

THE LAGUERRE TRANSFORM OF PRODUCT OF FUNCTIONS

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Abstract The Laguerre transform developed by Keilson, Nunn and Sumita (1979, 1981, 1981) provides an algorithmic basis for mechanizing various continuum operations such as multiple convolutions, integration, differentiation, and multiplication by polynomials and exponential functions. In this paper, we develop a numerical procedure for finding the Laguerre coefficients of a product of functions in terms of the Laguerre coefficients of individual functions. The procedure enables one to evaluate multiple convolutions and other continuum operations described above even when product form functions are involved, thereby further enhancing the power of the Laguerre transform.

1. Introduction

In certain reliability models, one often encounters lifetime distributions in the form of a product of functions. Consider, for example, a system consisting of two independent components with lifetimes X and Y . If the system fails when either one of the two components fails, the system lifetime T is given by $T = \min\{X, Y\}$ so that $P[T > x] = P[X > x]P[Y > x]$. Upon system failure, a repair action is taken place, requiring the random duration R . The system is renewed when the repair is completed. Clearly the model can be formulated as an alternating renewal process. In order to study the time dependent availability of the system and the associated renewal function, one has to calculate multiple convolutions of the probability densities associated with T and R .

The Laguerre transform [3, 4, 6, 8, 10, 11] has been introduced as an algorithmic basis for computer evaluation of various continuum operations such as multiple convolutions, differentiation, integration and multiplication by polynomials. Ample and useful operational properties are developed, which enable one to evaluate numerically many distribution results of interest in applied probability and statistics. The operational properties developed so far, however, heavily depend on the properties of Laplace transforms (see, §1.3 of [12]). In contrast, no operational properties of Laplace transforms exist for product of functions. Consequently, the approach taken in the literature is not available here.

In this paper we develop a numerical procedure for finding the Laguerre coefficients of a product of functions in terms of the Laguerre coefficients of individual functions. The procedure enables one to study the time dependent performance of stochastic models with product form lifetime distributions, thereby further enhancing the power of the Laguerre transform. A key tool for this purpose is the transition probability matrix governing the linear growth birth-death process of Karlin and McGregor [1, 2]. For numerical applications

demonstrating applicability and accuracy of the procedure, see Kijima [6].

2. The Laguerre Transform of Product of Functions

The Laguerre polynomials $L_n(x) = \frac{e^x}{n!} \left(\frac{d}{dx}\right)^n (x^n e^{-x})$ form a set of orthonormal polynomials with weighting function $w(x) = e^{-x}$ on $[0, \infty)$. The associated Laguerre functions $\ell_n(x) = e^{-x/2} L_n(x)$ constitute an orthonormal basis of a Hilbert space $L_2(0, \infty) = \{f \mid \int_0^\infty f^2(x) dx < \infty\}$ with inner product $\langle f, g \rangle = \int_0^\infty f(x)g(x)dx$, and the functions satisfy the recursion formula

$$\ell_{n+1}(x) = \frac{1}{n+1} [(2n+1-x)\ell_n(x) - n\ell_{n-1}(x)], \quad n \geq 1. \quad (2.1)$$

For any $f \in L_2(0, \infty)$, one has the Fourier-Laguerre expansion

$$f(x) = \sum_{n=0}^{\infty} f_n^\dagger \ell_n(x) \quad (2.2)$$

with $f_n^\dagger = \langle f, \ell_n \rangle$, where Equality (2.2) holds in the sense of the limit in the mean. The speed of convergence of the Laguerre dagger coefficients $(f_n^\dagger)_0^\infty$ to zero depends on the smoothness and the boundedness of $f(x)$. In particular, it has been shown in Keilson and Nunn [3] that if f belongs to the class of rapidly decreasing functions denoted by $C_1^\infty(0, \infty)$, then the set of the dagger coefficients $(f_n^\dagger)_0^\infty$ is also rapidly decreasing in the sense that $n^k |f_n^\dagger| \rightarrow 0$ as $n \rightarrow \infty$ for any positive integer k . Consequently for any $f \in C_1^\infty(0, \infty)$, the ordinary pointwise convergence in (2.2) is assured almost everywhere. In what follows, we restrict ourselves to this class to avoid inessential situations in practice. The reader is referred to [3, 4, 5, 8] for further detailed discussions.

For $f(x)$ defined on $(-\infty, \infty)$, let $f_+(x) = f(x)U(x)$ and $f_-(x) = f(x)U(-x)$ where $U(x) = 1$ for $x \geq 0$ and $U(x) = 0$ for $x < 0$. Then $f(x) = f_+(x) + f_-(x)$ except $x = 0$. One then easily sees that $f(x)g(x) = f_+(x)g_+(x) + f_-(x)g_-(x)$ for $x \neq 0$. Hence it will be sufficient to develop the operational property for f, g in $C_1^\infty(0, \infty)$. For $r(x) = f(x)g(x)$, it can be readily seen that

$$r(x) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} f_m^\dagger g_n^\dagger \ell_m(x) \ell_n(x). \quad (2.3)$$

We define the matrix $\mathbf{A}(k)$ by

$$\mathbf{A}(k) = [a_{mn}(k)]; \quad a_{mn}(k) = \int_0^\infty \ell_m(x) \ell_n(x) \ell_k(x) dx, \quad m, n, k \geq 0. \quad (2.4)$$

From (2.2) through (2.4), the following proposition is then immediate.

Proposition 1

$$r_k^\dagger = \mathbf{f}^{\dagger T} \mathbf{A}(k) \mathbf{g}^\dagger, \quad k \geq 0,$$

where $\mathbf{f}^\dagger = [f_0^\dagger, f_1^\dagger, \dots]^T$, $\mathbf{g}^\dagger = [g_0^\dagger, g_1^\dagger, \dots]^T$ and T denotes the transpose.

Let $N(\theta)$ be the linear growth birth-death process considered in Karlin and McGregor [1, 2]. It is shown there that the transition probabilities $p_{mn}(\theta)$ of $N(\theta)$ involve the Laguerre functions. Namely,

$$p_{mn}(\theta) = \int_0^\infty e^{-\theta x} \ell_m(x) \ell_n(x) dx. \quad (2.5)$$

It is known [8, 9] that

$$J(u, v; \theta) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} p_{mn}(\theta) u^m v^n = \frac{1}{1 - uv + (1 - u)(1 - v)\theta}. \quad (2.6)$$

To develop a numerical procedure for calculating $\mathbf{A}(\mathbf{k})$, we consider a truncated process $N_M(\theta)$ obtained from the original process $N(\theta)$ by making state $(M + 1)$ absorbing. The infinitesimal generator governing $N_M(\theta)$ is given by

$$\mathbf{Q}(M) = \begin{pmatrix} -1 & 1 & & & & & \mathbf{O} \\ 1 & -3 & 2 & & & & \\ & 2 & -5 & 3 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & (M-1) & -(2M-1) & M & \\ \mathbf{O} & & & & M & -(2M+1) \end{pmatrix}. \quad (2.7)$$

For convenience, we define $\mathbf{V}(M) = -\mathbf{Q}(M)$. We note that $\mathbf{V}(M)$ is symmetric and positive definite. Let $\phi_{M+1}(x) = |\mathbf{V}(M) - x\mathbf{I}|$, where \mathbf{I} is the identity matrix. By expanding it at the last row, one has $\phi_0(x) = 0$, $\phi_1(x) = 1 - x$, and

$$\phi_{M+1}(x) = (2M + 1 - x)\phi_M(x) - M^2\phi_{M-1}(x), \quad M \geq 1.$$

The characteristic polynomials $\phi_M(x)$ are then related to $\ell_M(x)$ by $\phi_M(x) = M!\ell_M(x)$. Let $\alpha_j(M)$ be positive eigenvalues of $\mathbf{V}(M)$ with corresponding eigenvectors $\mathbf{y}_j(M)$, $0 \leq j \leq M$. Without loss of generality, we assume that

$$0 < \alpha_0(M) < \alpha_1(M) < \cdots < \alpha_M(M).$$

Noting that $\phi_{M+1}(\alpha_j(M)) = \ell_{M+1}(\alpha_j(M)) = 0$, it follows from (2.1) and the fact $(\mathbf{V}(M) - \alpha_j(M)\mathbf{I})\mathbf{y}_j(M) = \mathbf{0}$ that

$$\mathbf{y}_j(M) = [\ell_0(\alpha_j(M)), \ell_1(\alpha_j(M)), \dots, \ell_M(\alpha_j(M))]^T, \quad 0 \leq j \leq M. \quad (2.8)$$

Let

$$\eta_j(M) = (\mathbf{y}_j^T(M)\mathbf{y}_j(M))^{-1} > 0, \quad 0 \leq j \leq M, \quad (2.9)$$

and define

$$\mathbf{J}_j(M) = \eta_j(M)\mathbf{y}_j(M)\mathbf{y}_j^T(M), \quad 0 \leq j \leq M. \quad (2.10)$$

We note that $\mathbf{J}_j(M)$ are idempotent and of rank one satisfying the matrix orthonormality, i.e. $\mathbf{J}_i(M)\mathbf{J}_j(M) = \delta_{ij}\mathbf{J}_j(M)$, $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. One then has the spectral representation of $\mathbf{V}(M)$ given by

$$\mathbf{V}(M) = \sum_{j=0}^M \alpha_j(M) \mathbf{J}_j(M); \quad \mathbf{I} = \sum_{j=0}^M \mathbf{J}_j(M). \quad (2.11)$$

Let $\mathbf{P}_M(\theta) = [p_{M;mn}(\theta)]$ be the transition probability matrix of $N_M(\theta)$. Since $\mathbf{P}_M(\theta) = \exp\{-\mathbf{V}(M)\theta\}$, one sees from (2.11) that

$$\mathbf{P}_M(\theta) = \sum_{j=0}^M e^{-\alpha_j(M)\theta} \mathbf{J}_j(M), \quad (2.12)$$

or componentwise

$$p_{M;mn}(\theta) = \sum_{j=0}^M e^{-\alpha_j(M)\theta} \ell_m(\alpha_j(M)) \ell_n(\alpha_j(M)) \eta_j(M), \quad 0 \leq m, n \leq M. \quad (2.13)$$

Recall that $\ell_n(x) = e^{-x/2} L_n(x)$, $n \geq 0$. Hence $p_{M;mn}(\theta)$ in (2.13) can be written as

$$p_{M;mn}(\theta) = \sum_{j=0}^M e^{-\alpha_j(M)\theta} L_m(\alpha_j(M)) L_n(\alpha_j(M)) e^{-\alpha_j(M)} \eta_j(M). \quad (2.14)$$

To convert the right hand side of (2.14) to a Stieltjes integral, we introduce a nondecreasing step function $\rho_M(x)$ by

$$\rho_M(x) = \begin{cases} 0, & \text{if } 0 \leq x < \alpha_0(M), \\ \sum_{j=0}^r e^{-\alpha_j(M)} \eta_j(M), & \text{if } \alpha_r(M) \leq x < \alpha_{r+1}(M), \\ \sum_{j=0}^M e^{-\alpha_j(M)} \eta_j(M), & \text{if } \alpha_M(M) \leq x. \end{cases} \quad (2.15)$$

One then sees from (2.14) and (2.15) that

$$p_{M;mn}(\theta) = \int_0^\infty e^{-\theta x} L_m(x) L_n(x) d\rho_M(x). \quad (2.16)$$

It is clear that $\rho_M(x)$ is nondecreasing in $x \geq 0$ and is uniformly bounded in $M \geq 0$ and $x \geq 0$. Hence from Lemma H of Ledermann and Reuter [7] there exists a subsequence $(\rho_{M_k}(x))$ such that $\rho_{M_k}(x) \rightarrow \rho(x)$ as $k \rightarrow \infty$ at all points of continuity of $\rho(x)$. Furthermore

$$\begin{aligned} \lim_{k \rightarrow \infty} p_{M_k;mn}(\theta) &= \lim_{k \rightarrow \infty} \int_0^\infty e^{-\theta x} L_m(x) L_n(x) d\rho_{M_k}(x) \\ &= \int_0^\infty e^{-\theta x} L_m(x) L_n(x) d\rho(x) \stackrel{\text{def}}{=} \hat{p}_{mn}(\theta). \end{aligned} \quad (2.17)$$

Note that $p_{M;mn}(0) = \delta_{mn}$ for any M . So, $\hat{p}_{mn}(0) = \delta_{mn}$. Since the Laguerre polynomials have the unique weight function e^{-x} , one concludes that $d\rho(x) = e^{-x} dx$, $x \geq 0$, and $\hat{p}_{mn}(\theta) = p_{mn}(\theta)$. The sequence $\rho_M(x)$ must converge as it stands. For, if not, there

would exist two subsequences with distinct limit functions. But, this is impossible from the arguments following (2.17). Therefore,

$$\lim_{M \rightarrow \infty} p_{M;mn}(\theta) = \lim_{M \rightarrow \infty} \int_0^\infty e^{-\theta x} L_m(x) L_n(x) d\rho_M(x) = \int_0^\infty e^{-\theta x} \ell_m(x) \ell_n(x) dx. \quad (2.18)$$

We are now in a position to develop a numerical procedure for calculating r_k^\dagger , $k \geq 0$ of Proposition 1. We define

$$\mathbf{A}(k; M) = [a_{mn}(k; M)] = \ell_k(\mathbf{V}(M)), \quad k \geq 0. \quad (2.19)$$

Then the following theorems can be established.

Theorem 2

- (a) For any $m, n \geq 0$, $\lim_{M \rightarrow \infty} a_{mn}(k; M) = a_{mn}(k)$.
 (b) $\mathbf{A}(k+1; M) = \frac{1}{k+1} [\{(2k+1)\mathbf{I} - \mathbf{V}(M)\} \mathbf{A}(k; M) - k \mathbf{A}(k-1; M)], k \geq 1$,
 where $\mathbf{A}(0; M) = \mathbf{P}_M(1/2)$ and $\mathbf{A}(1; M) = (\mathbf{I} - \mathbf{V}(M)) \mathbf{A}(0; M)$.

Proof. It should be noted from (2.11) that

$$\mathbf{A}(k; M) = \sum_{j=0}^M \ell_k(\alpha_j(M)) \mathbf{J}_j(M), \quad k \geq 0. \quad (2.20)$$

This can be rewritten componentwise as

$$a_{mn}(k; M) = \int_0^\infty \ell_k(x) L_m(x) L_n(x) d\rho_M(x), \quad 0 \leq m, n \leq M.$$

One then sees as in (2.18) that

$$\lim_{M \rightarrow \infty} a_{mn}(k; M) = \int_0^\infty \ell_k(x) \ell_m(x) \ell_n(x) dx = a_{mn}(k),$$

proving (a). Statement (b) follows immediately from (2.1) and (2.20). In particular, $\mathbf{A}(0; M) = \sum_{j=0}^M e^{-\alpha_j(M)/2} \mathbf{J}_j(M) = \mathbf{P}_M(1/2)$. \square

Theorem 3 Let $\mathbf{f}_M^\dagger = [f_0^\dagger, \dots, f_M^\dagger]^T$ and $\mathbf{g}_M^\dagger = [g_0^\dagger, \dots, g_M^\dagger]^T$. Define $r_k^\dagger(M) = \mathbf{f}_M^{\dagger T} \mathbf{A}(k; M) \mathbf{g}_M^\dagger$. Then $r_k^\dagger(M) \rightarrow r_k^\dagger$ as $M \rightarrow \infty$.

Proof. This follows immediately from Proposition 1 and Theorem 2(a) and the assumptions $f, g \in C_1^\infty(0, \infty)$. \square

Theorems 2 and 3 provide an efficient approximate numerical procedure for calculating r_k^\dagger , $k \geq 0$, provided that the initial matrix $\mathbf{A}(0; M) = \mathbf{P}_M(1/2)$ is found. Let $\mathbf{P}_M^*(1/2)$ be the $(M+1) \times (M+1)$ principal submatrix of $\mathbf{P}(1/2)$. Although $\mathbf{P}_M(1/2)$ closely approximates $\mathbf{P}_M^*(1/2)$ when M is large, we can develop an algorithm to obtain the exact matrix $\mathbf{P}_M^*(1/2)$ directly.

It can be seen from (2.6) that

$$J(u, v; 1/2) = \frac{2}{3} \left[1 - \frac{1}{3}(u + v + uv) \right]^{-1} = \frac{2}{3} \sum_{i=0}^{\infty} \left(\frac{1}{3} \right)^i (u + v + uv)^i.$$

One then finds

$$p_{mn}(1/2) = 2 \sum_{i=m \vee n}^{m+n} t(m, n; i); \quad t(m, n; i) = \frac{1}{3^{i+1}} \binom{i}{m} \binom{m}{i-n}, \quad (2.21)$$

where $m \vee n = \max\{m, n\}$. It can be shown after a little algebra that

$$t(m, m; m) = \frac{1}{3^{m+1}}, \quad (2.22)$$

$$t(m, n; i+1) = t(m, n; i) \left(\frac{m}{i-n} - 1 \right) / 3 \left(1 - \frac{m}{i+1} \right) \left(1 + \frac{1}{i-n} \right), \quad (2.23)$$

and

$$t(m, n+1; i+1) = t(m, n; i) / 3 \left(1 - \frac{m}{i+1} \right), \quad (2.24)$$

starting with $p_{0n}(1/2) = 2/3^{n+1}$. The matrix $P_M^*(1/2)$ may be calculated alternatively using Theorem 2.2 of Sumita [9].

Remark. The accuracy of the algorithm depends heavily on the tail behavior of functions $f(x)$ and $g(x)$. The matrix $A(k; M)$ may be taken as an $(M+1) \times (N+1)$ rectangular matrix ($M \leq N$) where the truncation points M and N should be determined by those of $(f_n^*)_0^M$ and $(g_n^*)_0^N$ using the moment formula (see §1.2 of [12]). In the recursion formula of Theorem 2(b), the matrix $V(M)$ should still be an $M \times M$ square matrix. Extensive numerical experiments suggest that the recursion formula is numerically stable for $0 \leq k \leq M-1$. When $M \geq 30$, the recursion formula should be modified for $A(k; M)/\sqrt{k!}$ to assure the numerical stability.

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References

- [1] Karlin, S. and McGregor, J. (1957), "The Differential Equations of Birth and Death Processes and the Stieltjes Moment Problem," *Transactions of the American Mathematical Society*, Vol. 85, pp. 489-546.
- [2] Karlin, S. and McGregor, J. (1958), "Linear Growth Birth and Death Processes," *Journal of Mathematics and Mechanics*, Vol. 7, pp. 643-662.
- [3] Keilson, J. and Nunn, W.R. (1979), "Laguerre Transformation as a Tool for the Numerical Solution of Integral Equations of Convolution Type," *Applied Mathematics and Computation*, Vol. 5, pp. 313-359.
- [4] Keilson, J., Nunn, W.R. and Sumita, U. (1981), "The Bilateral Laguerre Transform," *Applied Mathematics and Computation*, Vol. 8, pp. 137-174.

- [5] Keilson, J. and Sumita, U. (1986), "A General Laguerre Transform and a Related Distance Between Probability Measures," *Journal of Mathematical Analysis and Applications*, Vol. 113, pp. 288-308.
- [6] Kijima, M. (1986), "Development of the Bivariate Laguerre Transform for Numerical Study of Bivariate Distributions and Bivariate Processes," Ph.D. Thesis, William E. Simon Graduate School of Business Administration, University of Rochester.
- [7] Ledermann, W. and Reuter, G.E.H. (1954), "Spectral Theory for the Differential Equations of Simple Birth and Death Processes," *Philosophical Transactions of the Royal Society of London, Series A*, Vol. 246, pp. 321-369.
- [8] Sumita, U. (1981), "Development of the Laguerre Transform Method for Numerical Exploration of Applied Probability Models," Ph.D. Thesis, William E. Simon Graduate School of Business Administration, University of Rochester.
- [9] Sumita, U. (1984), "The Laguerre Transform and a Family of Functions with Non-negative Laguerre Coefficients," *Mathematics of Operations Research*, Vol. 9, pp. 510-521.
- [10] Sumita, U. (1984), "The Matrix Laguerre Transform," *Applied Mathematics and Computation*, Vol. 15, pp. 1-28.
- [11] Sumita, U. and Kijima, M. (1985), "The Bivariate Laguerre transform and Its Applications - Numerical Exploration of Bivariate Processes," *Advances in Applied Probability*, Vol. 17, pp. 683-708.
- [12] Sumita, U. and Kijima, M. (1988), "Theory and Algorithms of the Laguerre Transform, Part I: Theory," *Journal of the Operations Research Society of Japan*, Vol. 31, pp. 467-494.

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