Journal of the Operations Research
Society of Japan
Vol. 32, No. 2, June 1989

A STOCHASTIC PROGRAMMING MODEL FOR AGRICULTURAL PLANNING UNDER UNCERTAIN SUPPLY-DEMAND RELATIONS

Teruaki Nanseki
National Agriculture Research Center

(Received April 15, 1988; Revised October 28, 1988)

Abstract The main purpose of this paper is to present a stochastic programming model for agricultural planning under uncertain supply-demand relations and to propose an algorithm for the model. We first illustrate that several economic problems under undertainty must be formulated as a quadratic programming model in which the linear coefficients are stochastic variables. The model is considered as an extension of the linear stochastic programming model reported by Freund (1956) and Kataoka (1963, 1967).

The following alternative criteria for optimization are introduced to complete the model: (1) maximizing the expected value of utility; (2) maximizing aspiration level; (3) maximizing probability. The deterministic equivalents are then derived with the mathematical characteristics of the solutions. The equivalence relations among the equivalents are also clarified. We propose an iteration algorithm for the equivalents by taking advantage of the relations. The algorithm is based on the secant method and the convex quadratic programming method. Finally, an application of the model is illustrated with a procedure for application.

1. Introduction

The regional planning models in which demand functions are incorporated directly into the objective function are widely applied for planning and analyzing both pricing and allocation problems for the commercial sector of agriculture (see e.g. Judge and Takayama, 1973). However, the models have been constructed without consideration of prediction errors associated with statistical estimation of the model parameters. The effects of uncertainty, whose main source is the variability of prices and costs as well as yields, on the optimal solutions of the models have not been evaluated as a result. We therefore introduce the stochastic prediction errors into the current alternative deterministic regional and interregional planning models.

Various types of stochastic linear programming models (SLPM) have been developed and reported by many authors. Some of these models have been successfully applied in empirical studies. As considered by Nanseki(1986), however, certain types of agricultural planning problems with stochastic linear demand functions must be formulated as a quadratic programming model with stochastic coefficients in the linear term (abbreviated to SQPM).

In this paper, an algorithm is proposed and additional data are supplied with a view to further improving the model. In Section 2, we restate the model formulation described in the previous paper and illustrate how an interregional equilibrium model with stochastic linear demand and supply function must also be formulated as SQPM. In Section 3, alternative criteria for optimization are introduced and the mathematical characteristics of SQPM are analyzed. Equivalence relations among these criteria and an algorithm based on the relations for the solutions are considered in Section 4. A procedure for application of the model and a simple example based on actual data are illustrated in Section 5, followed by concluding remarks in Section 6.

2. Model Formulation

2.1 Regional production planning model

Consider an agricultural production planning model for a region with stochastic demand functions in the case of m commodities and single period (see e.g. Nanseki(1986) for details). Let

p : vector of the commodity price
x_d : vector of the consumption quantity (demand)
x_s : vector of the production activity level (supply)
d_0 : vector of the constant of the demand function
D : matrix of the coefficient of the demand function
ν : vector of the prediction error of p
t : vector of the unit transport cost
τ : vector of the prediction error of t
c : vector of the unit production cost
κ : vector of the prediction error of c.

The set of price-dependent demand functions may be written as

$$p = d_0 + Dx_d + \nu. \tag{2.1}$$

Then the regional aggregated profit functions can be defined as

$$\pi(x_s, x_d) = [-(c+\kappa) \quad d_0 + \nu - (t+\tau)] \begin{bmatrix} x_s \\ x_d \end{bmatrix} + [x_s \quad x_d] \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} x_s \\ x_d \end{bmatrix}. \tag{2.2}$$

For each commodity, we assume that the actual quantity consumed, x_d, is less than or equal to the effective supply, x_s, from all the production activities. The resource inputs for production and transport are assumed to be less than or equal to the resource quantity available. These constraints imposed on variables may be written as

$$\begin{bmatrix} -A_{11} & I \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_s \\ x_d \end{bmatrix} \leq \begin{bmatrix} 0 \\ b_2 \end{bmatrix} \tag{2.3}$$

where

A_{11} : matrix of the yield of production
A_{12} : matrix of the input-output coefficient of production
A_{22} : matrix of the input-output coefficient of transport
b_2 : vector of the resource availability
I : suitable unit matrix.

2.2 Interregional equilibrium model

Consider an interregional price equilibrium model for an n region single commodity with stochastic demand and supply functions (see e.g. Takayama and Judge(1971) for the deterministic version). Let

p_d : vector of the commodity price at the demand points
p_s : vector of the commodity price at the supply points
x_d : vector of the consumption quantity at the demand points
x_s : vector of the production activity level at the supply points
d_0 : vector of the constant of the demand function
D : matrix of the coefficient of the demand function
ν : vector of the prediction error of p_d
s_0 : vector of the constant of the supply function
S : matrix of the coefficient of the supply function
ξ : vector of the prediction error of p_s
x_t : vector of the transport activity level between the supply and demand points
t : vector of the unit transport cost
τ : vector of the prediction error of t.

The set of regional demand and supply functions may be written as

$$p_d = d_0 + Dx_d + \nu \tag{2.4}$$

$$p_s = s_0 + Sx_s + \xi. \tag{2.5}$$

Then the interregional net social payoff function can be defined as

$$nsp(x_s, x_d, x_t) = [-(s_0+\xi) \quad d_0+\nu \quad -(t+\tau)] \begin{bmatrix} x_s \\ x_d \\ x_t \end{bmatrix} + \frac{1}{2}[x_s \quad x_d \quad x_t] \begin{bmatrix} -S & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_s \\ x_d \\ x_t \end{bmatrix}. \tag{2.6}$$

For each region, we assume that the consumption quantity, x_d, is less than or equal to the quantity shipped into the region from the supply regions. The actual supply quantity,

x_s, is assumed to be greater than or equal to the effective supply from some region to all the regions. These constraints imposed on variables may be written as

$$(2.7)\qquad \begin{bmatrix} A_d \\ A_s \end{bmatrix} [x_t] \geq \begin{bmatrix} x_d \\ -x_s \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} -I & 0 & -A_s \\ 0 & I & -A_d \end{bmatrix} \begin{bmatrix} x_s \\ x_d \\ x_t \end{bmatrix} \leq 0$$

where both A_d and A_s are suitable matrices to complete the model and consist of 1, -1 and 0 element. I is a suitable unit matrix.

2.3 Quadratic programming model with stochastic coefficients in the linear term

The stochastic quadratic function such as (2.2) and (2.6) may be rewritten as

$$(2.8)\qquad d(x) = \gamma' x + x' Q x$$

where γ is a vector of stochastic coefficient and Q is a deterministic matrix. The set of linear constraints such as (2.3) and (2.7), and nonnegativity conditions of variables may be rewritten as

$$(2.9)\qquad C = \{x | Ax \leq b, \quad x \geq 0\}.$$

For simplicity, we, then, make the following assumptions on SQPM.

Assumption 1. *$\gamma' x$ is a random variable and has a normal distribution $N(r'x, x'\Sigma x)$, where r and Σ are a mean vector and a variance-covariance matrix of γ , respectively.*

The mean, μ, and variance, σ^2, of the objective function d are denoted as

$$(2.10)\qquad \mu = \mathrm{E}[d(x)] = r'x + x'Qx$$

$$(2.11)\qquad \sigma^2 = \mathrm{Var}[d(x)] = x'\Sigma x.$$

Assumption 1 obviously holds when each stochastic error vector (ν, ξ, κ, τ) is distributed as a multivariate normal distribution. Even if each stochastic vector is not a multivariate normal variable, the assumption is approximately met by the Central Limit Theorem (see e.g. Hoel et al. 1971) when the number of commodities or regions is sufficiently large.

From the definition of variance, it is obvious that $x'\Sigma x$ is positive semi-definite. For theoretical simplicity, however, we can make the following assumption without any loss in application for the time.

Assumption 2. *$x'\Sigma x > 0$ for all $x \neq 0$.*

In the regional planning model, D is a diagonal matrix with nonpositive elements when cross price flexibility coefficients between commodities are zero. The following assumption consequently holds in this case. The assumption also holds in the single commodity interregional equilibrium model, since D and S' are diagonal matrices with nonpositive or nonnegative elements, respectively.

Assumption 3. *$x'Qx \leq 0$ for all $x \neq 0$.*

We introduce the following deterministic quadratic programming problem and make the following assumptions for later discussion.

Problem 0.
maximize $\mu = r'x + x'Qx$
subject to $x \in C$

Assumption 4. *An optimal solution of Problem 0 exists and is finite.*

Assumption 5. *C does not include $x = 0$.*

Assumptions 1, 2 and 5 can be removed in the applied study (see Section 5).

3. Deterministic Equivalents and Properties of Solution

We have alternative criteria for optimization depending upon the purpose of the applied research. One of the well-known criteria in agricultural economics is to maximize the expected utility function specified by a suitable function form with constant absolute risk aversion. This criterion is often employed in empirical studies for modeling and simulating the economic behavior of a farm or a region. The criterion, however, is not appropriate to practical agricultural planning since it is difficult for policy makers to specify the risk aversion parameter. From the managerial view point of agriculture, the criteria corresponding to a satisfaction approach are more practical. Maximizing probability criterion and maximizing aspiration (or satisfactory) level criterion are two plausible criteria.

In this section, the three alternative criteria above are adopted for formulating problems. We call each model U-model, S-model and P-model, in this order, respectively.

Problem 1. U-model
maximize $\mathrm{E}[u(d(x))]$
subject to $x \in C$
where $u(d) = 1 - \exp(-ad), a \geq 0$.

Function $u(d)$ is the utility function employed by Freund(1956) and parameter a is a positive risk aversion constant, which may be considered as a measure of the aversion to risk .

Problem 2. S-model
maximize g
subject to $\mathrm{Prob}(g \leq d) \geq \eta, \quad x \in C$
where $0.5 \leq \eta < 1.0$.

The stochastic linear programming problems based on this criterion were considered in several papers (e.g. Kataoka, 1963). Parameter η is a reliability constant, which may be considered as a measure of the reliability of the planning .

Problem 3. P-model
maximize Prob$(l \leq d)$
subject to $x \in C'$
where parameter l is less than or equal to the optimal value of Problem 0.

The stochastic linear programming problems based on this criterion were considered in several papers (e.g. Charnes and Cooper, 1963; Kataoka, 1967). Parameter l is an aspiration level constant .

Because of the normality of the distribution of d, the deterministic equivalent (Charnes and Cooper, 1963) of Problem 1 is Problem 4 (see Appendix 1 for details).

Problem 4. U-model
maximize

$$f(x) = r'x + x'(Q - (\frac{a}{2})\Sigma)x \tag{3.1}$$

subject to $x \in C'$
where $a \geq 0$.

Since $f(x)$ is a strictly concave function of x for $a > 0$ by Assumptions 2 and 3, the optimal solution can be found by the known convex quadratic programming method (see e.g. Martos, 1975). When $a = 0$, Problem 4 is equivalent to Problem 0. Denote an optimal solution and an optimal functional value as a function of a by $\hat{x}(a)$ and $\hat{f}(a)$, respectively. The following properties can be derived (see Appendix 2 for the proofs).

Proposition (i) $\hat{x}(a)$ is unique for $a > 0$.
(ii) $\hat{f}(a)$ is a strictly monotone decreasing function of a.
(iii) $\hat{\sigma}(a) = \sqrt{\hat{x}(a)'\Sigma\hat{x}(a)}$ is a strictly monotone decreasing function of a.
(iv) $\hat{\mu}(a) = r'\hat{x}(a) + \hat{x}(a)'Q\hat{x}(a)$ is a strictly monotone decreasing function of a.

Now define

$$\Phi(k) = \int_{-k}^{+\infty} \frac{1}{\sqrt{2\pi}}\exp(-\frac{\rho^2}{2})d\rho \tag{3.2}$$

where $\rho \equiv N(0,1)$. The deterministic equivalent of Problem 2 is the following (see Appendix 3 for details).

Problem 5. S-model
maximize

$$g(x) = r'x + x'Qx - k\sqrt{x'\Sigma x} \tag{3.3}$$

subject to $x \in C$
where $k = \Phi^{-1}(\eta), k \geq 0$.

Parameter k is a nonnegative safety constant which corresponds to a reliability constant, η . Since $\sqrt{x'\Sigma x}$ is strictly convex (Kataoka, 1963), $g(x)$ is strictly concave for $k > 0$. The optimal solution therefore may be found by the known convex programming method. When $\eta = 0.5$, that is $k = 0$, Problem 5 is equivalent to Problem 0.

Denote an optimal solution and an optimal functional value as a function of k by $\tilde{x}(k)$ and $\tilde{g}(k)$, respectively. The following properties then can be derived. The proofs are similar to those of (i) $\sim$ (iv), respectively, and they are omitted.

Proposition (v) $\tilde{x}(k)$ is unique for $k > 0$.

(vi) $\tilde{g}(k)$ is a strictly monotone decreasing function of k.

(vii) $\tilde{\sigma}(k) = \sqrt{\tilde{x}(k)'\Sigma\tilde{x}(k)}$ is a strictly monotone decreasing function of k.

(viii) $\tilde{\mu}(k) = r'\tilde{x}(k) + \tilde{x}(k)'Q\tilde{x}(k)$is a strictly monotone decreasing function of k.

The deterministic equivalent of Problem 3 is the following (see Appendix 3 for details).

Problem 6. P-model.
maximize

$$h(x) = \frac{r'x + x'Qx - l}{\sqrt{x'\Sigma x}} \tag{3.4}$$

subject to $x \in C$
where $l \leq \max\{E[d(x)]|x \in C\}$.

Since the numerator of $h(x)$ is concave and the denominator is positive and strictly convex, an optimal solution of problem 6 may be found by applying the Dinkelbach's nonlinear fractional programming method (Dinkelbach, 1967). When parameter l is equal to the optimal functional value of Problem 0, the optimal solution of Problem 6 is equivalent to that of Problem 0 (see Theorem 2).

Denote an optimal solution and an optimal functional value as a function of l by $x^*(l)$ and $h^*(l)$, respectively. Then the following properties can be derived. The proofs are also similar to those of (i) $\sim$ (iv), respectively, and they are omitted.

Proposition (ix) $x^*(l)$ is unique for $l \leq \max\{E[d(x)]|x \in C\}$.

(x) $h^*(l)$ is a strictly monotone decreasing function of l.

(xi) $\sigma^*(l) = \sqrt{x^*(l)'\Sigma x^*(l)}$ is a strictly monotone increasing function of l .

(xii) $\mu^*(l) = r'x^*(l) + x^*(l)'Qx^*(l)$ is strictly monotone increasing function of l .

4. Equivalence Relations among the Criteria and Algorithm for P-model

In this section, we determine the equivalence relation between Problems 4 and 5, and that between Problems 5 and 6. Based on the relations, an algorithm for the P-model is proposed.

From a computational point of view, the equivalence relations are useful in applied research. Particularly in the case of a large model, it is difficult to solve Problems 6 and 5 directly by the known nonlinear optimizing method due to the nonlinearity of the objective function. Because Problem 4 is, however, a convex quadratic programming problem for which several efficient optimizing methods are known, an optimal solution of the problem is more easily found than in the case of Problems 5 and 6. The solution of Problem 4 then can be interpreted as that of Problems 5 and 6 by the equivalence relations. Furthermore, the relations give us more information and are useful to identify a wider implication of the optimal solutions for the applications.

4.1 Equivalence relations

We can derive the following equivalence relation between Problem 4 and Problem 5 with the same notation described as in the previous section.

Theorem 1. *An optimal solution of Problem 4, $\hat{x}(a)$, is that of Problem 5, $\tilde{x}(k)$, if the value of parameter k satisfies $k = a\sqrt{\hat{x}(a)'\Sigma\hat{x}(a)}$. An optimal solution of Problem 5, $\tilde{x}(k)$, is also that of Problem 4, $\hat{x}(a)$, if the value of parameter a satisfies $a = k/\sqrt{\tilde{x}(k)'\Sigma\tilde{x}(k)}$.*

Proof: Based on the convex programming theory, an optimal solution of Problem 4, $\hat{x}(a)$, is the x-part of the solution of the following Kuhn-Tucker condition KTL(a) (Martos, 1975).
KTL(a).

$$r + 2Q\hat{x}(a) - a\Sigma\hat{x}(a) - A'\hat{\lambda} + \hat{\psi} = 0 \quad , \quad A\hat{x}(a) - b + \hat{y} = 0 \tag{4.1}$$

$$\hat{r}'\hat{x}(a) + \hat{\lambda}'\hat{y} = 0 \quad , \quad \hat{x}(a), \hat{\lambda}, \hat{y}, \hat{\psi} \geq 0 \tag{4.2}$$

where $\hat{\lambda}$ is an m dimensional vector (Lagrange multiplier); $\hat{\psi}$ is an n dimensional slack vector; $\hat{y}$ is an m dimensional slack vector.

On the other hand, an optimal solution of Problem 5, $\tilde{x}(k)$, is the x-part of the solution of the following Kuhn-Tucker condition KTL(k).

$$r + 2Q\tilde{x}(k) - \frac{k\Sigma\tilde{x}(k)}{\sqrt{\tilde{x}(k)'\Sigma\tilde{x}(k)}} - A'\tilde{\lambda} + \tilde{\psi} = 0 \quad , \quad A\tilde{x}(k) - b + \tilde{y} = 0 \tag{4.3}$$

$$\tilde{\psi}'\tilde{x}(k) + \tilde{\lambda}'\tilde{y} = 0 \quad , \quad \tilde{x}(k), \tilde{\lambda}, \tilde{y}, \tilde{\psi} \geq 0 \tag{4.4}$$

where $\tilde{\lambda}$ is an m dimensional vector (Lagrange multiplier); $\tilde{\psi}$ is an n dimensional slack vector; $\tilde{y}$ is an m dimensional slack vector.

Therefore if $k = a\sqrt{\hat{x}(a)'\Sigma\hat{x}(a)}$ holds, $\hat{x}(a)$ is the optimal solution of Problem 5, because $\hat{x}(a)$ satisfies the KTL(k) equations. Similarly if $a = k/\sqrt{\tilde{x}(k)'\Sigma\tilde{x}(k)}$ holds, $\tilde{x}(k)$ is the optimal solution of Problem 4, because $\tilde{x}(k)$ satisfies the KTL(a) equations. ∎

Moreover we can find the following properties for the solution of Problem 4 (see Appendix 4 for the proofs).

Corollary 1. *$a(k)$ as a function of k is unique. $a(0) = 0$ and $\lim_{k\to\infty} a(k) = \infty$.*
Corollary 2. *$a(k)$ is a strictly monotone increasing function of k.*

It is obvious that $k(a) = a\sqrt{\hat{x}(a)'\Sigma\hat{x}(a)}$ as the inverse function of $a(k)$ has similar properties to Corollaries 1 and 2.

We can derive the following equivalence relation between Problems 5 and 6.

Theorem 2. *An optimal solution of Problem 5, $\tilde{x}(k)$, is that of Problem 6, if the value of parameter l satisfies*

$$l = r'\tilde{x}(k) + \tilde{x}(k)'Q\tilde{x}(k) - k\sqrt{\tilde{x}(k)'\Sigma\tilde{x}(k)}. \tag{4.5}$$

An optimal solution of Problem 6, $x^*(l)$, is also that of Problem 5, $\tilde{x}(k)$, if the value of parameter k satisfies

$$k = \frac{r'x^*(l) + x^*(l)'Qx^*(l) - l}{\sqrt{x^*(l)'\Sigma x^*(l)}} \quad . \tag{4.6}$$

Proof: Since $C = \{x|Ax \leq b, x \geq 0\}$ is compact and the denominator of the objective function of Problem 6, $\sqrt{x'\Sigma x}$, is positive for all $x \in C$ by Assumptions 2 and 4, the theorem of Dinkelbach (1967) for nonlinear fractional programming can be applied to Problem 6. A direct application of the theorem yields that x^* is an optimal solution of Problem 6 and the optimal value is k^*, if and only if x^* is an optimal solution of problem 5 with $k = k^*$ and the optimal value is l. The theorem therefore is obvious. ∎

We can now find the following properties (see Appendix 4 for the proofs).

Corollary 3. *$l(k)$ as a function of k is unique. $l(0) = \max\{\mathrm{E}[d(x)]|x \in C\}$ and $\lim_{k\to\infty} l(k) = -\infty$ holds, where $\max\{\mathrm{E}[d(x)]|x \in C\}$ is the optimal value of Problem 0.*
Corollary 4. *$l(k)$ is a strictly monotone decreasing function of k.*

We can prove that $k(l)$ as the inverse function of $l(k)$ has similar properties to Corollaries 3 and 4, and the proofs are omitted. Furthermore, we can find that $l(k)$ is a convex function of k , and so is $k(l)$ of l. The proof is similar to that of lemma 1 of Dinkelbach (1967) and it is omitted. By Theorems 1, 2 and Corollaries 1 $\sim$ 4, the following properties obviously hold.

Corollary 5. *$l(a)$ as a function of a is unique. $l(0) = \max\{\mathrm{E}[d(x)] | x \in C\}$ and $\lim_{a \to \infty} l(a) = -\infty$ holds.*

Corollary 6. *$l(a)$ is a strictly monotone decreasing function of a.*

4.2 Algorithm for P-model

We propose here an iteration algorithm for the P-model. It is obvious that an optimal solution of Problem 4, $\hat{x}(a)$, is that of Problem 6, $x^*(l)$, if the value of parameter a satisfies $l(a) = l$ by Theorems 1 and 2. We can evaluate the functional value of $l(a)$ by solving Problem 4 and find a solution of the equation $l(a) - l = 0$ by secant method (see e.g. Makinouchi and Torii, 1986). The solution is obviously nonnegative and unique by Corollaries 5 and 6.

Algorithm

[Step 0]

Solve Problem 4 with $i := 0$ and $a_0 := 0$, and continue.

[Step 1]

Solve Problem 4 with $i := 1$ and $a_1 := (\hat{l}(0) - l)/\hat{\sigma}(0)^2$, which is a lower bound of the value of a corresponding to the value of l, and continue.

[Step 2]

Set $i := i + 1$ and

$$a_i := a_{i-1} - (\hat{l}(a_{i-1}) - l)\frac{a_{i-1} - a_{i-2}}{\hat{l}(a_{i-1}) - \hat{l}(a_{i-2})}.$$

Solve Problem 4 with $a = a_i$ and continue.

[Step 3]

If $\hat{l}(a_i) = l$ and/or $|a_i - a_{i-1}| \leq \epsilon$ then $\hat{x}(a_i)$ is the optimal solution. Otherwise return to Step 2, where

- l : an aspiration level assumed to be less than or equal to the optimal value of Problem 0
- a_i : an estimated risk aversion constant at ith iteration
- $x(a_i)$: an optimal solution of Problem 4 with $a = a_i$
- $\hat{l}(a_i)$: the aspiration level calculated from $\hat{x}(a_i)$
- $\hat{\sigma}(a_i)^2$: the variance calculated from $\hat{x}(a_i)$
- i : iteration number

ϵ : tolerance parameter

Similar algorithm for S-model will be easily constructed.

5. Procedure for Application and Illustrative Example of P-model

5.1 Procedure for application

A procedure for application of the P-model, which is a more attractive formulation in practical planning, is outlined. In the applied problems, the propriety of the assumptions in Section 2 may not be verified apriori. Accordingly, we must verify the assumptions and then apply the algorithm described in the previous section. The procedure consists of the following five steps.

If Assumption 1 does not hold then the deterministic equivalents, Problems 4,5 and 6, can not be derived from Problems 1, 2 and 3. However Problem 6 can be interpreted as maximizing a lower bound for probability Prob$[d \geq l]$ by the application of the Chebyshev's Inequality (see e.g. Hoel et al. 1971 and Appendix 5). This interpretation is based upon the lack of assumption on the distribution of $\gamma' x$ other than that it has finite variance.

Procedure

[Step 1]

If Assumption 3 holds, then continue. Otherwise stop (failure).

[Step 2]

If we have a finite optimal solution of Problem 0 (Assumption 4 holds), then continue. Otherwise stop (failure).

[Step 3]

Solve Problem 6 by the algorithm and denote an optimal solution by x^*, and continue.

[Step 4]

Solve the following problem. If we have a zero optimal solution or a nonfeasible solution (Assumption 2 holds), then x^* is an optimal solution of the P-model. Otherwise denote the optimal solution of Problem 7 by $\bar{x}$ and continue.

Problem 7

maximize $r'x + x'Qx$

subject to $\Sigma x = 0, \quad Ax \leq b, \quad x \geq 0$

[Step 5]

If $g(\bar{x}) < l$ then x^* is an optimal solution of the P-model. Otherwise $\bar{x}$ is an optimal solution of the P-model.

$C = \{x | Ax \leq b, x \geq 0\}$ in an applied model may include $x = 0$. In this case we must theoretically add an inequality to exclude $x = 0$ in order to meet Assumption 5. When an optimal solution of the problem without such an inequality is not a zero vector, all the theorems and properties proved in this paper hold, however. We consequently need not add such an inequality in practice.

5.2 Illustrative example

In order to illustrate our algorithm and procedure, we consider the following simple example, which is based on an actual economic problem. One of the main issues in agricultural policies in developing countries is to determine the optimal allocation of resources of a region under uncertain supply-demand relations and production constraints. For evaluating the resource allocation in Indonesian agriculture, the regional production planning model is applied based upon data presented by the UN/ESCAP CGPRT Centre. The data cover commodity production (10^3 t), prices (10^6 Rp./10^3 t) expressed in 1985 Rp. from 1982 to 1985, average yields (10^3 t/10^3 ha), production cost (10^6 Rp./10^3 ha), transport cost (10^6 Rp./10^3 t), input-output coefficients (10^3 ha, 10^3 man-day/10^3 ha) and resource availability (10^3 ha, 10^3 man-day/10^3 ha per month) of upland and labor.

Main upland crops in Garut, Indonesia are maize, cassava, soybean and upland rice. Since an additional demand for rice, maize and cassava has not been generated recently, the increase in production leads to lower prices. On the other hand, prices of soybean are not sufficiently affected by the expanded production due to the increasing demand and price policy.

The set of the demand functions for markets in Garut is statistically estimated as

$$p = d_0 + Dx_d + \nu$$

where

$$p = \begin{bmatrix} p_{rice} \\ p_{maize} \\ p_{cassava} \\ p_{soybean} \end{bmatrix} \quad d_0 = \begin{bmatrix} 388.40 \\ 235.40 \\ 192.20 \\ 544.80 \end{bmatrix} \quad D = \begin{bmatrix} -1.2177 & 0 & 0 & 0 \\ 0 & -1.0242 & 0 & 0 \\ 0 & 0 & -0.6160 & 0 \\ 0 & 0 & 0 & -0.0001 \end{bmatrix}$$

$$\mathrm{E}[\nu] = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \mathrm{Var}[\nu] = \begin{bmatrix} 210.619 & 66.056 & 62.420 & -290.077 \\ 66.056 & 35.095 & 9.728 & -92.914 \\ 62.420 & 9.728 & 30.839 & -100.413 \\ -290.077 & -92.914 & -100.413 & 444.250 \end{bmatrix}.$$

We assume that the cost vector for production and transportation are deterministic and estimate the coefficients as

$$c' = [61.99 \quad 18.91 \quad 23.38 \quad 90.00] \quad t' = [154.00 \quad 15.00 \quad 25.00 \quad 15.00].$$

The yield matrix, input-output matrix and resource availability vector are also estimated as

$$A_{11} = \begin{bmatrix} 2.075 & 0 & 0 & 0 \\ 0 & 2.459 & 0 & 0 \\ 0 & 0 & 10.175 & 0 \\ 0 & 0 & 0 & 0.890 \end{bmatrix} \quad A_{21} = \begin{bmatrix} 1.0 & 1.0 & 1.0 & 1.0 \\ 32.1 & 35.7 & 57.1 & 37.5 \\ 34.3 & 17.9 & 0.0 & 30.3 \\ 30.3 & 10.7 & 0.0 & 16.1 \\ 14.3 & 0.0 & 0.0 & 16.1 \\ 14.3 & 21.4 & 0.0 & 21.4 \end{bmatrix}$$

$$b_2' = [73.3 \quad 3014.0 \quad 3014.0 \quad 3014.0 \quad 3014.0 \quad 3014.0].$$

Our problem is now restated as
maximize $\mathrm{Prob}[l \le d]$
subject to $Ax \le b$
where

$$d(x_s, x_d) = [-c \quad d_0 + \nu - t] \begin{bmatrix} x_s \\ x_d \end{bmatrix} + [x_s \quad x_d] \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} x_s \\ x_d \end{bmatrix}$$

$$A = \begin{bmatrix} -A_{11} & I \\ A_{21} & 0 \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ b_2 \end{bmatrix} \quad x = \begin{bmatrix} x_s \\ x_d \end{bmatrix} \quad \Sigma = \begin{bmatrix} 0 & 0 \\ 0 & \mathrm{Var}[\nu] \end{bmatrix}.$$

In this example, we selected $l = 33677$ as the aspiration level of the aggregated agricultural profit in the region. The corresponding Problem 6 and Problem 0 are solved by a hand-made program coded in C language with double precision on a personal computer NEC/PC-9801VX.

Since the matrix D is negative definite and Problem 0 has a finite optimal solution, Assumptions 3 and 4 obviously hold. We have also an optimal solution of Problem 6 at Iteration 11 by the algorithm with $\epsilon = 1.0\text{e-}6$. Table 1 presents the optimal solutions of both Problems 0 and 6 with a range of actual output levels from 1980 to 1985. The iteration process is presented in Table 2. Furthermore the corresponding Problem 7 has the only one feasible solution $\bar{x} = 0$. Assumption 2 therefore holds. We know as a result according to the procedure that the optimal solution of problem 6 is that of our problem.

Table 1 Optimal solution of Problems 0 and 6

Variable	Unit	Problem 0	Problem 6 $l = 33677$	Range of Actual Output 1980	~	1985
Area x_s						
rice	10^3ha	4.089	1.598	8.848	~	22.652
maize	10^3ha	11.427	23.847	10.972	~	32.909
cassava	10^3ha	10.164	8.775	18.262	~	24.186
soybean	10^3ha	47.620	39.080	9.950	~	32.471
Production x_d						
rice	10^3t	8.484	3.315	19.192	~	44.425
maize	10^3t	28.100	58.640	25.361	~	65.724
cassava	10^3t	103.415	89.285	180.445	~	276.282
soybean	10^3t	42.382	34.782	9.185	~	28.272
μ	10^6Rp.	35449.429	34338.658			
σ	10^6Rp.	241.046	34.689			
h	-	0.000	19.074			
l	10^6Rp.	35449.429	33677.000			

Table 2 Iteration process of the algorithm
(ϵ=1.0e-6, $l = 33677$)

i	a_i	$a_i - a_{i-1}$	$\hat{k}(a_i)$	$\hat{l}(a_i)$	$\hat{l}(a_i) - l$	$\hat{\mu}(a_i)$	$\hat{\sigma}(a_i)$
0	0.000000	-	0.000	35449.429	1.772e+3	35449.429	241.046
1	0.030505	3.050e−2	4.048	34787.987	1.111e+3	35325.240	132.711
2	0.081742	5.124e−2	7.830	34359.093	6.821e+2	35109.123	95.789
3	0.163227	8.149e−2	11.406	34065.371	3.884e+2	34862.368	69.877
4	0.270970	1.077e−1	14.467	33877.766	2.008e+2	34650.200	53.391
5	0.386271	1.153e−1	16.761	33767.061	9.006e+1	34494.390	43.393
6	0.480071	9.380e−2	18.189	33709.120	3.212e+1	34398.231	37.887
7	0.532071	5.200e−2	18.860	33684.493	7.493e+0	34353.033	35.447
8	0.547893	1.582e−2	19.051	33677.806	8.061e−1	34340.219	34.771
9	0.549800	1.907e−3	19.073	33677.023	2.277e−2	34338.702	34.691
10	0.549856	5.545e−5	19.074	33677.000	3.753e−5	34338.658	34.689
11	0.549856	9.152e−8	19.074	33677.000	2.470e−5	34338.658	34.689

6. Concluding Remarks

6.1 Extension of the equivalence relations of the alternative criteria

The equivalence relations of the alternative criteria may be extended to other criteria considered in several papers. The criteria are (1) maximizing the expected value of the objective function and (2) minimizing the variance of the objective function. The problems based on each criterion , which we call in order E-model and V-model, are the following, respectively;

Problem 8. E-model
maximize $\mathrm{E}[d(x)]$ subject to $\mathrm{Var}[d(x)] \le v, \quad x \in C$.

Problem 9. V-model
minimize $\mathrm{Var}[d(x)]$ subject to $\mathrm{E}[d(x)] \ge e, \quad x \in C$.
where parameters v and e are suitable constants. These formulations for which the normality of d (Assumption 1) is not a requirement may be useful in some applied fields.

6.2 Relation between SLPM and SQPM

One should note that in case of $Q = 0$, Problem 4 is reduced to the model presented in Freund(1956), and Problems 5 and 6 are reduced to the linear version considered by Kataoka (1963, 1967). The relations among optimal solutions of the linear version of the U-model, S-model and P-model have been discussed with numerical examples (without theoretical considerations) in several papers (e.g. Boussard, 1969, Sengupta, 1982). Theorems 1 and 2, which are extensions of Kataoka's theorem (1963, 1967), establish the equivalence relations among the U-model, S-model and P-model, however.

Kataoka (1967) has proposed an algorithm for solving the linear version of the S-model and P-model. The algorithm, which is more sophisticated than the previous algorithm (Kataoka, 1963), takes advantage of the parametric quadratic programming method of Wolfe (1959). Ishii et al.(1978) have also proposed a similar algorithm for solving a variant of the S-model. However the algorithm can not be applied to Problems 5 and 6 due to the nonlinearity of the objective function, μ. This paper therefore presents an iteration method which is capable of solving the problems.

6.3 Further applicability of SQPM

An application study of SQPM to agricultural development problems in South East Asia has been undertaken in cooperation with experts of developing economies. In the case study, a stochastic interregional planning model based on SQPM will be built for the economic analysis of production and marketing systems under uncertain demand conditions for selected commodities in Indonesia.

Acknowledgment

The author would like to thank Professor T. Kawaguchi of Kyushu University and Professor K. Tanabe of The Institute of Statistical Mathematics for their invaluable comments and suggestions. The author also wishes to thank two anonymous referees for their many helpful comments and Professor Y. Maruyama of Tsukuba University for his helpful comments on earlier drafts of this paper. The author is grateful to the Regional Coordination Centre for Research and Development of Coarse Grains, Pulses, Roots and Tuber Crops in the Humid Tropics of Asia and the Pacific (CGPRT Centre) under UN/ESCAP for providing the data.

References

[1] Boussard, J.M.: The Introduction of Risk into a Programming Model: Different Criteria and the Actual Behavior of Farmers. *European Economic Review,* Vol.1, No.1 (1969), 92-121.

[2] Charnes, A. and Cooper, W.W.: Deterministic Equivalents for Optimizing and Satisficing under Chance Constraints. *Operations Research,* Vol.11 (1963), 18-39.

[3] Dinkelbach, W.: On Nonlinear Fractional Programming. *Management Science,* Vol.13, No.7 (1967), 492-498.

[4] Freund, R.J.: The Introduction of Risk into a Programming Model. *Econometrica,* Vol.24, No.3 (1956), 253-263.

[5] Hoel, P.G., Port, S.C. and Stone, C.J.: *Introduction to Probability Theory.* Houghtone Mifflin, 1971.

[6] Ishii, H., Nishida, T. and Nanbu, Y.: A Generalized Chance Constraint Programming Problem. *Journal of the Operations Research Society of Japan,* Vol.21, No.1

(1978), 124-145.

[7] Judge G. and Takayama T. (ed.): *Studies in Economic Planning over Space and Time.* North-Holland, 1973.

[8] Kataoka, S.: A Stochastic Programming Model. *Econometrica,* Vol.31, No1-2 (1963), 181-196.

[9] Kataoka, S: Stochastic Programming - Maximum Probability Model - . *Hitotsubashi Journal of Arts Science,* Vol.8, No.1 (1967; Whole Number), 51-59.

[10] Makinouchi S. and Torii T.: *Numerical Analysis.* Ohmu-sha, 1986, 113-139.(in Japanese)

[11] Martos, B.: *Nonlinear Programming Theory and Methods.* North-Holland Publishing Company, 1975, 123-140.

[12] Nanseki, T.: The Introduction of Risk into a Regional Planning Model of Agriculture : Alternative Nonlinear Programming Models under Stochastic Demand Functions. *Journal of Rural Economics,* Vol.58, No.1 (1986), 19-29. (in Japanese)

[13] Sengupta, J.K.: *Decision Model in Stochastic Programming: Operational Methods of Decision Making under Uncertainty.* North-Holland, 1982, 37-38.

[14] Takayama T. and Judge G.: *Spatial and Temporal Price and Allocation Models.* North-Holland, 1971, 129-152, 275-298.

[15] Wolfe, P.: The Simplex Methods for Quadratic Programming. *Econometrica,* Vol.27 (1959), 382-398.

Appendix 1. Derivation of Problem 4

By integration,

$$\begin{aligned}\mathrm{E}[u(d))] &= \int_{-\infty}^{+\infty} u(d)\Phi(d)\mathrm{d}d = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{+\infty}\exp[-\frac{(d-\mu)^2}{2\sigma^2}]\mathrm{d}d \\ &- \exp[a(\frac{a}{2}\sigma^2)]\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{+\infty}\exp[-\frac{(d-\mu-a\sigma^2)^2}{2\sigma^2}]\mathrm{d}d \\ &= 1-\exp[a(\frac{a}{2})\sigma^2-\mu]\end{aligned}$$

Therefore an optimal solution for maximizing $\mathrm{E}[u(d))]$ is equivalent to that of maximizing $\mu-(a/2)\sigma^2$ for $a \geq 0$.

Appendix 2. Proofs of Propositions (i)$\sim$ (iv)

(i) Since the objective function $f(x)$ is strictly concave for $a > 0$, it is clear.

(ii) For $a_2 > a_1 \geq 0$,

$$\hat{f}(a_2) = \hat{\mu}(a_2) - \frac{a_2}{2}\hat{\sigma}(a_2)^2 < \hat{\mu}(a_2) - \frac{a_1}{2}\hat{\sigma}(a_2)^2 \quad (\text{from} \quad a_2 > a1 \geq 0)$$

$$< \hat{\mu}(a_1) - \frac{a_1}{2}\hat{\sigma}(a_1)^2 = \hat{f}(a_1) \quad (\text{from} \quad \text{optimality})$$

(iii) For $a_2 > \hat{a}_1 \geq 0$, from optimality of $\hat{x}(a_1)$ and $\hat{x}(a_2)$

$$\hat{f}(a_1) = \hat{\mu}(a_1) - \frac{a_1}{2}\hat{\sigma}(a_1)^2 > \hat{\mu}(a_2) - \frac{a_1}{2}\hat{\sigma}(a_2)^2$$

$$\hat{f}(a_2) = \hat{\mu}(a_2) - \frac{a_2}{2}\hat{\sigma}(a_2)^2 > \hat{\mu}(a_1) - \frac{a_2}{2}\hat{\sigma}(a_1)^2$$

By adding two inequalities,

$$(a_2 - a_1)[\hat{\sigma}(a_1)^2 - \hat{\sigma}(a_2)^2] > 0 \quad \text{or} \quad \hat{\sigma}(a_1)^2 > \hat{\sigma}(a_2)^2 \quad (\text{from} \quad a_2 > a_1 \geq 0)$$

(iv) For $a_2 > a_1 \geq 0$,

$$\hat{f}(a_1) = \hat{\mu}(a_1) - \frac{a_1}{2}\hat{\sigma}(a_1)^2 > \hat{\mu}(a_2) - \frac{a_1}{2}\hat{\sigma}(a_2)^2$$

$$> \hat{\mu}(a_2) - \frac{a_1}{2}\hat{\sigma}(a_1)^2 \quad (\text{from} \quad \text{Proposition(iii)})$$

holds. Therefore $\hat{\mu}(a_1) > \hat{\mu}(a_2)$.

Appendix 3. Derivation of Problems 5 and 6

By simple subtraction and derivation, the first constraint of Problem 2 becomes

$$\text{Prob}[g \leq d] = \text{Prob}[\frac{g-\mu}{\sigma^2} \leq \frac{d-\mu}{\sigma^2}] = \text{Prob}[-k \leq \rho] = \Phi(k) \geq \eta$$

$$\text{where} \quad k = -\frac{g-\mu}{\sigma^2} \quad , \quad \rho = \frac{d-\mu}{\sigma^2}.$$

Since ρ is distributed according to a standard normal distribution, $\text{Prob}[g \leq d] \geq \eta$ may be replaced by $k \geq \Phi^{-1}(\eta)$ or $g \leq \mu - \Phi^{-1}(\eta)\sigma^2$. An optimal solution of Problem 2 is consequently equivalent to that of Problem 5. Similar transformation of Problem 3 gives Problem 6.

Appendix 4. Proofs of Corollaries 1 $\sim$ 4

1. Since $\tilde{x}(k)$ is unique by Proposition (x), so is $a(k)$. Furthermore $\tilde{\sigma}(k)$ is bounded and the value is not zero by Assumption 4. Therefore the corollary is clear.

2. For $k_2 > k_1 \geq 0$,

$$a(k_1) = \frac{k_1}{\tilde{\sigma}(k_1)} \leq \frac{k_1}{\tilde{\sigma}(k_2)} \quad (\text{from} \quad \text{Proposition(vii)})$$

$$< \frac{k_2}{\tilde{\sigma}(k_2)} = a(k_2) \quad (\text{from} \quad k_2 > k_1 \geq 0)$$

3. Since $\tilde{x}(k)$ is unique, so is $l(k)$. Furthermore $\tilde{x}(k)$ is bounded and the value is not zero by Assumption 4. Therefore it is clear.

4. By Proposition (vi) and Theorem 2, it is clear.

Appendix 5. Derivation of Problem 6 without Assumption 1.

The Chebyshev's inequality may be written as $\text{Prob}[|d-\mu| \geq c] \leq \sigma^2/c^2$ for any real number $c > 0$ where d is a general random variable with mean, μ, and finite variance, σ^2. Furthermore

$$\text{Prob}[|d-\mu| \geq c] > \text{Prob}[d-\mu < -c] = \text{Prob}[d < l] = 1 - \text{Prob}[d \geq l]$$

holds where $c = \mu - l$. A lower bound of $\text{Prob}[d \geq l]$ is then $1 - \sigma^2/(\mu - l)^2$.

The optimal solution for maximizing the lower bound is equivalent to that of maximizing $(\mu - l)/\sigma$ since we assume $\mu \geq l$.

Teruaki NANSEKI: Department of
Agricultural Development,
National Agriculture Research Center,
Ministry of Agriculture, Forestry
and Fisheries,
Kannondai 3-1-1, Tsukuba 305, Japan