Journal of the Operations Research
Society of Japan
Vol. 32, No. 2, June 1989

DISCRETE-TIME STORAGE MODELS WITH NEGATIVE BINOMIAL INFLOW

Hideo Ōsawa
Tokoha Gakuen Hamamatsu University

(Received May 2, 1988; Revised September 14, 1988)

Abstract The present paper studies a discrete-time storage process with discrete states. This model has the inflow which is defined as independent random variables with a common negative binomial distribution and has the certain outflow discipline. Reversibility and quasi-reversibility for the process are investigated and the reversible measure is given. And thus, under a certain condition, it is shown that the process has time-reversibility with the stationary distribution constructed by the reversible measure. Also dynamic reversibility for the process is shown. As an application of the present results we consider an inventory model with a backlog for orders from substations. And the relationship between the Lindley process and this model is discussed. Moreover, we deal with tandem storage models of an open or a closed network whose each node has the outflow discipline of the certain form. For each model, the invariant measure of the product form is obtained.

1. Introduction

In general, it is said for an irreducible Markov chain $\{Y_n; n \in N\}$ to be reversible with respect to a measure $\{r(k), k \in S\}$ if $\{r(k)\}$ is strictly positive and satisfies

$$r(i)P(i,j) = r(j)P(j,i),$$

for $i, j \in S$ where N is the set of all non-negative integers, S is the countable state space and $P(i,j)$ denotes the transition probability from i to j. It is known that if $\{Y_n\}$ is reversible with respect to a bounded measure then $\{Y_n\}$ is ergodic and has time-reversibility, that is, $(Y_{n_1}, Y_{n_2}, \cdots, Y_{n_k})$ and $(Y_{m-n_1}, Y_{m-n_2}, \cdots, Y_{m-n_k})$ have the same distribution for any sequence of finite number of integers $n_1, n_2, \cdots, n_k$ and m such as $m \geq n_i$. Recently, a series of papers on reversibility of Markov processes have been published, for example Ōsawa ([4],[5],[6]) and Pollett ([7],[8]). Especially, Kelly [3] is well-known as a reference which dealt with reversibility of various stochastic phenomena. Moreover, in [3], Kelly introduced an important notion, quasi-reversibility, for continuous-time Markov chains associated with queues or queueing networks.

Suppose that $\{\nu(i), i \in S\}$ is an invariant measure for an irreducible Markov chain $\{Y_n\}$. If $\nu(i) > 0$ for all $i \in S$, then the transition probabilities defined by

$$P^{-}(i,j) = \nu(j)P(j,i)/\nu(i), \qquad i,j \in S,$$

construct an irreducible Markov chain $\{Y_n^-\}$, called time-reversed chain for $\{Y_n\}$. It is clear that $\{\nu(i)\}$ is invariant for $\{Y_n^-\}$. If $\{Y_n\}$ is reversible with respect to some measure, then $\{Y_n\}$ is the time-reversed chain of itself.

On the other hand assume that, to each state $j \in S$, there corresponds a conjugate state j' with $(j')' = j$ and a strictly positive measure $\{\nu(i)\}$ satisfies

$$\begin{aligned} &\nu(i) = \nu(i'); \\ &\nu(i)P(i,j) = \nu(j')P(j',i'), \qquad i,j \in S, \end{aligned}$$

then we call the Markov chain $\{Y_n\}$ dynamically reversible with respect to a measure $\{\nu(i)\}$. In this case $\{\nu(i)\}$ is clearly invariant. Moreover, if $\sum_i \nu(i) < \infty$ and the chain $\{Y_n; n \in Z\}$ is stationary then $\{Y_n\}$ is statistically equivalent to $\{(Y_{-n})'\}$, see Kelly [3], where Z is the collection of all integers. Dynamic reversibility for queueing networks was discussed by Disney and Kiessler [2].

In this paper we consider a discrete-time storage process with discrete states and study reversibility of this process and associated models. Let $\{X_n; n \in N\}$ be the storage process taking values on N which is defined by

$$X_{n+1} = X_n + A_n - D_{n+1}, \qquad n \in N, \tag{1.1}$$

where the inflow process $\{A_n; n \in N\}$ is a sequence of independent random variables with the common distribution

$$a(k) = \mathrm{P}[A_n = k], \qquad k = 0, 1, \cdots,$$

and the outflow process is controlled by

$$\mathrm{P}[D_{n+1} = j \mid X_m,\ D_m,\ 0 \le m \le n;\ A_k,\ X_n + A_n = i] = D(i,j), \tag{1.2}$$

for $0 \le j \le i$, $i \ge 0$ and $n \in N$. Further, assume that X_0 and D_0 are arbitrary. For an outflow discipline $D(i,j)$ of the certain form, Walrand [9] investigated such a model with a Poisson inflow and gave the quasi-reversibility of the model. Our question is "does such a property hold for a storage model with the other inflow process?". For the Lindley process

$$X_{n+1} = \max(0, X_n + A_n - D_{n+1}),$$

with discrete states and the inflow A_n according to a geometric law, some results on reversibility were obtained by Ōsawa [6]. This process has a close connection with the present model as seen in Section 4. Moreover, considering the inventory model having geometric inflows described in Section 3, we study the case where $\{a(k)\}$ is a negative binomial distribution.

In Section 2, for the above storage model (1.1) with a certain outflow discipline, it is shown that the bivariate Markov chain $\{(X_n, D_n)\}$ has an invariant measure of the product form and the trivariate chain $\{(A_n, X_n, D_n)\}$ is dynamically reversible. Moreover, it is seen that this model has quasi-reversibility and the process $\{X_n\}$ has reversibility. In Section 3 an inventory model which has motivated the present problem is considered. The Lindley process with discrete states is represented as an application of this model, hence, in Section 4 the relationship between two processes is investigated and the results are compared with ones in Ōsawa [6]. In Section 5 a discrete-time tandem storage model in which each node has the outflow discipline of the form (1.2) is discussed and the invariant measure of the product form is obtained. Also quasi-reversibility for this model is deduced. In the final section, a closed storage model with two nodes is considered and dynamic reversibility for the model is discussed.

2. Discrete-Time Storage Model with Discrete States

For the storage model described in the previous section, consider the bivariate Markov chain $\{(X_n, D_n); n \in N\}$. Its transition probabilities are given by

$$P_1(\underline{x}, \underline{y}) = a(j + e - i)D(j + e, e)1_B, \tag{2.1}$$

where $\underline{x} = (i, d)$, $\underline{y} = (j, e)$, $B = \{(\underline{x}, \underline{y}) : j + e \geq i\}$ and 1_B is the indicator of a set B. Assume that

$$D(i, j) = c(i)\alpha(i - j)\gamma(j) \prod_{k=i-j+1}^{i} \beta(k), \qquad 0 \leq j \leq i, \tag{2.2}$$

where $\alpha(k)$ and $\beta(k)$ are positive valued functions defined on N, $\beta(0) = 1$ and $c(i)$ is the normalizing constant such that $\sum_{j=0}^{i} D(i, j) = 1$ for each $i \geq 0$.

Theorem 2.1 *Let $\{a(k), k \geq 0\}$ be a negative binomial distribution*

$$a(k) = \binom{\eta + k - 1}{k} (1 - \lambda)^{\eta} \lambda^{k}, \qquad k = 0, 1, 2, \cdots, \tag{2.3}$$

where $\eta > 0$ and $0 < \lambda < 1$, and let

$$\gamma(j) = \binom{\eta + j - 1}{j}, \qquad j = 0, 1, 2, \cdots.$$

Then we have

(i) *the process $\{(X_n, D_n)\}$ has an invariant measure $\{r(i)a(d) : i, d \geq 0\}$ where*

$$r(i) = \alpha(i)\lambda^{i} / \prod_{k=0}^{i} \beta(k), \qquad i = 0, 1, 2, \cdots, \tag{2.4}$$

(ii) *trivariate Markov chains $\{(A_n, X_n, D_n)\}$ and $\{(D_n, X_n, A_n)\}$ are reversed in time each other.*

Proof: From (2.1)

$$P_1(\underline{x},\underline{y}) = \binom{\eta+j+e-i-1}{j+e-i}(1-\lambda)^\eta\lambda^{j+e-i} \cdot c(j+e)\alpha(j)\binom{\eta+e-1}{e}\prod_{k=j+1}^{j+e}\beta(k),$$

for $\underline{x} = (i,d)$, $\underline{y} = (j,e)$ and $j+e \geq i$. Then we have

$$\begin{aligned}\sum_{(i,d)} r(i)a(d)P_1(\underline{x},\underline{y}) &= \sum_i\{\alpha(i)\lambda^i/\prod_{k=0}^{i}\beta(k)\}\binom{\eta+j+e-i-1}{j+e-i} \\ &\quad\cdot(1-\lambda)^\eta\lambda^{j+e-i}c(j+e)\alpha(j)\binom{\eta+e-1}{e}\prod_{k=j+1}^{j+e}\beta(k) \\ &= \{\alpha(j)\lambda^j/\prod_{k=0}^{j}\beta(k)\}\binom{\eta+e-1}{e}(1-\lambda)^\eta\lambda^e \\ &\quad\cdot c(j+e)\sum_{i=0}^{j+e}\alpha(i)\binom{\eta+j+e-i-1}{j+e-i}\prod_{k=i+1}^{j+e}\beta(k) \\ &= r(j)a(e),\end{aligned}$$

for all $j, e \geq 0$. Hence $\{r(i)a(d)\}$ is invariant for $\{(X_n, D_n)\}$.

To prove the second part of the theorem, define a measure

$$\nu(\underline{u}) = a(a)r(i)a(d),$$

for $\underline{u} = (a,i,d)$ then the above result means that $\nu(\cdot)$ is invariant for $\{(A_n, X_n, D_n)\}$. The transition probabilities are given by

$$P_2(\underline{u},\underline{v}) = a(b)D(i+a,e)1_E,$$

where $\underline{u} = (a,i,d)$, $\underline{v} = (b,j,e)$ and $E = \{(\underline{u},\underline{v}) : j+e = i+a\}$. Write $\underline{u}' = (d,i,a)$ for $\underline{u} = (a,i,d)$ and let

$$P_2^-(\underline{u},\underline{v}) = \nu(\underline{v})P_2(\underline{v},\underline{u})/\nu(\underline{u}),$$

then it is readily seen that

$$P_2^-(\underline{u},\underline{v}) = P_2(\underline{u}',\underline{v}'), \quad (\underline{u},\underline{v}) \in E.$$

Thus the theorem is established.

Remark 2.2 Using the notation in Proof of Theorem 2.1, for the chain $\{(A_n, X_n, D_n)\}$, it follows that $\underline{u}'$ is a conjugate mapping and

$$\begin{aligned}&\nu(\underline{u}) = \nu(\underline{u}'), \\ &\nu(\underline{u})P_2(\underline{u},\underline{v}) = \nu(\underline{v}')P_2(\underline{v}',\underline{u}'), \qquad \underline{u},\underline{v} \in N^3.\end{aligned}$$

Hence $\{(A_n, X_n, D_n)\}$ is dynamically reversible with respect to $\{\nu(\cdot)\}$. In other words, under the condition that $\sum\nu(\underline{u}) < \infty$ and the process is stationary, $\{(A_n, X_n, D_n)\}$ is statistically indistinguishable from the reversed chain of the conjugate process $\{(A_n, X_n, D_n)'\}$. From this argument we can reach the following results: For the stationary storage process defined by Theorem 2.1,

(i) the outflow D_n has the same distribution $\{a(k)\}$ as the inflow A_n,

(ii) $\{A_k, k \geq n\}$, X_n and $\{D_k, k \leq n\}$ are independent for all n.

The stationary storage model (1.1) having these properties is said to be quasi-reversible. To analyze networks connecting some storage nodes, quasi-reversibility is a powerful property.

Theorem 2.1 also implies that $\{r(i)\}$ is invariant for a Markov chain $\{X_n\}$. Moreover, we can have the result on reversibility of $\{X_n\}$.

Theorem 2.3 *The Markov chain $\{X_n\}$ in Theorem 2.1 is reversible with respect to a measure $\{r(i), i \geq 0\}$. Thus if $C = \sum_{i=0}^{\infty} r(i) < \infty$, then $\{X_n\}$ has time-reversibility with the stationary distribution*

$$\pi(i) = r(i)/C, \qquad i = 0, 1, 2, \cdots.$$

Proof: The transition probabilities are written as

$$P(i,j) = \sum_{k=\max(0,i-j)}^{\infty} a(j+k-i)D(j+k,k), \qquad i,j \geq 0.$$

Then we have

$$\begin{aligned} r(i)P(i,j) &= \frac{\alpha(i)\lambda^i}{\prod_{k=0}^{i}\beta(k)} \sum_{k=i-j}^{\infty} \binom{\eta+j+k-i-1}{j+k-i} (1-\lambda)^{\eta}\lambda^{j+k-i} \\ &\qquad \cdot c(j+k)\alpha(j) \binom{\eta+k-1}{k} \prod_{l=j+1}^{j+k} \beta(l) \\ &= \frac{\alpha(j)\lambda^j}{\prod_{k=0}^{j}\beta(k)} \sum_{k=0}^{\infty} \binom{\eta+i+k-j-1}{i+k-j} (1-\lambda)^{\eta}\lambda^{i+k-j} \\ &\qquad \cdot c(i+k)\alpha(i) \binom{\eta+k-1}{k} \prod_{l=i+1}^{i+k} \beta(l) \\ &= r(j)P(j,i), \qquad i \geq j \geq 0. \end{aligned}$$

The proof is completed.

Remark 2.4 When the system has a Poisson inflow

$$a(k) = \exp(-\lambda)\lambda^k/k!, \qquad k = 0, 1, 2, \cdots,$$

and an outflow discipline

$$D(i,j) = c(i)\alpha(i-j) \prod_{k=i-j+1}^{i} \beta(k)/j!, \qquad 0 \leq j \leq i,$$

then the similar results as given in Theorem 2.1 have been obtained by Walrand [9]. In this case, we can also see that $\{X_n\}$ is reversible with respect to $\{r(i)\}$ of the same form as (2.4).

3. An Inventory Model

The problem dealt with in the previous section has been motivated by the following simple inventory model. There is a factory which manufactures some kinds of goods by using machines. One of these, which is expensive, is made to order from two branch offices of the factory. Each branch gives an order for the article each day, the number of which is approximately distributed according to a geometric distribution $(1-\lambda)\lambda^k$. The orders are sent to the factory at the next morning and we denote by A_n the total number of these at the beginning of nth day. Let X_n be the number of backlog of orders. Then the factory produces the articles as follows. Suppose that $X_n + A_n = i$, then it produces the first article at rate $\beta(i)\delta(1)$, the second article at rate $\beta(i-1)\delta(2)$, $\cdots$, jth article $\beta(i-j+1)\delta(j)$, and so on. In this case, $\beta(k)$ are the rate depending on the number of backlog and $\delta(j)$ depend on the number of articles which are produced in the day. Then the rate at which j articles are produced is given by

$$\beta(i)\delta(1)\beta(i-1)\delta(2)\cdots\beta(i-j+1)\delta(j).$$

Therefore, letting D_n be the number of articles which the factory produces in nth day, D_n is stochastically determined by

$$D(i,j) = c(i)\alpha(i-j)\prod_{k=i-j+1}^{i}\beta(k)\prod_{k=1}^{j}\delta(k), \quad 0 \le j \le i, \ i \ge 0,$$

where $c(i)$ is the normalizing function and $\alpha(i-j)$ is a factor with respect to backlog of orders.

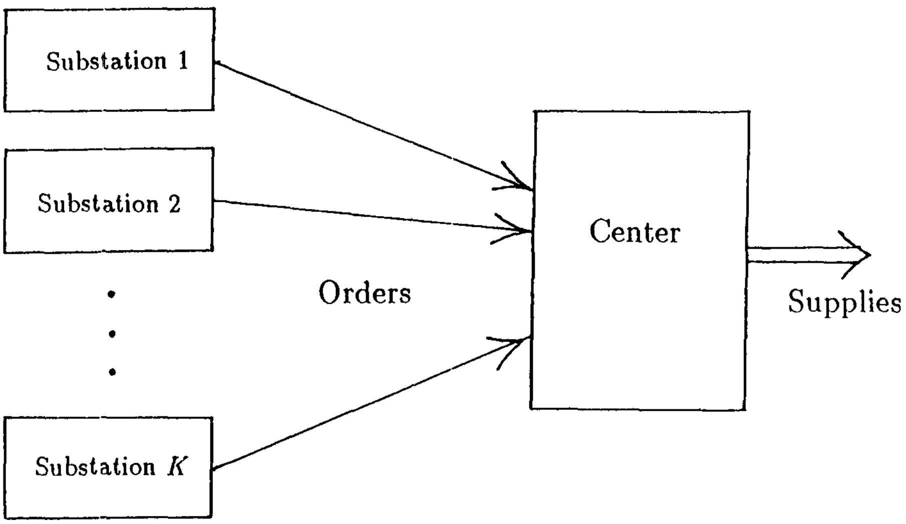

Figure 3.1 An Inventory Model

Here note that the empty product is unity. Under the above situation, in order to keep the balance between the manufacture of the articles and orders, what kind of the production policy should the factory take? Interpreting the balance as the matter that the distributions of the inflow (orders) and the outflow (production and shipment for orders) are almost the same, we get an answer from the arguments in the previous section.

For the above problem, consider an inventory model represented in Figure 3.1. It contains the central station, called the center, and K substations and an inventory level is the total amount of backlog of articles ordered in the center. In the discrete time each substation orders the center k articles with the probability $(1-\lambda)\lambda^k$ for non-negative integer k. Then the total number of articles ordered, say A_n, is distributed according to the negative binomial law

$$a(k) = \binom{K+k-1}{k}(1-\lambda)^K\lambda^k, \qquad k = 0, 1, 2, \cdots.$$

For the above orders the center supplies the articles in accordance with the outflow discipline of the form (2.2). In this model the total number of backlog of orders in the center at time n, say X_n, are represented by (1.1). Note that $\gamma(j) = \binom{K+j-1}{j}$.

Case (i): If the center choose the outflow discipline with $\beta(k) = \mu$ for $k \geq 1$, that is,

$$D(i,j) = c(i)\alpha(i-j)\gamma(j)\mu^j, \qquad j = 0, 1, \cdots, i,$$

then the reversible measure is given by

$$r(i) = \alpha(i)\rho^i, \qquad i = 0, 1, 2, \cdots,$$

where $\rho = \lambda/\mu$. When $\alpha(i) = 1$ and $\rho < 1$, then $\{X_n\}$ has time-reversibility with the stationary distribution

$$\pi(i) = (1-\rho)\rho^i, \qquad i = 0, 1, 2, \cdots,$$

which corresponds to one of the queue-length in the (discrete-time) $M/M/1$ queue.

Case (ii): If $\alpha(i) = i+1$ in Case (i) and $\rho < 1$ then $\{X_n\}$ is time-reversible with respect to the stationary distribution (a negative binomial distribution)

$$\pi(i) = (i+1)(1-\rho)^2\rho^i, \qquad i = 0, 1, 2, \cdots.$$

Case (iii): Choosing $\beta(k) = k\mu$ for $k \geq 1$, that is,

$$D(i,j) = c(i)\alpha(i-j)\gamma(j)\binom{i}{j}j!\mu^j, \qquad j = 0, 1, \cdots, i,$$

then we have the reversible measure

$$r(i) = \alpha(i)\rho^i/i!, \qquad i = 0, 1, 2, \cdots,$$

where $\rho = \lambda/\mu$. Let $\alpha(i) = 1$ and $\rho < 1$, then $\{X_n\}$ has time-reversibility with the stationary distribution

$$\pi(i) = \exp(-\rho)\rho^i/i!, \qquad i = 0, 1, 2, \cdots,$$

which equals to one of the queue length in an $M/M/\infty$.

For the inventory model described in the first part of this section, it seems desirable that the factory has a little backlog of orders. Case (i) is a model in which the production rate is a constant regardless of backlog of orders. In Case (iii) the factory can improve manufacturing capacity in response to orders how much it has backlog. Comparing the means of backlog for these two cases, in equilibrium, it is found that

$$\{\text{the mean of a backlog in Case (iii)}\} = (1 - \rho)\{\text{the mean of a backlog in Case (i)}\}.$$

Suppose that a Markov chain $\{X_n\}$ is irreducible and recurrent, let $\#_k(n)$ be the number of visits to state k during $0 \leq t \leq n$, then we have

$$\lim_{n\to\infty} \#_k(n)/\#_i(n) = r(k)/r(i), \quad \text{with probability 1},$$

from the ergodic theorem for Markov chains (see Chung [1]). The left-hand side of this equation, denote by $R(k, i)$, is regarded as the ratio of the time spent in state k to that spent in state i on an infinitely long time interval. Therefore, it is expected that $R(k, i) < 1$ for $k > i$. For that purpose, in Case (i), it suffices that $\rho < 1$. However, in Case (ii), this is rewritten as

$$\alpha(i + 1)\rho < \alpha(i), \qquad \text{for all } i,$$

and thus we have $2\rho < 1$. Since Case (ii) is a model in which the work takes much time in proportion of backlog of orders, the factory has to improve a production rate as compared with Case (i). In fact, for the stationary models, to keep up the mean of state X_n in Case (ii) less than that in Case (i) it is required that

$$\mu_2 \geq 2\mu_1 - \lambda,$$

where μ_1 and μ_2 are the production rates in Case (i) and Case (ii), respectively.

For a Poisson inflow, we can also consider the similar model; that is, in the discrete time, substation j, $1 \leq j \leq K$, orders k articles from the center with probability $\exp(-\lambda_j)\lambda_j^k/k!$ for $k \geq 0$. Then the total number of articles ordered is distributed according to the Poisson law

$$a(k) = \exp(-\lambda)\lambda^k/k!, \qquad k = 0, 1, 2, \cdots,$$

where $\lambda = \sum_{j=1}^{K} \lambda_j$. As seen in Remark 2.4, by rewriting as $\gamma(j) = 1/j!$ for $j \geq 0$ the similar arguments in this section are available. So we should note that the assemble results are obtained in two cases, Poisson and negative binomial inflow models.

4. The Lindley Process

Reversibility of Lindley processes was dealt with Ōsawa [6]. In this section, we see the relationship between the present storage models and Lindley processes.

Let $\{\tilde{X}_n\}$ be a Lindley process taking values on N: that is,

$$\tilde{X}_{n+1} = \max(0, \tilde{X}_n + A_n - \tilde{D}_{n+1}),$$

where $\{A_n\}$ and $\{D_n\}$ are sequences of mutually independent random variables with common distributions $\{a(k), k \in N\}$ and $\{d(k), k \in N\}$, respectively. Let

$$D_{n+1} = \begin{cases} \tilde{D}_{n+1} & \text{if } \tilde{X}_n + A_n > \tilde{D}_{n+1}; \\ \tilde{X}_n + A_n, & \text{if } \tilde{X}_n + A_n \le \tilde{D}_{n+1}, \end{cases}$$

then $\{\tilde{X}_n\}$ is equivalent to the process $\{X_n\}$ defined by (1.1) with

$$D(i,j) = \begin{cases} d(j), & 0 \le j < i; \\ \bar{d}(i) = \sum_{k=i}^{\infty} d(k), & j = i. \end{cases}$$

Assume that $\{a(k)\}$ is the geometric distribution, i.e., $\eta = 1$ in (2.3), and moreover $D(i,j)$ is of the form (2.2), then we have

$$d(j) = c(i)\alpha(i-j) \prod_{k=i-j+1}^{i} \beta(k), \qquad 0 \le j < i; \tag{4.1}$$

$$\bar{d}(i) = c(i)\alpha(0) \prod_{k=1}^{i} \beta(k), \qquad i \ge 1. \tag{4.2}$$

By dividing (4.1) for $j = i-1$ term by term by (4.2) it follows that

$$\frac{d(i-1)}{\bar{d}(i)} = \frac{\alpha(1)}{\alpha(0)\beta(1)} = \kappa(= \text{constant}).$$

It is therefore found that

$$d(i) = \{1/(1+\kappa)\}^i d(0),$$

hence also $\{d(i)\}$ is a geometric distribution. This fact accords with the well-known result that, for a geometric inflow, the Lindley process $\{\tilde{X}_n\}$ is reversible if and only if $\{d(k)\}$ is a geometric distribution (see Ōsawa [6]). As examples of such a reversible model, there are the waiting-time process in the discrete-time $M/M/1$ and the queue-length process in the bulk-arrival queue $M^A/M/1$.

5. Tandem Storage Model

Consider a discrete-time tandem storage model with J nodes illustrated in Figure 5.1. At the beginning of the nth slot of time, the inflow from the external into node 1, denoted by A_n^1, is assumed to be a random variable with the distribution $\{a(k), k \in N\}$. At the end of nth slot, the outflow from node k to $k+1$ is denoted by D_{n+1}^k which equals to the inflow A_{n+1}^{k+1} into node $k+1$ at the beginning of $(n+1)$th slot for each $1 \leq k < J$. Thus D_{n+1}^J denotes the number of customers who depart from the system at the end of nth slot. Assume that node k has the outflow discipline of the form (2.2): that is,

$$\begin{aligned} D_k(i,j) &= \mathrm{P}[D_{n+1}^k = j \mid X_m^k, D_m^k, 0 \leq m \leq n; A_t^k, X_n^k + A_n^k = i] \\ &= c_k(i)\alpha_k(i-j)\gamma(j) \prod_{l=i-j+1}^{i} \beta_k(l), \qquad 0 \leq j \leq i,\ 1 \leq k \leq J. \end{aligned}$$

where $\sum_{j=0}^{i} D_k(i,j) = 1$.

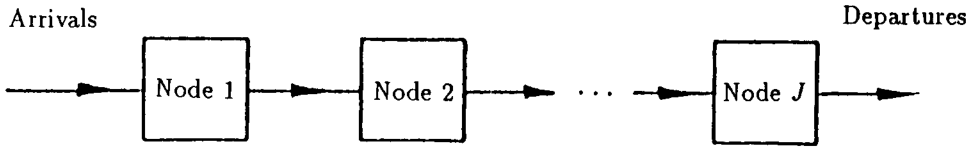

Figure 5.1 A Tandem Storage Model

Define three random vectors as

$$\begin{aligned} \underline{\boldsymbol{X}}_n &= (X_n^1,\ X_n^2,\ \cdots,\ X_n^J), \\ \underline{\boldsymbol{A}}_n &= (A_n^1,\ A_n^2,\ \cdots,\ A_n^J), \\ \underline{\boldsymbol{D}}_n &= (D_n^1,\ D_n^2,\ \cdots,\ D_n^J), \end{aligned}$$

then the state of this tandem model is described by

$$\underline{\boldsymbol{X}}_{n+1} = \underline{\boldsymbol{X}}_n + \underline{\boldsymbol{A}}_n - \underline{\boldsymbol{D}}_{n+1}.$$

Theorem 5.1 *Suppose that $\{a(k)\}$ is the negative binomial distribution (2.3) and $\gamma(j) = \binom{\eta+j-1}{j}$. Then the process $\{(\underline{\boldsymbol{X}}_n, \underline{\boldsymbol{D}}_n)\}$ has an invariant measure of the product form*

$$\prod_{k=1}^{J} r_k(i_k)a(d_k),$$

where $r_k(i) = \alpha_k(i)\lambda^i / \prod_{l=0}^{i} \beta_k(l)$.

Proof: Transition probabilities are given by

$$P(\underline{\boldsymbol{x}},\underline{\boldsymbol{y}}) = a(j_1+e_1-i_1)\prod_{k=1}^{J} D_k(j_k+e_k,e_k)1_F,$$

where $\underline{\boldsymbol{x}} = (i_1,\cdots,i_J,d_1,\cdots,d_J)$, $\underline{\boldsymbol{y}} = (j_1,\cdots,j_J,e_1,\cdots,e_J)$ and

$$F = \{(\underline{\boldsymbol{x}},\underline{\boldsymbol{y}}) | j_1+e_1 \geq i_1; j_k+e_k = i_k+d_{k-1}, 2\leq k\leq J\}.$$

After some calculations we have

$$\begin{aligned}
&\sum_{\underline{\boldsymbol{x}}}\prod_{k=1}^{J} r_k(i_k)a(d_k)P(\underline{\boldsymbol{x}},\underline{\boldsymbol{y}}) \\
&= \prod_{k=1}^{J}\sum_{i_k=0}^{j_k+e_k} r_k(j_k)a(e_k)c_k(j_k+e_k)\alpha_k(i_k)\gamma(j_k+e_k-i_k)\prod_{m=i_k+1}^{j_k+e_k}\beta_k(m) \\
&= \prod_{k=1}^{J} r_k(j_k)a(e_k).
\end{aligned}$$

Thus the theorem is established.

When $\{\underline{\boldsymbol{X}}_n\}$ is not stationary $\underline{\boldsymbol{X}}_{n+1}$ is not stochastically determined without the information of $\underline{\boldsymbol{D}}_n$ which depends on $\underline{\boldsymbol{X}}_{n-1}$. Therefore, since $\{\underline{\boldsymbol{X}}_n\}$ is not a Markov chain we can not have the argument on reversibility of $\{\underline{\boldsymbol{X}}_n\}$ which corresponds to one obtained in Theorem 2.3. However, associated with quasi-reversibility for the model, consider the process $\{(A_n^1,\underline{\boldsymbol{X}}_n(k),D_n^k)\}$ for $1\leq k\leq J$ where we put

$$\underline{\boldsymbol{X}}_n(k) = (X_n^1,X_n^2,\cdots,X_n^k,D_n^1,D_n^2,\cdots,D_n^{k-1}).$$

For the above $\underline{\boldsymbol{X}}_n(k)$ we also use the notation

$$\underline{\boldsymbol{X}}_n^-(k) = (X_n^k,X_n^{k-1},\cdots,X_n^1,D_n^{k-1},D_n^{k-2},\cdots,D_n^1).$$

Theorem 5.2 *The process $\{(A_n^1,\underline{\boldsymbol{X}}_n^-(k),D_n^k)\}$ is the time-reversed chain of $\{(A_n^1,\underline{\boldsymbol{X}}_n(k),D_n^k)\}$ for any $1\leq k\leq J$.*

Proof: From Theorem 2.1 this argument is available for $k=1$. Consider the case $2\leq k\leq J$. The transition probabilities for $\{(A_n^1,\underline{\boldsymbol{X}}_n(k),D_n^k)\}$ are

$$P(\underline{\boldsymbol{x}},\underline{\boldsymbol{y}}) = a(e_0)\prod_{m=1}^{k} D_m(i_m+d_{m-1},e_m)1_G,$$

where $\underline{\boldsymbol{x}} = (d_0,i_1,\cdots,i_k,d_1,\cdots,d_{k-1},d_k)$, $\underline{\boldsymbol{y}} = (e_0,j_1,\cdots,j_k,e_1,\cdots,e_{k-1},e_k)$ and $G = \{(\underline{\boldsymbol{x}},\underline{\boldsymbol{y}}) | j_m = i_m+d_{m-1}-e_m, 1\leq m\leq k\}$. Let $r_A(\underline{\boldsymbol{x}}) = \prod_{m=1}^{k} r_m(i_m)a(d_m)$ and define

$$P^-(\underline{\boldsymbol{y}},\underline{\boldsymbol{x}}) = \frac{a(d_0)r_A(\underline{\boldsymbol{x}})}{a(e_0)r_A(\underline{\boldsymbol{y}})}P(\underline{\boldsymbol{x}},\underline{\boldsymbol{y}}).$$

Then, for $(\underline{\boldsymbol{x}},\underline{\boldsymbol{y}})\in G$, we can easily have

$$P^-(\underline{\boldsymbol{y}},\underline{\boldsymbol{x}}) = P(\underline{\boldsymbol{y}}^-,\underline{\boldsymbol{x}}^-),$$

where we write $\underline{\boldsymbol{x}}^- = (d_k,i_k,\cdots,i_1,d_{k-1},\cdots,d_1,d_0)$. This means the theorem holds.

Remark 5.3 Under the condition that $\sum_i r_m(i) < \infty$ for $1 \leq m \leq k$ the stationary chain $\{(A_n^1, \underline{\boldsymbol{X}}_n(k), D_n^k)\}$ is quasi-reversible for $1 \leq k \leq J$. This implies that, in equilibrium, D_n^k has the same distribution as A_n^1 and $\{A_m^1; m \leq n\}$, $\underline{\boldsymbol{X}}_n(k)$ and $\{D_m^k; m \geq n\}$ are mutually independent for all n. Moreover, it is found that the system described by the reversed chain $\{(D_n^k, \underline{\boldsymbol{X}}_n^-(k), A_n^1)\}$ is also a tandem storage process where every nodes are in reversed order for the original model.

6. Closed Storage Model

A discrete-time storage model of the closed type represented in Figure 6.1 is considered. The outflow discipline from node k is assumed to be of the form

$$D_k(i,j) = c_k(i)\alpha_k(i-j) \prod_{l=i-j+1}^{i} \beta_k(l), \qquad 0 \leq j \leq i, \; k = 1, 2.$$

Let L be the total storage level then the state of the system is described by (X_n^1, X_n^2, D_n) where X_n^k denotes the storage level in the node k at time n and D_n the level of outflow from node 1 to node 2. The state space of the process is $S = \{(i_1, i_2, d) \in N^3 | \; i_1 + i_2 + d \leq L\}$.

The transition probabilities are written as

$$P(\underline{x}, \underline{y}) = D_1(L - i_2 - d, e)D_2(i_2 + d, L - j_1 - j_2 - e)1_H,$$

where $\underline{x} = (i_1, i_2, d)$, $\underline{y} = (j_1, j_2, e)$ and $H = \{(\underline{x}, \underline{y}) \in S^2 | i_2 + j_1 + d + e = L\}$. Then we have

$$P(\underline{x}, \underline{y}) = c(j_1 + e)\alpha_1(j_1) \prod_{l=j_1+1}^{j_1+e} \beta_1(l) c(i_2 + d)\alpha_2(j_2) \prod_{l=j_2+1}^{i_2+d} \beta_2(l) 1_H.$$

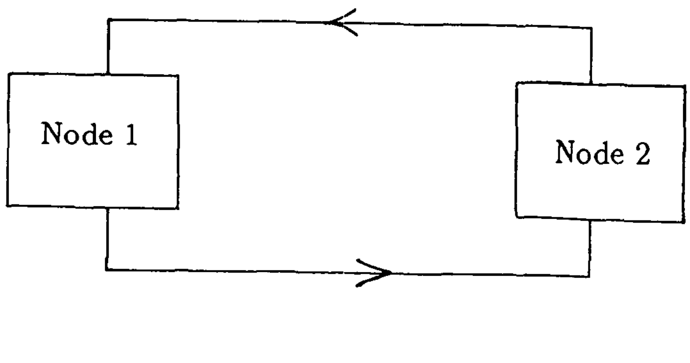

Figure 6.1 A Closed Model

Theorem 6.1 $\{(X_n^1, X_n^2, D_n)\}$ *has an invariant measure*

$$\prod_{k=1}^{2} r_k(i_k), \qquad \underline{x} = (i_1, i_2, d) \in S,$$

where $r_k(i) = \alpha_k(i)/\prod_{l=0}^{i} \beta_k(l)$.

Proof: In a similar manner as the previous theorems, the present result is proved directly. In fact, we have

$$\begin{aligned}
&\sum_{(i_1,i_2,d)} r_1(i_1)r_2(i_2)c_1(j_1+e)\alpha_1(j_1)\prod_{l=j_1+1}^{j_1+e} c(i_2+d)\alpha_2(j_2)\prod_{l=j_2+1}^{i_2+d}\beta_2(l)1_H \\
&= r_1(j_1)r_2(j_2)c_1(j_1+e)\sum_{i_1=0}^{j_1+e}\alpha_1(i_1)\prod_{i_1+1}^{j_1+e}\beta_1(l) \\
&\qquad \cdot c_2(L-j_1-e)\sum_{i_2=0}^{L-j_1-e}\alpha_2(i_2)\prod_{l=i_2+1}^{L-j_1-e}\beta_2(l) \\
&= r_1(j_1)r_2(j_2),
\end{aligned}$$

for any $(j_1, j_2, e) \in S$.

Since the Markov chain $\{(X_n^1, X_n^2, D_n)\}$ takes values only on finitely many states, it is found that, from this theorem, there exists the stationary distribution given by

$$\pi(\underline{x}) = r_1(i_1)r_2(i_2)/C,$$

for $\underline{x} = (i_1, i_2, d)$ where $C = \sum_{\underline{x}\in S} r_1(i_1)r_2(i_2)$. If the process $\{(X_n^1, X_n^2, D_n)\}$ is stationary then the outflow is distributed according to the distribution

$$P[D_n = d] = \sum_{i_1+i_2\le L-d} r_1(i_1)r_2(i_2)/C.$$

Let D_n' be the number of outflow from node 2 to node 1 at time n, then $D_n' = L - X_n^1 - X_n^2 - D_n$. We write $\underline{x}' = (i_1, i_2, d')$ for $\underline{x} = (i_1, i_2, d) \in S$ where $d' = L - i_1 - i_2 - d$. Since $(d')' = d$ hence $\underline{x}'$ is the conjugate state for $\underline{x}$. Then we have the following.

Theorem 6.2 $\{(X_n^1, X_n^2, D_n')\}$ *is reversed in time for* $\{(X_n^1, X_n^2, D_n)\}$.

Proof: Define

$$P^{\sim}(\underline{y}, \underline{x}) = r_1(i_1)r_2(i_2)P(\underline{x}, \underline{y})/\{r_1(j_1)r_2(j_2)\},$$

then we get

$$\begin{aligned}
P^{\sim}(\underline{y}, \underline{x}) &= c_1(j_1+e)\alpha_1(i_1)\prod_{l=i_1+1}^{j_1+e} c_2(i_2+d)\alpha_2(i_2)\prod_{l=i_2+1}^{i_2+d}\beta_2(l) \\
&= c_1(i_1+d')\alpha_1(i_1)\prod_{l=i_1+1}^{i_1+d'}\beta_1(l)c_2(j_2+e')\alpha_2(i_2)\prod_{l=i_1+1}^{j_2+e'}\beta_2(l) \\
&= D_1(j_1+e, d')D_2(j_2+e', d) \\
&= P(\underline{y}', \underline{x}'),
\end{aligned}$$

for $\underline{x} = (i_1, i_2, d)$, $\underline{y} = (j_1, j_2, e) \in H$. Thus the theorem is proved.

Remark 6.3 In view of Proof in the theorem, it is seen that $\{(X_n^1, X_n^2, D_n)\}$ is dynamically reversible with respect to a measure $\{r_1(i_1)r_2(i_2),\ (i_1, i_2, d) \in S\}$. Therefore, the reversed process for $\{(X_n^1, X_n^2, D_n)\}$ represents the closed storage model in which the flow goes in the opposite direction of the original model.

Acknowledgments

The author wishes to acknowledge his grateful thanks to Professor Takehisa Fujisawa, the University of Electro-Communications, for helpful discussions on this paper. He also thanks the referees for valuable suggestions on the earlier draft of the paper. This research was partly supported by the Grant-in-Aid for Scientific Research from the Ministry of Education, Science and Culture of Japan.

References

[1] Chung, K. L.: *Markov Chains with Stationary Transition Probabilities*, Second Edition. Springer. New York, 1967.

[2] Disney, R. L. and Kiessler, P. C.: *Traffic Processes in Queueing Networks.* Johns Hopkins University Press. Baltimore, 1986.

[3] Kelly, F. P.: *Reversibility and Stochastic Networks.* John Wiley & Sons, New York, 1979.

[4] Ōsawa, H.: Reversibility of Markov Chains with Applications to Storage Models, *Journal of Applied Probability*, Vol.**22**, pp.123-137, 1985.

[5] Ōsawa, H.: Reversibility of First-Order Autoregressive Processes, *Stochastic Processes and their Applications*, Vol.**28**, pp.61-69, 1988.

[6] Ōsawa, H.: Reversibility of Lindley Processes with Discrete States, (submitted).

[7] Pollett, P. K.: Connecting Reversible Markov Processes, *Advances in Applied Probability*, Vol.**18**, pp.880-900, 1986.

[8] Pollett, P. K.: Preserving Partial Balance in Continuous-Time Markov Chains, *Advances in Applied Probability*, Vol.**19**, pp.431-453, 1987.

[9] Walrand, J.: A Discrete-Time Queueing Network, *Journal of Applied Probability*, Vol.**20**, pp.903-909, 1983.

Hideo ŌSAWA : Department of Business Administration and Information Science, Tokoha Gakuen Hamamatsu University, 1230 Miyakoda-cho, Hamamatsu, Shizuoka 431-21, JAPAN.