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THE PATH FOLLOWING ALGORITHM FOR STATIONARY POINT PROBLEMS ON POLYHEDRAL CONES

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Abstract Given a set Ω of R^n and a function $f : \Omega \to R^n$ the problem of finding a point $x^s \in \Omega$ such that $(x - x^s)^t f(x^s) \geq 0$ for any $x \in \Omega$ is referred to as a stationary point problem and x^s is called a stationary point. For the problem with conical Ω and strongly copositive f we propose a system of equations whose solution set contains a path connecting a trivial starting point to a stationary point. We also develop an algorithm to trace the path when f is an affine function with a copositive plus matrix. Starting with an appropriate point the algorithm provides a stationary point or shows that there exist no stationary points.

1. Introduction

The linear complementarity problem (LCP in short) is:

(1.1) given an $n \times n$ matrix Q and an n-dimensional vector c, find a point x of the n-dimensional Euclidean space R^n such that

$$x \geqq 0,\ Qx + c \geqq 0 \ \text{ and } \ x^t(Qx + c) = 0,$$

where t means transposition. This problem arises in various fields, such as economics, game theory and mathematical programming and so a good many researchers have been working for this problem, including Cottle and Dantzig [2], Eaves [3] and Lemke [19]. The problem (1.1) with the affine function $Qx + c$ replaced by a general nonlinear function $f(x)$ is called the nonlinear complementarity problem. It has also been attracting attention of many researchers, e.g. Cottle [1], Karamardian [11], Eaves [4] and Kojima [12]. Habetler and Price [9] have considered a generalization of the nonlinear complementarity problem (GCP in short) and provided a condition under which a solution exists. In their genernalization the nonnegative orthant in (1.1) is replaced by a closed convex cone and its negative polar. Namely,

(1.2) given a closed convex cone $\Omega \subset R^n$ and a continuous function $f: \Omega \to R^n$, find a point x^c such that

$$x^c \in \Omega,\ f(x^c) \in \Omega^+ \ \text{ and } \ (x^c)^t f(x^c) = 0,$$

where Ω^+ is the negative polar of Ω, i.e.,

$$\Omega^+ = \{ y \mid y \in R^n,\ y^t x \geqq 0 \ \text{ for any } \ x \in \Omega \}.$$

Their result was further extended by Karamardian [10] and Saigal [20]. On the other hand, the variable dimension algorithm for fixed point problems proposed by van der Laan and Talman [15,16] has been extended for the complementarity problem by Talman and Van der Heyden [22], van der Lann, Talman and Van der Heyden [18] and further by Talman and Yamamoto [23] and Yamamoto [25] for stationary point problems (SPP in short):

(1.3) given a convex set $\Omega \subset R^n$ and a continuous function $f: \Omega \to R^n$, find a point x^s such that

$$x^s \in \Omega, \ (x - x^s)^t f(x^s) \geqq 0 \quad \text{for any} \quad x \in \Omega.$$

Especially Yamamoto [25] dealt with a problem with an affine function f on a compact set Ω. By exploiting the linearity of f the algorithm uses Dantzig-Wolfe decomposition method for the linear programming to trace the piecewise linear path of solutions which will lead to a stationary point. Note that an SPP on a polyhedral cone Ω and a GCP were proved to be equivalent by Karamardian [10]. Hence Yamamoto's algorithm has paved the way for solving GCP or SPP with conical Ω. The main purpose of this paper is to generalize his algorithm for the above problem. We show that the new algorithm provides a stationary point after a finitely many iterations under some mild condition.

The organization of this paper is as follows: In Section 2 we give a brief review of subdivided manifolds, a basic theorem for fixed point algorithm and the primal-dual pair of subdivided manifolds. In Section 3 and 4, we make primal and dual subdivided manifolds for a polyhedral cone and Section 5 gives the primal-dual pair of subdivided manifolds for the polyhedral cone. Our main theorems are given in Section 6. In Section 7 we describe the algorithm and show that it traces the path of solutions of a certain system of equations and then terminates after a finite number of iterations with a stationary point under some condition.

2. Basic Theorem for Fixed Point Algorithms and Primal-Dual Pair of Subdivided Manifolds

We give a brief review of the primal-dual pair of subdivided manifolds introduced by Kojima and Yamamoto [14].

We call a convex polyhedral set a cell or an ℓ-cell to clarify its dimension. When a cell B is a face of a cell C, we write $B \prec C$. Especially when C is an ℓ-cell and B is an $(\ell-1)$-dimensional face of C, we say that B is a facet of C and write $B \prec\!\!\!\cdot\; C$.

Let M be a finite or countable collection of ℓ-cells. We write $\bar{M} = \{ B \mid B \prec C \text{ for some } C \in M \}$ and $|M| = \cup \{ C \mid C \in M \}$. We call M a subdivided ℓ-manifold if and only if

(2.1) for any $B, C \in M$, $B \cap C = \emptyset$, or $B \cap C \prec B$ and C.

(2.2) for each $(\ell-1)$-cell B of $\bar{M}$ at most two ℓ-cells of M have B as a facet.

(2.3) M is locally finite: each point $x \in |M|$ has a neighborhood which intersects only a finite number of cells of M.

We write $\partial M = \{ B \mid B \in \bar{M}, B \prec C$ for exactly one ℓ-cell C of $M \}$, and call it the boundary of M.

A function $g: |M| \subset R^k \to R^n$ is said to be piecewise continuously differentiable (PC^1 for short) on M if it is continuous on $|M|$ and the restriction of g to each cell C of M can be extended to a continuously differentiable function (C^1-function in short) defined on an open subset of R^k which contains C. Let f be a C^1-function from an open subset U of R^k into R^n and V be a subset of U. We denote the Jacobian matrix of f at each $x \in U$ by $Df(x)$. A point $r \in R^n$ is said to be a regular value of $f: U \to R^n$ on V if $\operatorname{rank} Df(x) = n$ for every $x \in V$ satisfying $f(x) = r$. Obviously, if $k < n$ then any $r \in R^n$ can not be a regular value of the function f on V provided $f^{-1}(r) \cap V \neq \emptyset$. Let $C \subset R^k$ be an ℓ-cell contained in U. Then there exists a $k \times \ell$ matrix B with rank $B = \ell$ and $d \in R^k$ such that

$$\operatorname{aff} C = \{ Bz + d: z \in R^\ell \},$$

where aff C is the affine subspace spanned by C. The set of the columns of B forms a basis of the ℓ-dimensional subspace $\{ Bz: z \in R^\ell \}$ of R^k. Let

$$W = \{ z \in R^\ell: Bz + d \in U \},$$
$$D = \{ z \in R^\ell: Bz + d \in C \}.$$

Then we see that W is an open subset of R^ℓ and $D \subset R^\ell$ is an ℓ-cell. Define the C^1-function $F: W \to R^n$ by

$$F(z) = f(Bz + d) \quad \text{for every } z \in W.$$

We say that a point $r \in R^n$ is a regular value of $f|C$, the restriction of the function f to the cell C, if r is a regular value of the function $F: W \to R^n$ on D, i.e.,

$$\operatorname{rank} Df(x)B = n \quad \text{for every } x \in X,$$

where $X = \{ x \in C: f(x) = r \}$. Let M be a subdivided ℓ-manifold in R^k, and $g: |M| \to R^n$ a PC^1-function on M. A point $r \in R^n$ is said to be a regular value of the PC^1-function $g: |M| \to R^n$ if $r \in R^n$ is a regular value of $g|C$ for every cell C of $\bar{M}$.

We now state one of the basic theorems for the fixed point algorithms (see Kojima [13]).

Theorem 2.1. Let M be a subdivided $(n+1)$-manifold in R^k and $g: |M| \to R^n$ be a PC^1-function on M. Suppose $r \in R^n$ is a regular value of g and $g^{-1}(r) \neq \emptyset$. Then $g^{-1}(r)$ is a disjoint union of paths and loops, where a path is a 1-dimensional subdivided manifold homeomorphic to one of the intervals $(0,1)$, $(0,1]$ and $[0,1]$, and a loop is one homeomorphic to the 1-dimensional sphere. Furthermore they satisfy the following conditions.

(2.4) $g^{-1}(r) \cap C$ is either empty or a smooth curve for each $C \in M$.

(2.5) A loop of $g^{-1}(r)$ does not intersect $|\partial M|$.

(2.6) If a path S of $g^{-1}(r)$ is compact, ∂S consists of two distinct points in $|\partial M|$.

Let P and D be subdivided manifolds. If P and D satisfy the following conditions with positive integer ℓ and an operator $d: \bar{P} \cup \bar{D} \to \bar{P} \cup \bar{D} \cup \{\emptyset\}$, we say that $(P,D;d)$ is a primal-dual pair of subdivided manifolds (PDM for short) with degree ℓ.

(2.7) For each $X \in \bar{P}$, $X^d \in \bar{D} \cup \{\emptyset\}$ and for each $Y \in \bar{D}$, $Y^d \in \bar{P} \cup \{\emptyset\}$.

(2.8) If $Z \in \bar{P} \cup \bar{D}$ and $Z^d \neq \emptyset$, then $(Z^d)^d = Z$ and $\dim Z^d + \dim Z = \ell$.

(2.9) If $Z_1, Z_2 \in \bar{P}$ (or $\bar{D}$), $Z_1 \prec Z_2$, $Z_1^d \neq \emptyset$ and $Z_2^d \neq \emptyset$, then $Z_2^d \prec Z_1^d$.

We call the operator d the dual operator and Z^d the dual of Z.

For a PDM $(P,D;d)$ with degree ℓ , let

$$\langle P,D;d\rangle = \{ X \times X^d \mid X \in \bar{P},\ X^d \neq \emptyset \}$$

or equivalently

$$\langle P,D;d\rangle = \{ Y^d \times Y \mid Y \in \bar{D},\ Y^d \neq \emptyset \}.$$

Then we have the following theorem. See Kojima and Yamamoto [14] for the proof.

Theorem 2.2. Let $(P,D;d)$ be a PDM with degree ℓ. Then $M = \langle P,D;d\rangle$ is a subdivided ℓ-manifold and

$$\partial M = \{ X \times Y \mid X \times Y \text{ is an } (\ell-1)\text{-cell of } \bar{M},\ X \in \bar{P},\ Y \in \bar{D}, \text{ and either } X^d = \emptyset \text{ or } Y^d = \emptyset \}.$$

3. The Primal Subdivided Manifold over Convex Cone

Let A be an $n\times m$ matrix and $\Omega = \{ x \mid x \in R^n,\ A^t x \leqq 0 \}$ be a nonempty convex polyhedral cone. Let $\mathcal{F}$ be the family of all faces of Ω. For each face $F \in \mathcal{F}$, let $I(F)$ be the index set of binding constraints at F, i.e.,

$$I(F) = \{ i \mid 1 \leqq i \leqq m,\ (a^i)^t x = 0 \ \text{ for any point } x \text{ of } F \}$$

where a^i is the i^{th} column of A. Let F^* be the cone generated by a^i's for $i \in I(F)$, i.e.,

$$F^* = \{ y \mid y = \sum_{i\in I(F)} \mu_i a^i \ \text{ for some } \mu_i \geqq 0 \ \ i \in I(F) \},$$

where we assume that $F^* = \{0\}$ when $I(F) = \emptyset$. Cone F^* is called the dual cone or normal cone of face F. Note that $\dim F^* = n - \dim F$. By these definitions the stationary point problem is a problem of finding a point $x \in \Omega$ and a face $F \in \mathcal{F}$ such that

$$(3.1) \quad x \in F \ \text{ and } \ -f(x) \in F^*.$$

Let $w \in \Omega$ be an initial guess of a stationary point. We do not require the point w to lie in the relative interior of Ω. For each $F \in \mathcal{F}$ with $w \notin F$, let

$$[0, w] + F = \{ x \mid x = \alpha w + z \ \text{ for some } z \in F \text{ and } 0 \leqq \alpha \leqq 1 \}.$$

and let

$$\{w\} + \Omega = \{ x \mid x = w + z \ \text{ for some } z \in \Omega \}.$$

We define

$$(3.2) \quad P = \{ [0, w] + F \mid w \notin F \prec \Omega \} \cup \{\{w\} + \Omega \}.$$

Two-dimensional examples of P are shown in Fig.1 for different starting points.

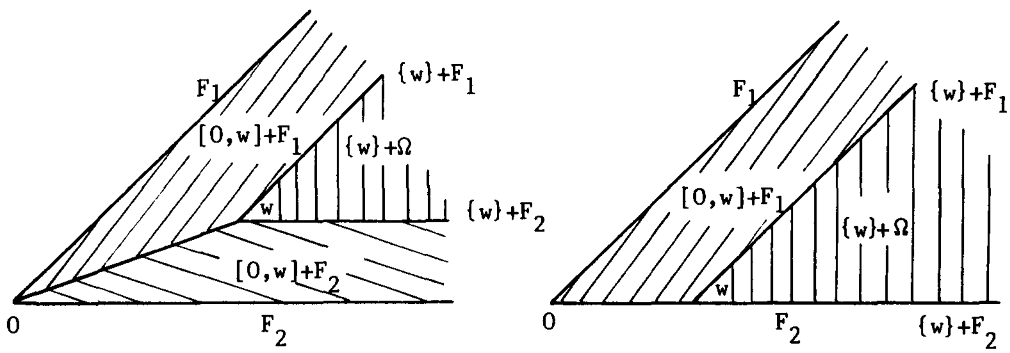

Fig.1

Before proving that P is a subdivided manifold, we give the following two lemmas.

First let $K \subset R^k$ be a polyhedron and let $\psi(x) = Bx + b$ be an affine function from R^k to R^k with a nonsingular matrix $B \in R^{k\times k}$ and a vector $b \in R^k$. Then the image K' of K under ψ is also a polyhedron of R^k and furthermore the image of a face of K is also a face of K'. Next, let $X \subset R^n$ and $Y \subset R^m$ be polyhedra and consider the cross product $X \times Y \subset R^{n+m}$. It is readily seen that the boundary $\partial(X \times Y)$ of $X \times Y$ is the union of $\partial X \times Y$ and $X \times \partial Y$.

Lemma 3.1. Let F be a nonempty convex polyhedral cone. If $w \notin \text{tng } F$, then we have

$$\partial([0, w] + F) = (\{0\} + F) \cup (\{w\} + F) \cup \{[0, w] + G \mid G \lhd F\},$$

where $\text{tng } F = \text{aff } F - x$ for some point $x \in F$.

proof: Since $w \notin \text{tng } F$, $[0, w] + F \subset R^n$ is the image of the cross product of some polyhedron $X \subset R^{n-1}$ and $\{ x \mid 0 \leqq x \leqq 1 \} \subset R$ under an affine function $\psi(x) = Bx + b$ defined by a nonsingular $n\times n$ matrix B. Therefore the lemma follows from the above two comments. □

Lemma 3.2. Suppose $w \in \Omega$ and $F \prec \Omega$. If $w \notin F$, then $w \notin \text{aff } F = \text{tng } F$.

Proof: Since F is a face of Ω, $F = \Omega \cap H \supset \Omega \cap \text{aff } F$ for some hyperplane H. From $w \notin F$, we immediately have $w \notin \text{aff } F$. Since F is itself a cone and contains the origin, aff F coincides with tng F. □

Now we are ready to prove that P is a subdivided manifold.

Lemma 3.3. P is a subdivided manifold with the same dimension as Ω.

Proof: We suppose that $w \neq 0$ since $P = \{\Omega\}$ and nothing to prove when $w = 0$. First, let us see the dimension of the cells of P. It is clear that $\dim (\{w\} + \Omega) = \dim \Omega$. To show that $\dim ([0, w] + F) = \dim \Omega$ when $w \notin F \lhd \Omega$, let δ be the dimension of F and let $b^1,\dots,b^\delta$ be vectors of F which forms a basis of tng F. From $w \notin F$ which gives $w \notin \text{tng } F$, we have linearly independent vectors $w, b^1,\dots,b^\delta$ in $[0, w] + F$. Therefore $\dim ([0, w] + F) \geqq \delta + 1 = \dim \Omega$. Meanwhile, since every x of $[0, w] + F$ is a linear combination of $w, b^1,\dots,b^\delta$, $\dim ([0, w] + F) \leqq \delta + 1$. Thus we have seen that P is a finite collection of cells with $\dim \Omega$.

Second, choose any two cells of P and consider their intersection. We must consider two cases. Case 1: The two cells chosen are $[0, w] + F$

and $[0, w] + G$. Each point $x \in ([0, w] + F) \cap ([0, w] + G)$ can be represented by $x = \lambda w + y$ and $x = \lambda' w + z$, where $0 \leqq \lambda \leqq 1$, $0 \leqq \lambda' \leqq 1$, $y \in F$ and $z \in G$. Suppose $\lambda > \lambda'$, then $y + (\lambda - \lambda')w = z$. By the definition of $\Omega = \{ x \mid A^t x \leqq 0 \}$, we have $G = \{x \mid x \in \Omega, C^t x = 0 \}$, where C is a submatrix of A consisting of several columns of A. By $w \notin G$ and $y \in \Omega$, $C^t z = C^t(y + (\lambda - \lambda')w) = C^t y + (\lambda - \lambda')C^t w < 0$, which means $z = y + (\lambda - \lambda')w \notin G$, a contradiction. Therefore $\lambda = \lambda'$, and $y = z$, that is $x \in [0, w] + (F \cap G)$. Therefore we have seen that $([0, w] + F) \cap ([0, w] + G) = [0, w] + (F \cap G)$. Since $[0, w] + (F \cap G)$ is the facet of both $[0, w] + F$ and $[0, w] + G$, the two cells in this case intersect in their common face. Case 2: The two cells chosen are $[0, w] + F$ and $\{w\} + \Omega$. Choose $x \in ([0, w] + F) \cap (\{w\} + \Omega)$. We have $x = \lambda w + y = w + z$, where $0 \leqq \lambda \leqq 1$, $y \in F$ and $z \in \Omega$. Since $B^t y = 0$ and $B^t w \leqq 0$ for the submatrix B of A defining the facet F, $B^t z = B^t(z - y) = (\lambda - 1)B^t w \geqq 0$. On the other hand, $B^t z \leqq 0$ by $z \in \Omega$, therefore $B^t z = 0$, i.e., $z \in F$. Then $x = w + z \in \{w\} + F$. Thus we have seen that $([0, w] + F) \cap (\{w\} + \Omega) = \{w\} + F$, which is a face of both $[0, w] + F$ and $\{w\} + \Omega$.

Third, for each $(\dim \Omega-1)$-cell of $\bar{P}$, we must make sure that at most two $\dim \Omega$-cells of P contain it as a facet. Case 1: Let C be a facet of $[0, w] + F$, i.e., $C = F$, $\{w\} + F$ or $[0, w] + G$, where F is a facet of Ω and G is a facet of F. When $C = F$, it lies on the boundary of Ω and there is no other cell of P containing it. When $C = \{w\} + F$, only $\{w\} + \Omega$ contains it because F is a facet of Ω. When $C = [0, w] + G$, there is a unique facet F' of Ω different from F such that $[0, w] + F'$ contains $[0, w] + G$ as a facet because $\partial\Omega$ itself is a subdivided manifold without boundary. Case 2: Let C be a facet of $\{w\} + \Omega$, then $C = \{w\} + F$ for some facet F of Ω. In this case C is also a facet of $[0, w] + F$ and there is no other cell of P which contains C as a facet. Now we have shown that P has all the properties required for a subdivided manifold. □

Lemma 3.4.

$$\bar{P} = \{[0, w] + F \mid w \notin F \prec \Omega \} \cup \{ F \mid w \notin F \prec \Omega \} \cup \{ \{w\} + F \mid F \prec \Omega \}.$$

Proof. This lemma is readily obtained from Lemma 3.1. □

Lemma 3.5. $|P| = \Omega$.

Proof. For each point $x \in \Omega$, let $\alpha = \max \{ \lambda \geqq 0 \mid x \in \lambda w + \Omega \}$. When $\alpha \geqq 1$, x belongs to $\{w\} + \Omega$, otherwise it belongs to $\{\alpha w\} + F \subset [0, w]$

+ F for some facet of Ω. Since clearly $|P| \subset \Omega$, we have shown the lemma. □

4. Dual Subdivided Manifold

In the following we assume that the interior $\operatorname{int} \Omega^+$ is not empty and let h be a point of $\operatorname{int} \Omega^+$. Define Ω_η as Ω truncated by $h^t x \leqq \eta$, i.e., $\Omega_\eta = \Omega \cap \{ x \mid h^t x \leqq \eta \}$ for $\eta > 0$. Then we obtain the following lemma.

Lemma 4.1. Ω_η is bounded.

Proof: Suppose Ω_η is unbounded, then there are $x \in \Omega_\eta$ and some $\bar{x} \neq 0$ such that $x + \beta\bar{x} \in \Omega_\eta$ for any $\beta \geqq 0$. This means

$$A^t x + \beta A^t \bar{x} \leqq 0 \quad \text{and} \quad 0 < h^t x + \beta h^t \bar{x} \leqq \eta \quad \text{for any} \quad \beta \geqq 0.$$

Since β takes any nonnegative value, we see $A^t \bar{x} \leqq 0$ and hence $\bar{x} \in \Omega$. By the choice of h, $h^t \bar{x} > 0$, which causes a contradiction. □

The following lemma gives us a clear description of the vertices of Ω_η.

Lemma 4.2. Let $\eta > 0$, $U(\Omega_\eta)$ be the vertex set of Ω_η and $H = \{ x \mid h^t x = \eta \}$. Then

$$U(\Omega_\eta) = \{0\} \cup \{ L \cap H \mid L \text{ is an extreme ray of } \Omega \}.$$

Let D be the collection of dual cones of all vertices of Ω_η, i.e.,

$$(4.1) \quad D = \{\{v\}^* \mid v \in U(\Omega_\eta) \}.$$

We define $\langle h \rangle = \{ y \mid y = \alpha h \text{ for } \alpha \geqq 0 \}$, and $L^* = \{ y \mid y = Bu, \text{ for } u \geqq 0 \}$ for an extreme ray $L = \{ x \mid x \in \Omega, B^t x = 0 \}$ of Ω. Then we have another representation of D.

Lemma 4.3. $D = \{0\}^* \cup \{ L^* + \langle h \rangle \mid L \text{ is an extreme ray of } \Omega \}$.

Proof: Let v be a vertex of Ω_η and suppose $v \neq 0$. By Lemma 4.2, $\{v\} = L \cap H$, where $L = \{ x \mid x \in \Omega, B^t x = 0 \}$ is an extreme ray of Ω and $H = \{x \mid h^t x = \eta \}$. Therefore

$$\{v\}^* = \{ y \mid y = Bu + \alpha h \text{ for } u \geqq 0, \alpha \geqq 0 \} = L^* + \langle h \rangle. \quad □$$

It is obvious that D is a subdivided n-manifold and

$$(4.2) \quad \bar{D} = \{ F^* \mid F \prec \Omega \} \cup \{ F^* + \langle h \rangle \mid \{0\} \neq F \prec \Omega \}$$

$$(4.3) \quad |D| = R^n.$$

5. The Primal-Dual Pair of Subdivided Manifolds

We define the dual operator d as follows according to the location of w.

When $w \neq 0$,

$$
(5.1)\qquad
\begin{array}{lll}
([0,\ w] + F\ )^d = F^* & & \text{if } \ w \notin F \prec \Omega \\
(\ \ F\ \ )^d = \emptyset & & \text{if } \ w \notin F \prec \Omega \\
(\ \{w\} +\ F\ )^d = F^* + \langle\ h\ \rangle & & \text{if } \ \{0\} \neq F \prec \Omega \\
(\ \ \{w\}\ \ )^d = \emptyset & & \\
(\ \ F^*\ \ )^d = [0,\ w] + F & & \text{if } \ w \notin F \prec \Omega \\
\qquad\quad\ = \emptyset & & \text{if } \ w \in F \prec \Omega \\
(\ F^* + \langle\ h\ \rangle)^d = \{w\} + F & & \text{if } \ \{0\} \neq F \prec \Omega .
\end{array}
$$

When $w = 0$,

$$
(5.2)\qquad
\begin{array}{ll}
(\ \ F\ \ )^d = F^* + \langle\ h\ \rangle & \text{if } \ \{0\} \neq F \prec \Omega \\
(\ \ \{0\}\ \ )^d = \{0\}^* & \\
(\ F^* + \langle\ h\ \rangle)^d = F & \text{if } \ \{0\} \neq F \prec \Omega \\
(\ \ \{0\}^*\ \ )^d = \{0\}. &
\end{array}
$$

Fig.2 shows the case where w lies in face F_2.

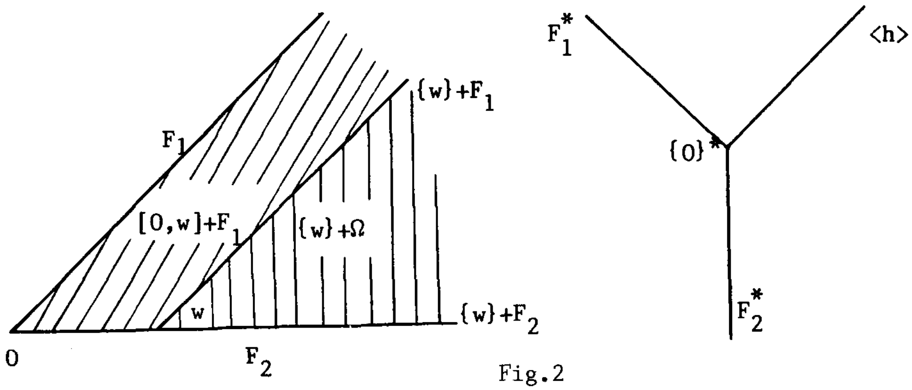

Fig.2

Let $(P,D;d)$ be a PDM with degree $n+1$ defined by (3.2), (4.1), (5.1) and (5,2), and let $M = \langle P,D;d\rangle$. Then as a direct consequence of Theorem 2.2, we have that M is a subdivided $(n+1)$-manifold. As for the boundary of M we obtain the following lemma.

Lemma 5.1. Suppose $w \neq 0$. Then

$$
(5.3)\qquad
\begin{array}{rl}
\partial M = & \{\{w\} \times \{0\}^*\} \\
& \cup\ \{\{w\} \times (L^* + \langle\ h\ \rangle)\ |\ L \ \text{ is an extreme ray of } \ \Omega\ \} \\
& \cup\ \{\ F \times F^*\ |\ w \notin F \prec \Omega\ \} \\
& \cup\ \{(\{w\} + F) \times F^*\ |\ w \in F \prec \Omega\ \}
\end{array}
$$

$\cup \{([0, w] + G) \times F^* \mid w \in F \prec \Omega,\ w \notin G \lhd F \}$.

and

(5.4) $\quad |\partial M| = (\{w\} \times R^n) \cup (\ \cup \{ F \times F^* \mid F \prec \Omega \})$.

Proof: Since the cells on the right side of (5.3) have dimension n and $\{w\}^d = \emptyset$, $F^d = \emptyset$, and $(F^*)^d = \emptyset$ if $w \in F$, they are clearly members of ∂M. To see the reverse relation note that if $X \times Y \in \partial M$ then one of X^d and Y^d is empty. If $X^d = \emptyset$ (resp. $Y^d = \emptyset$), then the unique cell of M containing $X \times Y$ is $Y^d \times Y$ (resp. $X \times X^d$). Suppose first $X^d = \emptyset$, then $X = \{w\}$ or F such that $w \notin F \prec \Omega$ by (5.1). Case 1: When $X = \{w\}$, we have $\dim Y = n$ and $\{w\} \lhd Y^d$. This implies by the definition of the dual operator d that either $Y = \{0\}^*$ or $Y = L^* + \langle h \rangle$ for some ray L of Ω. Then $X \times Y$ is either $\{w\} \times \{0\}^*$ or $\{w\} \times (L^* + \langle h \rangle)$. Case 2: When $X = F$ with $w \notin F \prec \Omega$, we have $\dim Y = n - \dim F$ and $F \lhd Y^d$. Therefore $Y = F^*$ and then $X \times Y = F \times F^*$. Next, suppose $Y^d = \emptyset$, i.e., $Y = F^*$ for some face of Ω with $w \in F$. Then $\dim X = n - \dim F^* = \dim F$ and $F^* \lhd X^d$. Therefore we obtain that $X = \{w\} + F$ or $X = ([0, w] + G)$ for some facet G of F not containing w. That is $X \times Y$ is either $(\{w\} + F) \times F^*$ or $([0, w] + G) \times F^*$. Now we have proved (5.3). Note that

$$F = (\{w\} + F) \cup \{[0, w] + G \mid w \notin G \lhd F \} \quad \text{if } w \in F, \text{ and}$$

$$R^n = \{0\}^* \cup \{ L^* + \langle h \rangle \mid L \text{ is a extreme ray of } \Omega \},$$

then we obtain (5.4) from (5.3). □

When $w = 0$, we have the following lemma by the similar argument.

Lemma 5.3. When $w = 0$,

$$\partial M = \{\{0\} \times (L^* + \langle h \rangle) \mid L \text{ is an extreme ray of } \Omega \} \cup \{ F \times F^* \mid \{0\} \neq F \prec \Omega \}.$$

and

$$|\partial M| = \{0\} \times \cup \{L^* + \langle h \rangle \mid L \text{ is an extreme ray of } \Omega \} \cup (\cup \{ F \times F^* \mid \{0\} \neq F \prec \Omega \}).$$

6. Basic System of the Algorithm and Convergence Condition

Now let a function $g\colon |M| \to R^n$ be defined by

$$g(x,y) = y + f(x),$$

for $(x,y) \in |M|$ and consider the system of n equations in 2n variables

(6.1) $\quad g(x,y) = 0, \quad (x,y) \in |M|$.

Though our main purpose of this paper is to develop an algorithm for SPP with an affine function f, we will first discuss the condition for the existence of a path leading to a stationary point without the linearlity assumption. It should be noted that the path is generally a nonlinear curve and so we have to resort to simplicial algorithm as in Talman and Yamamoto [23] or the well-known predictor-corrector method to trace it.

For a bounded subset W of Ω we say that $U \subset \Omega$ separates W from ∞ if $U \cap W = \emptyset$ and any unbounded connected subset of Ω which intersects W also intersects U.

Condition 6.1. There exists a subset U of Ω which separates the starting point w from ∞ and

$$(x - w)^t f(x) > 0 \quad \text{for any} \quad x \in U.$$

Lemma 6.2. Suppose $w \in \Omega$ is not a stationary point and $0 \in R^n$ is a regular value of PC^1-function $g: |M| \to R^n$. Then

(6.2) $(x^o, y^o) = (w, -f(w))$ lies in $g^{-1}(0) \cap |\partial M|$.

(6.3) the connected component of $g^{-1}(0)$ containing (x^o, y^o) is a path.

Proof: When $w \neq 0$, $(w, -f(w))$ is contained in $|\partial M|$ by (5.4). When $w = 0$, $(w, -f(w)) \in \{0\} \times (L^* + \langle h \rangle)$ for some L because w is not a stationary point and therefore also in $|\partial M|$. If $0 \in R^n$ is a regular value of g, by Theorem 2.1, we have that the connected component of $g^{-1}(0)$ having (x^o, y^o) is a path. □

In the following we denote the path by S.

Lemma 6.3. Let (x,y) be an arbitrary solution of (6.1). If point (x,y) is neither the starting point nor a stationary point, then $(x - w)^t y \geqq 0$.

Proof: Since (x,y) is a solution of (6.1), then $y = -f(x)$ and there is a cell $Z \in M$ containing (x,y). Assume first $Z = ([0, w] + F) \times F^*$ for $w \notin F$. If $(x,y) \in (\{0\} + F) \times F^* = F \times F^*$, it means that x is a stationary point, a contradiction. Therefore $(x,y) \in ((0, w] + F) \times F^*$. Let B be a submatrix of A such that $F = \{ x \mid x \in \Omega, B^t x = 0 \}$. Each point (x,y) of $((0, w] + F) \times F^*$ can be written as $x = \lambda w + z$, where $0 < \lambda \leqq 1$, $z \in F$ and $y = Bu$ for some $u \geqq 0$. Since $w \in \Omega$, then $B^t w \leqq 0$ and so $w^t y = u^t B^t w \leqq 0$. Therefore $(x - w)^t y = ((\lambda - 1)w + z)^t y = (\lambda - 1)w^t y \geqq 0$. Next suppose $Z = (\{w\} + F) \times (F^* + \langle h \rangle)$. If $(x,y) \in (\{w\} + F) \times F^*$, that is $x = \lambda w + z$ for $\lambda = 1$ and $z \in F$, this is included in the above case. So we only consider the case where $x = w + z$

and $y = z^* + \alpha h$ for $z \in F$, $z^* \in F^*$ and $\alpha > 0$. Then $(x - w)^t y = z^t(z^* + \alpha h) = z^t z^* + \alpha z^t h > 0$ by the choice of h. □

Theorem 6.4. Suppose $f\colon \Omega \to R^n$ is a C^1-function and satisfies Condition 6.1. Suppose also $0 \in R^n$ is a regular value of $g(x,y) = y + f(x)$. Then there exists a path of $g^{-1}(0)$ leading from $(x^o,y^o) = (w,-f(w))$ to a stationary point.

Proof: By the regular value assumption Lemma 6.2 yields the existence of a path S starting from (x^o,y^o). We suppose S is unbounded, then $S_x = \{ x \mid (x,y) \in S$ for some $y \}$ is also unbounded since f is continuous, and so S_x intersects U. For a point $x \in S_x \cap U$, by Lemma 6.3, we have $(x - w)^t(-f(x)) \geq 0$. This contradicts condition 6.1, and we have obtained that S is bounded. Obviously S is compact, therefore S has another end-point, say (x,y) in $|\partial M|$ by (2.6). Suppose first $(x,y) \in \{w\} \times R^n$, then $x = w$. Since $(x,y) \in g^{-1}(0)$, $y = -f(x) = -f(w) = y^o$. Hence we have $(x,y) = (x^o,y^o)$, which contradicts (2.6). Therefore by (5.4) we have $(x,y) = (x,-f(x)) \in F \times F^*$, which means that x is a stationary point by (3.1). □

Definition 6.5. The function $f\colon \Omega \to R^n$ is called strongly copositive on Ω with respect to $w \in \Omega$ if there is an $\alpha > 0$ such that

$$(x - w)^t(f(x) - f(w)) \geq \alpha \|x - w\|^2.$$

This is different from the conventional definition of strongly copositive function in which the point w is replaced by $0 \in R^n$. See for example Habetler and Price [9].

Lemma 6.6. If the function $f\colon \Omega \to R^n$ is strongly copositive on Ω with respect to $w \in \Omega$, then Conditon 6.1 holds.

Proof: Let $C = \{ x \mid x \in \Omega,\ \|x - w\| \leq \|f(w)\| / \alpha \}$, then C is clearly bounded and $w \in C$. Therefore $U = \Omega \setminus C$ separates w from ∞. For an arbitrary point x of U we also obtain

$$\begin{aligned}(x - w)^t f(x) &\geq \alpha \|x - w\|^2 + (x - w)^t f(w) \\ &\geq \alpha \|x - w\|^2 - \|x - w\| \|f(w)\| \\ &\geq \|x - w\| (\alpha \|x - w\| - \|f(w)\|) \\ &> 0.\end{aligned}$$ □

Theorem 6.7. Suppose f is a C^1-function and $0 \in R^n$ is a regular value of g. If $f\colon \Omega \to R^n$ is strongly copositive on Ω with respect to the starting point $w \in \Omega$, then there is a path of $g^{-1}(0)$ connecting (x^o,y^o)

$= (w,-f(w))$ to a stationary point.

As mentioned before, the path is generally a nonlinear curve and some technique such as simplicial approximation is needed to trace it. We do not go further into this subject and change to the case where f is an affine function $Qx + c$.

Definition 6.8. Q is said to be copositive plus on Ω if $x^tQx \geqq 0$ for any $x \in \Omega$ and $x \in \Omega$ and $x^tQx = 0$ imply $(Q + Q^t)x = 0$.

Lemma 6.9. There is not an $x \in \Omega$ such that $Qx + c \in \Omega^+$ if and only if there exists a $v \in \Omega$ such that $Q^tv \in -\Omega^+$ and $v^tc < 0$.

Proof: There is not an $x \in \Omega$, such that $Qx + c \in \Omega^+$, if and only if there is not a solution $x_1 \geqq 0$, $x_2 \geqq 0$, $\alpha \geqq 0$ of the system

$$A^t(x_1 - x_2) \leqq 0$$
$$Q(x_1 - x_2) + c = -A\alpha,$$

where $\Omega = \{ x \mid A^tx \leqq 0 \}$. By Farkas' Alternative Theorem, this is equivalent to that the following system (6.4) has a solution.

$$(6.4) \qquad \begin{aligned} &A\bar{u} - Q^t\bar{v} = 0 \\ &-A^t\bar{v} \leqq 0 \\ &\bar{u} \leqq 0 \\ &\bar{v}^tc > 0. \end{aligned}$$

Using u, v instead of $-\bar{u}$ and $-\bar{v}$, respectively, we have

$$(6.5) \qquad \begin{aligned} &Q^tv = Au \\ &A^tv \leqq 0 \\ &u \geqq 0 \\ &v^tc < 0, \end{aligned}$$

which means $v \in \Omega$, $Q^tv \in -\Omega^+$ and $v^tc < 0$. □

Lemma 6.10 Suppose that Q is copositive plus and the starting point w is chosen so that $Qw \in \text{int } \Omega^+$. If the path S in Lemma 6.2 is unbounded and does not contain a point which provides a stationary point, then the stationary point problem has no solutions.

Proof: Suppose S is unbounded, then there are $(x,y) \in S$ and $(\bar{x},\bar{y}) \neq 0$ such that $(x,y) + \beta(\bar{x},\bar{y}) \in S$ for any $\beta \geqq 0$. Then

$$\bar{y} + Q\bar{x} = 0$$

and for some $F \prec \Omega$

$$(x,y) + \beta(\bar{x},\bar{y}) \in ([0, w] + F) \times F^* \quad \text{or} \quad (\{w\} + F) \times (F^* + \langle h \rangle).$$

Here note that $\bar{x} \neq 0$ because the contrary would yield $(\bar{x},\bar{y}) = 0$. Case 1: Consider the case where $(x,y) + \beta(\bar{x},\bar{y}) \in (\{w\} + F) \times (F^* + \langle h \rangle)$. In this case

$$x = w + x' \quad \text{for some} \quad x' \in F$$
$$y = y' + \lambda h \quad \text{for some} \quad y' \in F^* \quad \text{and some} \quad \lambda \geqq 0.$$

and

$$\bar{x} \in F$$
$$\bar{y} = \bar{y}' + \alpha h \quad \text{for some} \quad \bar{y}' \in F^* \quad \text{and some} \quad \alpha \geqq 0.$$

Then we obtain

$$\bar{x}^t Q\bar{x} = \bar{x}^t(-\bar{y}) = \bar{x}^t(-\bar{y}' - \alpha h) = -\alpha\bar{x}^t h.$$

Suppose $\alpha > 0$. Since $h \in \text{int}\ \Omega^+$ and $\bar{x} \in \Omega$ we have $\bar{x}^t Q\bar{x} = -\alpha\bar{x}^t h < 0$. This contradicts that Q is copositive plus on Ω. Therefore α should be equal to 0. If $\lambda = 0$, then the ray is entirely contaied in $(\{w\} + F) \times F^*$. This is excluded by the regular value assumption. Then $\lambda > 0$ in the following. Since $\alpha = 0$, we also have $\bar{x}^t Q\bar{x} = 0$, which implies that $(Q + Q^t)\bar{x} = 0$. Then

$$Q^t\bar{x} = -Q\bar{x} = \bar{y} = \bar{y}' + \alpha h = \bar{y}' \in F^* \subset -\Omega^+. \tag{6.6}$$

We also have

$$\begin{aligned} \bar{x}^t(-y) &= \bar{x}^t(Qx + c) = \bar{x}^t Qx + \bar{x}^t c = x^t Q^t\bar{x} + \bar{x}^t c \\ &= x^t(-Q\bar{x}) + \bar{x}^t c = x^t\bar{y} + \bar{x}^t c. \end{aligned} \tag{6.7}$$

On the other hand since $\bar{x} \in F$ and $y' \in F^*$

$$\bar{x}^t(-y) = -\bar{x}^t(y' + \lambda h) = -\bar{x}^t y' - \lambda\bar{x}^t h = -\lambda\bar{x}^t h < 0. \tag{6.8}$$

From (6.7) and (6.8), we have $x^t\bar{y} + \bar{x}^t c < 0$, i.e.,

$$\begin{aligned} \bar{x}^t c < -x^t\bar{y} &= -(w + x')^t\bar{y} = -w^t\bar{y} - x'^t\bar{y} \\ &= -w^t\bar{y} - x'^t\bar{y}' = -w^t\bar{y} = -w^t(-Q\bar{x}) \\ &= -w^t(Q^t\bar{x}) = -\bar{x}^t Qw < 0. \end{aligned} \tag{6.9}$$

The last inequality comes from $Qw \in \text{int}\ \Omega^+$. From (6.6), (6.9) and Lemma 6.9, there are no stationary points.

Case 2: Suppose $(x,y) + \beta(\bar{x},\bar{y}) \in ([0, w] + F) \times F^*$. In the same way as Case 1, we have $\lambda \geqq 0$ and $x' \in F$ such that $x = \lambda w + x'$. Note also that $y \in F^*$, $\bar{x} \in F$ and $\bar{y} \in F^*$. When $\lambda = 0$, $(x,y) \in F \times F^*$, i.e., x is a stationary point. Since we have assumed that S does not contain such a point, we see $\lambda > 0$. Since $\bar{y} \in F^*$, by the same argument as in Case 1 we obtain $(Q + Q^t)\bar{x} = 0$ and

(6.10) $\quad Q^t\bar{x} = -Q\bar{x} = \bar{y} \in F^* \subset -\Omega^+.$

Note that (6.7) holds also in this case and $\bar{x}^t y = 0$. Then

(6.11) $\quad \bar{x}^t c = -x^t\bar{y} = -(\lambda w + x')^t\bar{y} = -\lambda w^t\bar{y} - x'^t\bar{y}$

$\qquad\qquad = \lambda w^t(-Q\bar{x}) = -\lambda w^t(Q^t\bar{x}) = -\lambda\bar{x}^t Qw < 0.$

From (6.10), (6.11) and Lemma 6.9, we have seen that there are no stationary points in this case either. □

7. The Algorithm

In this section we confine ourselves to the affine function $f(x) = Qx + c$, where Q is an $n\times n$ matrix and c is an n-dimensional vector. We appropriate the letter A to denote the matrix defining Ω with the $m+1^{st}$ column $a^{m+1} = h$ added. Then Ω_η given in Section 4 is written as

$$\Omega_\eta = \{ x \mid A^t x \leq b \},$$

where $b = (0,\ldots,0,\eta)^t \in R^{m+1}$ and $\eta > 0$. Letting $R(F)$ be the set of the directions of all extreme rays which lie in face F of Ω, F is written as

$$F = \{ x \mid x = \textstyle\sum_{u\in R(F)}\lambda_u u,\ \lambda_u \geq 0 \ \text{ for } \ u \in R(F) \}.$$

Then $g^{-1}(0)$ intersects a cell $C = ([0, w] + F) \times F^*$ if and only if the system (7.1.a) has a solution (λ,μ):

$$\text{(7.1.a)}\qquad \begin{aligned} &\textstyle\sum_{i\in I(F)}\mu_i a^i + \sum_{u\in R(F)}\lambda_u Qu + \lambda_w Qw = -c. \\ &\mu_i \geq 0 \ \text{ for all } \ i \in I(F) \\ &\lambda_u \geq 0 \ \text{ for all } \ u \in R(F) \\ &0 \leq \lambda_w \leq 1. \end{aligned}$$

In exactly the same way $g^{-1}(0)$ intersects a cell $C = (\{w\} + F) \times (F^* + \langle h \rangle)$ if and only if there is a solution of

$$\text{(7.1.b)}\qquad \begin{aligned} &\textstyle\sum_{i\in I(F)}\mu_i a^i + \mu_{m+1}a^{m+1} + \sum_{u\in R(F)}\lambda_u Qu + Qw = -c \\ &\mu_i \geq 0 \ \text{ for all } \ i \in I(F) \\ &\mu_{m+1} \geq 0 \\ &\lambda_u \geq 0 \ \text{ for all } \ u \in R(F). \end{aligned}$$

In the following we put upper bar, e.g. $\bar{F}$, to denote faces of Ω_η in distinction from faces of Ω. For each face $\bar{F}$ of Ω_η, let $U(\bar{F})$ be the set of all vertices of $\bar{F}$. Then $U(\bar{F}) \setminus \{0\}$ coincides with $R(F)$

normalized by $h^t x = \eta$ for the corresponding face F of Ω when $U(\bar{F})$ contains the origin. Let $\bar{I}(\bar{F})$ be the index set of binding constraints of $\bar{F}$, that is

$$\bar{I}(\bar{F}) = \{\, i \mid 1 \leq i \leq m+1,\ (a^i)^t x = b_i \ \text{ for any point } x \text{ of } \bar{F} \,\}.$$

The systems (7.1.a) and (7.1.b) then can be combined and yield the following system

$$(7.2)\qquad \begin{aligned} &\textstyle\sum_{i\in\bar{I}(\bar{F})}\mu_i a^i + \sum_{u\in U(\bar{F})}\lambda_u Qu + \lambda_w Qw = -c \\ &\mu_i \geq 0 \ \text{ for all } \ i \in \bar{I}(\bar{F}) \\ &\lambda_u \geq 0 \ \text{ for all } \ u \in U(\bar{F}) \\ &\lambda_w \geq 0,\ \lambda_o \geq 0 \\ &\lambda_o + \lambda_w = 1 \\ &\lambda_o = 0 \ \text{ when } \ 0 \notin U(\bar{F}). \end{aligned}$$

The point $(x,y) \in g^{-1}(0) \cap C$ corresponding to the solution (λ,μ) of (7.2) is clearly given by

$$(7.3)\qquad \begin{aligned} x &= \textstyle\sum_{u\in U(\bar{F})}\lambda_u u + \lambda_w w \\ y &= \textstyle\sum_{i\in\bar{I}(\bar{F})}\mu_i a^i. \end{aligned}$$

Since f is an affine function, $g^{-1}(0) \cap C$ is either a line segment, a half line or a line under the regular value assumption. Thus we obtain the following lemma, which was observed by Yamamoto [25].

Lemma 7.1. Suppose $0 \in R^n$ is a regular value of g and $g^{-1}(0) \cap C \neq \emptyset$ for a cell $C \in M$. Suppose further two linear programming problems of minimizing and maximizing

$$(7.4)\qquad p^t x + q^t y \quad \text{subject to (7.2) and (7.3)}$$

have optimum solutions $(\lambda^1,\mu^1,x^1,y^1)$ and $(\lambda^2,\mu^2,x^2,y^2)$ respectively. If $p^t x^1 + q^t y^1 \neq p^t x^2 + q^t y^2$, then (x^1,y^1) and (x^2,y^2) are two distinct end-points of the line segment $g^{-1}(0) \cap C$.

The next lemma shows how to find $\bar{I}(\bar{F})$ for face $\bar{F}$. Here we abbreviate $\bar{I}(\{u\})$ by $\bar{I}(u)$ when u is a single point.

Lemma 7.2. Let $u^1,\ldots,u^k$ be points of face $\bar{F}$ such that the affine hull $\mathrm{aff}(\{u^j \mid j=1,\ldots,k\})$ contains $\bar{F}$. Then

$$\bar{I}(\bar{F}) = \bigcap_{j=1}^{k} \bar{I}(u^j).$$

Proof. See Lemma 3.2 of Yamamoto [25]. □

Lemma 7.3 gives us the termination condition of the algorithm.

Lemma 7.3. Let (λ,μ) be a solution of (7.2) and let

$$\bar{I}_+ = \{ i \mid i \in \bar{I}(\bar{F}) \text{ and } \mu_i > 0 \}.$$

If either

(7.5.a) $\lambda_w = 0$ or

(7.5.b) $\bar{I}_+ \subset I(w)$ where $I(w) = \{ i \mid i \in \{1,\dots,m\},\ (a^i)^t w = b_i \}$,

then $x = \sum_{u\in U(\bar{F})}\lambda_u u + \lambda_w w$ is a stationary point.

Proof: If $\lambda_w = 0$, then $\lambda_o = 1$ by (7.2), which implies $0 \in U(\bar{F}) \subset \bar{F}$. Therefore $m+1 \notin \bar{I}(\bar{F})$. By the definition of $\bar{I}_+$, $\bar{I}_+ \subset \bar{I}(\bar{F})$, and hence $\bar{F} \subset \bar{F}(\bar{I}_+)$, where $\bar{F}(\bar{I}_+)$ is a face of Ω_η defined by $\bar{I}_+$. Then $x = \sum_{u\in U(\bar{F})}\lambda_u u + \lambda_w w = \sum_{u\in U(\bar{F})}\lambda_u u \in \bar{F} \subset \bar{F}(\bar{I}_+)$ and $-f(x) = \sum_{i\in\bar{I}_+}\mu_i a^i \in \bar{F}(\bar{I}_+)^*$. Furthermore since $m+1 \notin \bar{I}_+$, then $\bar{F}(\bar{I}_+) \subset F(\bar{I}_+)$ and $\bar{F}(\bar{I}_+)^* = F(\bar{I}_+)^*$, where F is the face of Ω having the same index set $\bar{I}_+$ of binding constraints. So $x \in F(\bar{I}_+)$ and $-f(x) \in F(\bar{I}_+)^*$. This means that x is a stationary point. If $\bar{I}_+ \subset I(w)$, then $w \in \bar{F}(\bar{I}_+)$. On the other hand, $\bar{I}_+ \subset \bar{I}(\bar{F})$ yields $U(\bar{F}) \subset U(\bar{F}(\bar{I}_+))$. Then the point $x = \sum_{u\in U(\bar{F})}\lambda_u u + \lambda_w w$ lies in $\bar{F}(\bar{I}_+)$. Note that $I(w)$ does not contain $m+1$ and hence $m+1 \notin \bar{I}_+$. Let F be a face of Ω whose binding constraint index set is $\bar{I}_+$. Then $\bar{F}(\bar{I}_+) \subset F(\bar{I}_+)$ and $\bar{F}(\bar{I}_+)^* = F(\bar{I}_+)^*$. Therefore also in this case x lies in $F(\bar{I}_+)$ and $-f(x)$ in $F(\bar{I}_+)^*$, and hence x is a stationary point. □

Now we give the algorithm. Here $A(\bar{I},U)$ denote the set of (λ,μ) satisfying (7.2) with $U(\bar{F}(\bar{I}))$ replaced by U. Let $U(\bar{I})$ be the set of all vertices of the face of Ω_η having the index set $\bar{I}$ of binding constraints, and we denote the linear subspace spanned by a set X of vectors by $\mathrm{spc}(X)$. We define for the sake of simplicity of notation

$$\begin{aligned} w \oplus v &= w + v && \text{if } v \neq 0 \\ &= 0 && \text{if } v = 0 \\ w \oplus X &= \{ w \oplus x \mid x \in X \} \end{aligned}$$

for a set X of R^n. The variable δ keeps the dimension of the current face.

```
      begin
{1}      read (w);
{2}      solve    min.  μ_{m+1}
                  s.t.  Σ_{i=1}^{m+1} μ_i a^i = -f(w)
```

```
                    μ_i ≧ 0    1 ≦ i ≦ m+1;
{3}     μ := the optimal solution;
        v := the vertex of Ω_η which is optimal of the dual problem
             of the above linear programming;
        Ī := Ī(v);
        Ī_+ := { i | μ_i > 0  for  1 ≦ i ≦ m+1 };
{4}     if  Ī_+ ⊂ I(w)  then
          writeln('The starting point  w  is a stationary point')
        else
        begin
{5}         U := {v};  δ := 0;
{6}         solve   max. λ_v
                    s.t. (λ,μ) ∈ A(Ī,U);
{7}         (λ,μ) := the optimal solution;
            Ī_+ := { i | μ_i > 0, i ∈ Ī };
            U_+ := { u | λ_u > 0, u ∈ U };
{8}         while  λ_w ≠ 0  and  Ī_+ ⊄ I(w)  do
            begin
                if  |U_+| = δ + 1  then
                begin
{9}                 choose  k ∈ Ī \ Ī_+  such that  a^k  is linearly
                    independent of  { a^i | i ∈ Ī_+ };
{10}                find  v ∈ U(Ī_+)  such that  (a^k)^t v < b_k;
{11}                Ī := Ī ∩ Ī(v); δ := δ + 1; U := U(Ī);
{12}                find  p ∈ aff((w ⊕ (U_+ ∪ {v})) ∪ {w})  such that
                    p^t(u-w) = 0  for  u ∈ w ⊕ U_+  and  p^t(w ⊕ v - w) < 0;
{13}                solve    min. p^t x
                             s.t. x = λ_w w + Σ_{u∈U} λ_u u
                                  (λ,μ) ∈ A(Ī,U);
{14}                if there are no optimal solutions then go to unbounded
                    else
                      begin
                        (λ,μ) := the optimal solution;
                        Ī_+ := { i | μ_i > 0  for  i ∈ Ī };
                        U_+ := { u | λ_u > 0  for  u ∈ U }
                      end
                end
                else
```

```
                    begin
{15}                   Ī' := ⋂_{u∈U} Ī(u);
{16}                   choose  k ∈ Ī' \ Ī;
{17}                   Ī := Ī'; δ := δ - 1; U := U(Ī);
{18}                   find  q ∈ spc({ a^i | i ∈ Ī_+ } ∪ { a^k })  such that
                       q^t a^i = 0  for  i ∈ Ī_+  and  q^t a^k < 0;
{19}                   solve   min.  q^t y
                                s.t.  y := Σ_{i∈Ī} μ_i a^i
                                      (λ,μ) ∈ A(Ī,U);
{20}                   if there is no optimal solution then go to unbounded
                       else
                         begin
                           Ī_+ := { i | μ_i > 0  for  i ∈ Ī };
                           U_+ := { u | λ_u > 0  for  u ∈ U }
                         end
                    end
                end
{21}            x := Σ_{u∈U} λ_u u + λ_w w;
                writeln('A stationary point is found ', x );
                unbounded
            end
          end.
```

We wil give several lemmas to show that the algorithm traces the path S. We assume that the system (7.2) is nondegenerate and that the linear programming problems to be solved in the algorithm have a finite solution. Through the following lemmas we assume that $\lambda_w > 0$. We abbreviate $U(\bar{F}(\bar{J}))$ by $U(\bar{J})$ and employ the notations

$$
\begin{aligned}
x &= \textstyle\sum_{u\in U(\bar{J})}\lambda_u u + \lambda_w w \\
y &= \textstyle\sum_{i\in \bar{J}} \mu_i a^i \\
U_+ &= \{\, u \mid u \in U(\bar{J}),\ \lambda_u > 0 \,\} \\
\bar{J}_+ &= \{\, i \mid i \in \bar{J},\ \mu_i > 0 \,\}.
\end{aligned}
\tag{7.6}
$$

For any solution $(\lambda,\mu) \in A(\bar{J},U(\bar{J}))$, (x,y) defined by (7.6) is on the line segment $S \cap C$ for some cell C of M, which could be denoted by $X(\bar{J}) \times Y(\bar{J})$. In fact, the cell C is $([0, w] + F(\bar{J})) \times F(\bar{J})^*$ when $m+1 \notin \bar{J}$ and $(\{w\} + F(\bar{J}\setminus\{m+1\})) \times (F(\bar{J}\setminus\{m+1\})^* + \langle h \rangle)$ when $m+1 \in \bar{J}$.

We will omit the proofs of the following Lemma 7.4 to 7.10 since they can be seen in almost the same way as Lemma 4.4 to 4.10 in Yamamoto [25], respectively.

Lemma 7.4 Let (λ,μ) be a solution of $A(\bar{J},U(\bar{J}))$. Suppose

(a) (x,y) defined by (7.6) is an end-point of the line segment $S \cap C$, where $C = X(\bar{J}) \times Y(\bar{J}) \in M$ and (x,y) lies in the common facet $X(\bar{J}) \times Y(\bar{I})$ of both cells $X(\bar{J}) \times Y(\bar{J})$ and $X(\bar{I}) \times Y(\bar{I})$, where $Y(\bar{I}) \lhd Y(\bar{J})$.

(b) $X(\bar{J}) \subset \mathrm{aff}((w \oplus U_+) \cup \{w\})$.

(c) $p \in R^n$ satisfies the condition in step 12 for some vertex $v \in U(\bar{I}) \setminus U(\bar{J})$.

Let $(\lambda^1,\mu^1,x^1,y^1)$ be a minimizer of p^tx under $(\lambda,\mu) \in A(\bar{I},U(\bar{I}))$ and (7.6). Then (x^1,y^1) is the opposite end-point of the line segment $S \cap C$.

Lemma 7.5. Let (λ,μ) be a solution of $A(\bar{J},U(\bar{J}))$. Suppose

(a) (x,y) defined by (7.6) is an end-point of line segment $S \cap C$, where $C = X(\bar{J}) \times Y(\bar{J}) \in M$ and (x,y) lies in the facet $X(\bar{I}) \times X(\bar{J})$ of both cells $X(\bar{J}) \times Y(\bar{J})$ and $X(\bar{I}) \times Y(\bar{I})$, where $X(\bar{I}) \lhd X(\bar{J})$.

(b) $Y(\bar{J}) \subset \mathrm{spc}(\{\ a^i \mid i \in \bar{J}_+\ \})$.

(c) $q \in R^n$ satisfies the condition in Step 18 for some $k \in \bar{I} \setminus \bar{J}$.

Let $(\lambda^1,\ \mu^1,\ x^1,\ y^1)$ be a minimizer of q^ty under $(\lambda,\mu) \in A(\bar{I},U(\bar{I}))$ and (7.6). Then (x^1,y^1) is the opposite end-point of the line segment $S \cap C$.

Lemma 7.6. Let (λ,μ) be a basic solution of $A(\bar{J},U(\bar{J}))$ such that (x,y) defined by (7.6) is an end-point of the line segment $S \cap C$, where $C = X(\bar{J}) \times Y(\bar{J}) \in M$. If $|U_+| = \dim \bar{F}(\bar{J}) + 1$, then the point (x,y) lies in $X(\bar{J}) \times Y(\bar{I})$ for some $Y(\bar{I}) \lhd Y(\bar{J})$. If $|U_+| < \dim \bar{F}(\bar{J}) + 1$, then the point (x,y) lies in $X(\bar{I}) \times Y(\bar{J})$ for some $X(\bar{I}) \lhd X(\bar{J})$.

In fact, in the above lemma when $|U_+| = \dim \bar{F}(\bar{J}) + 1$, if $m+1 \in \bar{J}$, we see that $X(\bar{J}) \times Y(\bar{I})$ is either $(\{w\} + F(\bar{J}\setminus\{m+1\})) \times F(\bar{J}\setminus\{m+1\})^*$ or $(\{w\} + F(\bar{J}\setminus\{m+1\})) \times (F(\bar{J}\setminus\{i\})^* + \langle h \rangle)$ for some $i \in \bar{J} \setminus \{m+1\}$. If $m+1 \notin \bar{J}$, then $X(\bar{J}) \times Y(\bar{I}) = ([0,\ w] + F(\bar{J})) \times F(\bar{J}\setminus\{i\})^*$ for some $i \in \bar{J}$. When $|U_+| < \dim \bar{F}(\bar{J}) + 1$, if $m+1 \in \bar{J}$, we see that $X(\bar{I}) \times Y(\bar{J}) = (\{w\} + F(J')) \times (F(\bar{J}\setminus\{m+1\})^* + \langle h \rangle)$ for some $F(J') \lhd F(\bar{J}\setminus\{m+1\})$. If $m+1 \notin \bar{J}$, then $X(\bar{I}) \times Y(\bar{J})$ is either $(\{w\} + F(\bar{J})) \times F(\bar{J})^*$ or $([0,\ w] + F(J')) \times F(\bar{J})^*$ for some $F(J') \lhd F(\bar{J})$.

Lemma 7.7. Let (λ,μ) be a basic solution of $A(\bar{J},U(\bar{J}))$ such that (x,y) defined by (7.6) is an end-point of the line segment $S \cap C$, where $C = X(\bar{J}) \times Y(\bar{J}) \in M$. Suppose

(a) $\bar{J} = \bar{I}(\bar{F}(\bar{J}))$, i.e., $\bar{J}$ is the maximum index set defining $\bar{F}(\bar{J})$;
(b) $|U_+| = \dim \bar{F}(\bar{J}) + 1$. Then
(7.7) there exists a $k \in \bar{J} \setminus \bar{J}_+$ such that a^i's $(i \in \bar{J}_+)$ and a^k are linearly independent,
(7.8) there is a vertex $v \in U(\bar{J}_+)$ such that $(a^k)^t v < b_k$,
(7.9) let $\bar{I} = \bar{J} \cap \bar{I}(v)$, then $\bar{I} = \bar{I}(\bar{F}(\bar{I}))$, i.e., $\bar{I}$ is the maximum index set defining $\bar{F}(\bar{I})$ and $\bar{F}(\bar{J}) \lhd \bar{F}(\bar{I})$.

Lemma 7.8. Let (λ,μ) be a basic solution of $A(\bar{J},U(\bar{J}))$ such that (x,y) defined by (7.6) is an end-point of the line segment $S \cap C$, where $C = X(\bar{J}) \times Y(\bar{J}) \in M$. Suppose
(a) $\bar{J} = \bar{I}(\bar{F}(\bar{J}))$, and
(b) $|U_+| < \dim \bar{F}(\bar{J}) + 1$.
Let $\bar{I}' = \bigcap_{u \in U_+} \bar{I}(u)$, then
(7.10) $\bar{I}' \setminus \bar{J} \neq \emptyset$,
(7.11) a^i's $(i \in \bar{J}_+)$ and a^k are linearly independent for any $k \in \bar{I}' \setminus \bar{J}$,
(7.12) $\bar{I}' = \bar{I}(\bar{F}(\bar{I}'))$ and $\bar{F}(\bar{I}') \lhd \bar{F}(\bar{J})$.

Lemma 7.9. Let $w \oplus U_+ = \{u^1,\ldots,u^\ell\}$ and $P = [u^1-w,\ldots,u^\ell-w,w\oplus v-w]$. If $w \oplus U_+$, $w \oplus v$ and w are affinely independent then $p \in R^n$ in Step 12 is obtained by $P = P(P^tP)^{-1}e$, where $e = (0,\ldots,0,-1)^t \in R^{\ell+1}$.

Lemma 7.10. Let $\bar{I}_+ = \{i_1,\ldots,i_\ell\}$. If a^i's $(i \in \bar{I}_+)$ and a^k are linearly independent, then the vector q in Step 18 is obtained by $q = Q(Q^tQ)^{-1}e$, where $Q = [a^i_1,\ldots,a^i_\ell,a^k]$ and $e = (0,\ldots,0,-1)^t \in R^{\ell+1}$.

Lemma 7.4 and 7.5 show that we move from an end-point to the opposite end-point of a linear piece of the path in step 13 and 19. Lemma 7.6 shows in which facet of the current cell $X(\bar{I}) \times Y(\bar{I})$ the end-point we reached lies. Lemma 7.7 to 7.10 guarantee the steps 9 to 12 and 15 to 18.

Now we have seen that each step of the algorithm can be performed and that we trace the piecewise linear path of $g^{-1}(0)$ until the path ends up either at a point on the boundary of M providing a stationary point or in a ray. The next theorem shows that terminating in a ray implies that the problem has no stationary points under some conditions on the matrix Q and the starting point w.

Theorem 7.11. Suppose Q is a copositive plus matrix. If a starting point is chosen so that $Qw \in \text{int } \Omega^+$, then after finitely many iterations the algorithm provides a stationary point or ends in a ray and shows that the problem has no stationary points.

Proof: If $\bar{I}_+ \subset I(w)$ in Step 4, the starting point w is a stationary point. Otherwise $(x^o,y^o) = (w,-f(w))$ lies in $S \cap |\partial M|$ by (6.2). Note that $\delta = \dim \bar{F}(\bar{I})$ in Step 5 and the algorithm traces path S in $(\{w\} + F(\bar{I})) \times (F(\bar{I})^* + \langle h \rangle)$ or $([0, w] + \{0\}) \times \{0\}^*$ by maximizing λ_v in step 6. From Lemma 7.2 to 7.10 we see by induction that δ is the dimension of face $\bar{F}(\bar{I})$ throughout the algorithm and the algorithm traces the path S. Therefore within a finite number of iterations it ends at a stationary point or in a ray. The latter case implies that the problem has no solutions by lemma 6.10. □

8. Remark

Replacing the polyhedral cone Ω and Ω^+ in (1.2) with the nonnegative orthant of R^n, the GCP (1.2) reduces to the conventional nonlinear complementarity problem and further to the LCP when f is an affine function. Talman and Van der Heyden [22] proposed algorithms for LCP which allow an arbitrary starting point. The algorithm in this paper is very close to one of their algorithms with the parameters $k = 1$ and $P_1 = \{1,2,...,n\}$. In fact, when $w > 0$, both of them have $n+1$ directions along which the algorithms can move away from the starting point. Their algorithm however may continue after it reaches the origin by following the line segment $[0, w]$, while our algorithm stops at the origin. This is due to the difference between the two algorithms in the structure of the dual subdivided manifold. Namely, we assign the dual cone

$$\{ y \mid y \in R^n, y \leqq 0 \}$$

to the origin, whereas they assign

$$\{ y \mid y \in R^n, \sum_{j=1}^{n} y_j \leqq -y_i \text{ for any } i \},$$

which contains the former one properly. The important feature of their algorithm is that starting with any point it gives a solution of the LCP or shows that the problem has no solutions if the matrix Q is copositive plus. The convergence is independent of the choice of a starting point.

One of the possible subjects of further research is to improve on the algorithm in this paper to make its convergence property independent of the starting point.

A general polyhedral set is the "sum" of a compact polyhedral set and a polyhedral cone (see, for example Stoer and Witzgall [21], Theorem 2.5.8). The algorithm for SPP on a compact polyhedral set was developed by Yamamoto [25] and we have proposed an algorithm on a polyhedral cone.

Combining these two and developing an algorithm for SPP on general polyhedral sets is also an important possible subject.

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