithm presented has been coded in PASCAL and all the computational tests were made on IBM-PC AT. For solving primal subproblems we used the revised simplex method. We report 40 test problems with the computational results in *cpu* time and the number of iterations.

We tested problem (P) with a special structure $\begin{pmatrix} A \\ D \end{pmatrix} = (B, I)$, where B is an arbitrary matrix and I is an identity matrix. Nonnegative right-hand-side vectors q and d, and nonnegative objective coefficients c have been used. This special structure has been adapted to ensure the feasibility and boundedness of the test problems.

Two different densities of nonzero elements in the constraint matrix (B, I) have been tested for each size of the test problems. Computational results of 20 tested problems are summarized in Table 1. The numbers of iterations required have been remarkably small compared to $|Q|$ and the algorithm seems insensitive to $|Q|$ and the density of constraint matrix. Our method terminates on the optimality test 1 except for a few extraordinary cases which have no duality gap by chance.

(TABLE 1) Computational results for general problems

Problem No.	# of Var.	# of Constraints	Density of $[B, I]$	$\|Q\|$	# of Iterations	CPU time**
1	15	6(3)	0.4	20	2	1.04
2	15	6(3)	0.6	20	3	1.60
3	40	12(4)	0.4	20	4	21.31
4	40	12(4)	0.6	20	2	16.04
5	50	14(4)	0.4	20	3	32.96
6	50	14(4)	0.6	20	4	49.87
7	100	22(8)	0.4	20	3	204.54
8	100	22(8)	0.6	20	3	221.07
9	150	35(10)	0.4	20	6	1858.41
10	150	35(10)	0.6	20	4	1235.33
11	15	6(3)	0.4	40	4	1.49
12	15	6(3)	0.6	40	1*	.50
13	40	12(4)	0.4	40	4	29.55
14	40	12(4)	0.6	40	5	27.18
15	50	14(4)	0.4	40	5	61.63
16	50	14(4)	0.6	40	3*	27.96
17	100	22(8)	0.4	40	3	142.26
18	100	22(8)	0.6	40	4	284.84
19	150	35(10)	0.4	40	4	853.37
20	150	35(10)	0.6	40	5	1494.90

(*i*) Numbers in the parentheses indicate the dimension of q.
(*ii*) * indicates the termination at test 2.
(*iii*) ** is in seconds on IBM-PC AT.

We also tested our method for a class of very special problems whose corresponding (MIP) problems have no duality gap. We expect that our algorithm will terminate on the optimality test 2, in fact at step 3 (*i*) for more problems in this case. This is important because termination at step 3 (*i*) implies that an optimal right hand choice can be detected even before the corresponding primal subproblem has been solved in the algorithm. Table 2 shows the computational results for 20 problems with no duality gap. In this case we observed 7 problems terminating at step 3 (*i*).

(TABLE 2) Computational results under no duality gap

Problem No.	# of Var.	# of Constraints	Density of $[B, I]$	$\|Q\|$	# of Iterations	CPU time**
1	15	6(3)	0.4	20	1*	.55
2	15	6(3)	0.6	20	1*	.27
3	40	12(4)	0.4	20	2*	15.33
4	40	12(4)	0.6	20	2	16.04
5	50	14(4)	0.4	20	3	28.61
6	50	14(4)	0.6	20	4*	48.61
7	100	22(8)	0.4	20	4	223.05
8	100	22(8)	0.6	20	2	156.32
9	150	35(10)	0.4	20	5	1304.97
10	150	35(10)	0.6	20	3	822.89
11	15	6(3)	0.4	40	2	.77
12	15	6(3)	0.6	40	1*	.27
13	40	12(4)	0.4	40	4	22.19
14	40	12(4)	0.6	40	2	12.36
15	50	14(4)	0.4	40	2*	21.75
16	50	14(4)	0.6	40	1*	7.80
17	100	22(8)	0.4	40	2	127.16
18	100	22(8)	0.6	40	4	307.64
19	150	35(10)	0.4	40	3	618.41
20	150	35(10)	0.6	40	6	1717.68

(i) Numbers in the parentheses indicate the dimension of q.
(ii) * indicates the termination at test 2.
(iii) ** is in seconds on IBM-PC AT.

As a whole, we have shown that our simplified version of the cross decomposition method works well in the MRCLP which arises in many real situations where the decision maker has a finite set of alternative policies. It is clear that our method is a significant improvement over the simple enumeration method.

References

[1] Dantzig G.B.: *Linear Programming and Extensions.* Princeton University Press, Princeton, N.J., 1963.

[2] Geoffrion A.M.: Lagrangean Relaxation for Integer Programming. *Mathematical Programming Study*, Vol. 2, (1974), 82–114.

[3] Granot D., Granot F. and Johnson E.L.: Duality and Pricing in Multiple Right Hand Choice Linear Programming Problems. *Mathematics of Operations Research*, Vol. 7, (1982), 545–556.

[4] Johnson E.L.: Integer Programming in Continuous Variables. *Report No. 7418-OR*, Institut für Ökonometrie und Operations Research (Nassestrasse 2, D5300 Bonn, West Germany), 1974.

[5] Johnson E.L.: Characterization of Facets for Multiple Right Hand Choice Linear Programs. *Mathematical Programming Study*, Vol. 14, (1981), 112–142.

[6] Van Roy T.J.: Cross Decomposition for Mixed Integer Programming. *Mathematical Programming*, Vol. 25, (1983), 46–63.

[7] Van Roy T.J.: A Cross Decomposition Algorithm for Capacitated Facility Location. *CORE Discussion Paper No. 8320*, Center for Operations Research and Econometrics, (Universite Catholique de Louvain), 1983.

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