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ON AN EXTENSION OF BARLOW-WU SYSTEMS — BASIC PROPERTIES

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Abstract The present paper considers multi-state systems which are regarded as a generalization of binary systems, and generalizes the concepts of BW systems proposed by R. E. Barlow and Alexander S. Wu. Since the structure function of a BW system is defined by using the minimal path sets of a binary coherent system, the BW system has a strict restriction that the form of minimal path sets should be identical for any states of the system. In this paper we relax the strict condition and propose a class of multi-state systems which we call Extended BW systems (EBW systems). We present some properties of EBW systems and clarify the relationship among multi-state, EBW and binary coherent systems. As the main results of this paper, we show that any EBW systems are equivalent to a class of binary coherent systems satisfying some conditions. Furthermore, we provide the characterization that the class of EBW systems is the maximum class of the multi-state systems which can be transformed into binary coherent systems whose number is identical to the number of the states of the multi-state system.

1. Introduction

Multi-state systems have been studied by several authors, e.g., R.E. Barlow and Alexander S.Wu [2], E.Nl-Neweihi, F.Proschan and J.Sethuraman [3], W.S.Griffith [4], F.Ohi and T.Nishida [5,6,7] and J.I.Ansell and A.Bendell [8]. In considering multi-state systems, the state spaces of systems and their componets are assumed to have at least two states including both perfect functioning and complete failure. For this reason, multi-state systems are regarded as the generalization of binary systems. Binary coherent systems have naturally been extended to finite multi-state systems by R.E.Barlow and A.S.Wu [2], which are called BW (Barlow-Wu) systems. On the other hand, F.Ohi and T.Nishida [5,6] have discussed the multi-state systems on the basis of more generalized conditions such that state spaces of systems and their components are not necessarily the same.

In the present paper we generalize the concepts of BW systems. Since the structure function of a BW system is defined by using minimal path sets of a binary coherent system, the BW system has a strict restriction that the form of minimal path sets should be identical for any states of the system. In this paper, we relax the strict condition and propose a class of multi-state systems which we call Extended BW systems (EBW systems). We present some properties of EBW systems and clarify the relationship among multi-state, EBW and binary coherent systems.

As the main results of this paper, we show that any EBW systems are equivalent to a class of binary coherent systems satisfying some conditions. Furthermore, we provide the interesting characterization that the class of EBW systems is the maximum class of the multi-state systems which can be transformed into the class of binary coherent systems whose number is identical to the number of states of the multi-state system. In more recent work, J.I.Ansell and A.Bendell [8] have established a hierarchical definition of multi-state systems. Concerning this work, we see that the class of EBW systems includes the multi-state systems with 'well-defined binary image' proposed in [8]. The results of this paper will be useful for the structural analysis of multi-state systems and the evaluation of system reliability. Moreover, since the class of EBW systems is the maximum class which can be transformed into binary systems, its concept may be applied to the study of the multiple-valued logic [9].

In Section 2 we present the definition of multi-state systems, BW systems and EBW systems, and several notations. Section 3 provides some properties of EBW systems. In Section 4 we discuss the transformation method of EBW systems into the binary coherent systems. In Section 5 we present a relation between EBW systems and binary systems satisfying some conditions.

2. Multi-state systems

We assume that a system is composed of n components, and the state spaces of the system and its components can be represented as $\{0,1,\ldots,m\}$. Let $C = \{1,2,\ldots,n\}$ be the set of the components and let Ω_i ($i \in C$) and S be finite totally ordered sets with m+1 elements. The state spaces of the system and the components are the same, i.e., $\Omega_i = S = \{0,1,\ldots,m\}$. Physically, the elements of the state space '0' and 'm' correspond to the completely failed state and the perfectly functioning state respectively, and any other elements, i.e., 1,2,... and m-1, mean intermediate states between the state 0 and m. For j and k of Ω_i (or S), $j<k$ implies that the state j is better than the state k. The combination of states of all the components is described by the state vector $\underline{x} = (x_1,x_2,\ldots,x_n)$, where x_i, the state of the i-th component of the system, is an element of Ω_i. It is assumed that the state of the system is determined by the states of all the components, so that the state of the system is described by a structure function $\phi(\underline{x})$ of the state vector $\underline{x}$ with range $S = \{0,1,\ldots,m\}$.

In this paper, we use the following notations, where $s\varepsilon S=\Omega_i$, $\underline{x}\varepsilon\Pi_{i\varepsilon C}\Omega_i$ and $\underline{y}\varepsilon\Pi_{i\varepsilon C}\Omega_i$.

(1) $\underline{s} = (s,s,\ldots,s)\varepsilon\Pi_{i\varepsilon C}\Omega_i$, $\underline{0} = (0,0,\ldots,0)\varepsilon\Pi_{i\varepsilon C}\Omega_i$ and $\underline{m} = (m,m,\ldots,m)\varepsilon\Pi_{i\varepsilon C}\Omega_i$.
(2) $\underline{x} > \underline{y}$ if and only if $x_i \geq y_i$ for every $i\varepsilon C$, and $x_i > y_i$ for some $i\varepsilon C$.
(3) $(s_i,\underline{x}) = (x_1,x_2,\ldots,x_{i-1},s,x_{i+1},\ldots,x_n)$.
(4) $\phi^{-1}(s\leq) = \{\underline{x};\phi(\underline{x})\geq s\}$. (5) $\phi^{-1}(s) = \{\underline{x};\phi(\underline{x})=s\}$.
(6) $M_{s\leq}$ is the set of all the minimal elements of $\phi^{-1}(s\leq)$.
(7) M_s is the set of all the minimal elements of $\phi^{-1}(s)$.
(8) $C_{s\leq}(\underline{x}) = \{i;x_i\geq s\}$. (9) $C_s(\underline{x}) = \{i;x_i=s\}$.
(10) $A_{s\leq} = \{C_s(\underline{x});\underline{x}\varepsilon M_{s\leq}\}$. (11) $A_s = \{C_s(\underline{x});\underline{x}\varepsilon M_s\}$.

In (6) and (7) if $\underline{x}$ belongs to $M_{s\leq}(M_s)$, we have $\phi(\underline{x}) \geq s$ ($\phi(\underline{x}) = s$), and $\underline{x} < \underline{y}$ implies $\phi(\underline{y}) < s$ for any $\underline{y}$.

We consider the definitions for several multi-state systems in the following. Suppose that all the components and the system have the same finite totally ordered set, i.e., $\{0,1,\ldots,m\}$, then a multi-state system may be defined as follows:

Definition 2.1. The triplet $(\Pi_{i\varepsilon C}\Omega_i,S,\phi)$ is called a multi-state system if and only if $\phi(\underline{x})$ is a non-decreasing surjection from $\Pi_{i\varepsilon C}\Omega_i$ to S.

From a physical point of view, the above multi-state system ϕ is considered as follows: To begin with, the non-decreasing property means that the state of a system does not deteriorate unless the state of a component comes down. Considering surjection of ϕ, if all the components are perfectly functioning, then the system will be so, too, i.e., $\underline{x} = \underline{m}$ implies $\phi(\underline{x}) = m$. Similarly, if all the components are completely failed, then the system will be so, too, i.e., $\underline{x} = \underline{0}$ implies $\phi(\underline{x}) = 0$. Furthermore, suppose that more than one components do not deteriorate simultaneously, and that the state of each component does not come down by more than one at a time, then the state of the system will take all the value from m to 0. Consequently, we may consider a multi-state system as a non-decreasing surjection.

Next we define a multi-state system which are proposed in [2].

Definition 2.2. A multi-state system $(\Pi_{i\in C}\Omega_i, S, \phi)$ is called a BW system if and only if

$\phi(\underline{x}) = \max_{1\leq j\leq p} \min_{i\in P_j} x_i$ holds for every $\underline{x}$ of $\Pi_{i\in C}\Omega_i$,

where $\{P_j\}_{j=1}^{p}$ is the set of all the minimal path sets of a binary coherent system (see [1] on the definitions of minimal path sets and coherent systems).

Finally, the multi-state system which we propose are defined as follows:

Definition 2.3. A multi-state system $(\Pi_{i\in C}\Omega_i, S, \phi)$ is called an EBW system if and only if $M_{s\leq} \subset \{0,s\}^n$ holds for every s of S.

Since we assume that a structure function ϕ is non-decreasing, giving a function ϕ (i.e., determining the correspondence relation of ϕ) will be identical with obtaining all the sets of the minimal elements $\{M_1, M_2, \ldots, M_m\}$. To illustrate the distinction among those multi-state systems, we provide simple examples of multi-state systems which are represented by the sets of minimal elements.

Example 2.1. Let $C = \{1,2,3\}$ and $S = \Omega_i = \{0,1,2\}$, $(i\in C)$.

(1) Multi-state system:

$M_2 = \{(2,0,0),(0,2,1),(1,1,2)\}$, $M_1 = \{(1,0,0),(0,1,1),(0,0,2)\}$, and $M_0 = \{(0,0,0)\}$.

(2) BW system:

$M_2 = \{(2,0,2),(0,2,2)\}$, $M_1 = \{(1,0,1),(0,1,1)\}$ and $M_0 = \{(0,0,0)\}$.

(3) EBW system:

$M_2 = \{(2,2,0),(2,0,2),(0,2,2)\}$, $M_1 = \{(1,0,0),(0,1,1)\}$ and $M_0 = \{(0,0,0)\}$.

3. Characterizations of EBW systems

In this section we investigate some basic properties of EBW systems.

Theorem 3.1. A BW system is an EBW system.

Proof: Let $(\Pi_{i\in C}\Omega_i, S, \phi)$ be a BW system and let $\{P_j\}_{j=1}^{p}$ be the set of all the minimal path sets. For an arbitrarily fixed s of S, we define $\underline{x}_k$ $(1\leq k\leq p)$ as follows: $C_s(\underline{x}_k) = P_k$, $C_0(\underline{x}_k) = C\backslash P_k$. Since $\phi(\underline{x}_k) = s$, it follows that $\underline{x}_k$ is a minimal element of $\phi^{-1}(s\leq)$ from Definition 2.2. Therefore for every s of S, we have $\{\underline{x}_1, \ldots, \underline{x}_p\} \subset M_{s\leq}$.

Suppose that for some s of S, there is a minimal element $\underline{y}$ of $\phi^{-1}(s\leqq)$ such that $\underline{y} \notin \{\underline{x}_1,\underline{x}_2,\ldots,\underline{x}_p\}$ holds. Since $\min_{i\in P_t} y_i \geqq s$ for some P_t and $\underline{y} \neq \underline{x}_t$, we have $\underline{y} > \underline{x}_t$. This contradicts the assumption that $\underline{y}$ is a minimal element of $\phi^{-1}(s\leqq)$. Then for every s of S, $M_{s\leqq} = \{\underline{x}_1,\underline{x}_2,\ldots,\underline{x}_p\} \subset \{0,s\}^n$ holds.

Q.E.D.

In the following, we show some results on EBW systems. Since the structure function ϕ of an EBW system is a surjection. It is immediate that for each $s(s\neq 0)$ of S, $\underline{x}\varepsilon M_{s\leqq}$ implies $\underline{x} \neq \underline{0}$.

Property 3.1. If $(\Pi_{i\varepsilon C}\Omega_i, S, \phi)$ is an EBW system, then $\phi(\underline{s}) = s$ for each s of S.

Proof: Let $\underline{x}$ be an element of $M_{s\leqq}$. Since $\underline{x} \leqq \underline{s}$ and ϕ is non-decreasing, it follows that $\phi(\underline{s}) \geqq \phi(\underline{x}) \geqq s$. If $\phi(\underline{s}) = t > s$, then there exists $\underline{y}$ of $M_{t\leqq}$ such that $\underline{y} \leqq \underline{s}$, which contradicts $\underline{y}\varepsilon\{0,t\}^n$ and $\underline{y} \neq \underline{0}$. Consequently, for each s of S, $\phi(\underline{s}) = s$ holds.

Q.E.D.

Theorem 3.2. If $(\Pi_{i\varepsilon C}\Omega_i, S, \phi)$ is an EBW system, then $M_{s\leqq} = M_s$ for each s of S.

Proof: (1) (Proof of $M_{s\leqq} \supset M_s$) Let $\underline{x}$ be an element of M_s and consider $\underline{y}$ of $\phi^{-1}(s\leqq)$ such that $\underline{y} \leqq \underline{x}$. From the non-decreasing property of ϕ,

$$s \leqq \phi(\underline{y}) \leqq \phi(\underline{x}) = s,$$

so that $\phi(\underline{y}) = s$, i.e., $\underline{y}\varepsilon\phi^{-1}(s)$. Then $\underline{x} = \underline{y}$ holds because $\underline{x}$ is a minimal element of $\phi^{-1}(s)$. Hence $\underline{x}$ is the minimal element of $\phi^{-1}(s\leqq)$.

(2) (Proof of $M_{s\leqq} \subset M_s$) Since $M_{s\leqq} \subset \{0,s\}^n$, it follows that $\underline{x} \leqq \underline{s}$ for each $\underline{x}$ of each $\underline{x}$ of $M_{s\leqq}$. By Property 3.1, $\phi(\underline{x}) = s$ holds for each $\underline{x}$ of $M_{s\leqq}$, so that there exists $\underline{y}\varepsilon M_s$ such that $\underline{y} \leqq \underline{x}$. Then $\underline{x} = \underline{y}$ holds because $\underline{y}$ belongs to $\phi^{-1}(s\leqq)$ and $\underline{x}$ is the minimal element of $\phi^{-1}(s\leqq)$. Hence $\underline{x}$ is a minimal element of $\phi^{-1}(s)$.

Q.E.D.

With regard to the notations $\mathcal{A}_{s\leqq}$ and $\mathcal{A}_s$, the following corollary holds from Theorem 3.2.

Corollary 3.1. For each s of S, $\mathcal{A}_{s\leqq} = \mathcal{A}_s$ holds.

Theorem 3.3. Let $(\Pi_{i\varepsilon C}\Omega_i, S, \phi)$ be an EBW system. For each s and t ($s<t$, $s,t\varepsilon S$), and for each $\underline{x}$ of $M_{t\leqq}$, there exists $\underline{y}$ of $M_{s\leqq}$ such that $\underline{x} > \underline{y}$.

Proof: From $M_{t\leqq} \subset \phi^{-1}(t\leqq) \subset \phi^{-1}(s\leqq)$, for each $\underline{x}\varepsilon M_{t\leqq}$, there exists $\underline{y}\varepsilon M_{s\leqq}$ such that $\underline{x} \geqq \underline{y}$. Since from Theorem 3.2 $\phi(\underline{x}) = t$ for each $\underline{x}$ of $M_{t\leqq}$ and $\phi(\underline{y}) = s$ for each $\underline{y}$ of $M_{s\leqq}$, then we have $\underline{x} \neq \underline{y}$.

Q.E.D.

Corollary 3.2. For each s and t ($s<t$, $s,t\varepsilon S$) and for each A of $\mathcal{A}_t$, there exists B of $\mathcal{A}_s$ such that $A \supset B$.

Proof: From Theorem 3.3, this corollary is easily proved by using the relation in Corollary 3.1, i.e.,

$A_s = \{C_s(\underline{x});\underline{x}\in M_s\} = \{C_s(\underline{x});\underline{x}\in M_{s\leq}\} = A_{s\leq}$. Q.E.D.

Theorem 3.3 tells us a relationship between the minimal elements of $\phi^{-1}(s\leq)$ and those of $\phi^{-1}(s+1\leq)$ for $s=0,1,\ldots,m-1$. We define functions f_s from $\Pi_{i\in C}\Omega_i$ to S for every s of S as follows:

$f_s(\underline{x}) = \max_{A\in A_s} \min_{i\in A} x_i$ for every $\underline{x}$ of $\Pi_{i\in C}\Omega_i$.

In Definition 2.2, the structure function of a BW system is formulated by a given family of sets $\{P_j\}_{j=1}^{p}$ having no connection with any states. On the other hand, the structure function of an EBW system is defined by giving families of sets $A_1, A_2, \ldots, A_m$. Therefore, the above function f_s is identical to the structure function of a BW system if $\{P_j\}_{j=1}^{p}$ is regarded as A_s, in other words, if for every s, $A_s = \{P_j\}_{j=1}^{p}$ holds. The following Theorems hold for the functions ϕ and f_s.

Theorem 3.4. Let $(\Pi_{i\in}\Omega_i, S, \phi)$ be an EBW system. For each $\underline{x}$ of $\Pi_{i\in C}\Omega_i$, $\phi(\underline{x}) \geq s$ and $f_s(\underline{x}) \geq s$ are equivalent.

Proof: $\phi(\underline{x}) \geq s$ implies that there exists $\underline{y}$ of $M_{s\leq}$ such that $\underline{x} \geq \underline{y}$. Since $\underline{y}\in M_{s\leq} \subset \{0,s\}^n$, it follows that

$\min_{i\in C_s(\underline{y})} x_i \geq \min_{i\in C_s(\underline{y})} y_i = s$,

and that $C_s(\underline{y})\in A_s$, so that $f_s(\underline{x}) = \max_{A\in A_s} \min_{i\in A} x_i \geq s$ holds.

On the other hand, $f_s(\underline{x}) \geq s$ implies that there exists A of A_s such that $\min_{i\in A} x_i \geq s$ holds. Here we fix $\underline{x}$ and A satisfying the above relation. From the definiton of A_s, it follows that there exists $\underline{y}$ of $M_{s\leq}$ such that $A = C_s(\underline{y})$. Since $\underline{y}\in\{0,s\}^n$, we have $\underline{x} \geq \underline{y}$. Therefore $\phi(\underline{x}) \geq s$ holds because of $\phi(\underline{y}) = s$ and the non-decreasing property of ϕ. Q.E.D.

Theorem 3.5. Let $(\Pi_{i\in C}\Omega_i, S, \phi)$ be an EBW system. For every $\underline{x}$ of $\Pi_{i\in C}\Omega_i$, the following relations holds.

(i) $\phi(\underline{x}) = s$ if and only if $f_r(\underline{x}) \geq r$ for every $r(\leq s)$, and
$f_t(\underline{x}) < t$ for every $t(>s)$.

(ii) $\phi(\underline{x}) = \max_{s\in S} \{s; f_s(\underline{x})\geq s\}$.

Proof: Since the necessity ('only if' part) of (i) is evident from Theorem 3.4, we show the sufficiency('if' part) of (i) as follows: If $f_s(\underline{x}) \geq s$, then $\phi(\underline{x}) \geq s$ holds from Theorem 3.4. Suppose that $\phi(\underline{x}) \geq t > s$, then $f_t(\underline{x}) \geq t$, which contradicts the condition. Thus we have $\phi(\underline{x}) = s$.

The proof of (ii) is easily shown from (i). Q.E.D.

In the above discussion, we have considered multi-state systems apart from the concept of coherency which presents whether each state of all the components contirbutes to the state of the system or not. In the following of this section we discuss the relations between an EBW system and the coherent property of multi-state systems proposed by F.Ohi and T.Nishida [6].

Definition 3.1. A multi-state system is said to be coherent if each i of C and for each s and t of Ω_i $(s\neq t)$, there exists $\underline{x}$ such that $\phi(s_i,\underline{x})\neq\phi(t_i,\underline{x})$.

The coherent property of this definition is identical with 'FHN-regularity' in F.Ohi and T.Nishida [6]. On a coherent EBW system, the following theorem holds.

Theorem 3.6. Let ϕ be a structure function of EBW systems. Then ϕ is coherent if and only if $\cup_{A\in A_s} A = C$ holds for every s of S.

Proof: (1) (Necessity) Since $A_0 = \{C\}$, the condition clearly holds for $s=0$. Suppose that for some s $(\neq 0)$ of S and for some i of C,

$$i \notin \cup_{A\in A_s} A.$$

Let s' be a predecessor element of s, i.e., $s'< s$ and for any r of S, $r < s$ implies $r \leq s'$. If $\phi(s_i,\underline{x}) = s$, we have $\phi(s_i,\underline{x}) = \phi(s'_i,\underline{x})$ since $i\notin\cup_{A\in A_s} A$ and $M_s \subset \{0,s\}^n$. If $\phi(s_i,\underline{x}) = u$ $(s\neq u)$, there exists $\underline{y}$ of M_u such that $(s_i,\underline{x}) \geq \underline{y}$. Since $s\geq y_i$ and $\underline{y}\in\{0,u\}^n$, $y_i<s$ holds. so that we have $(s_i\ \underline{x}) > (s'_i\ \underline{x}) \geq \underline{y}$ from $s>s'\geq y_i$. This implies that

$$u = \phi(s_i,\underline{x}) \geq \phi(s'_i,\underline{x}) \geq \phi(\underline{y}) = u.$$

Thus we have $\phi(s_i,\underline{x}) = \phi(s'_i,\underline{x})$. Therefore for any $\underline{x}$ $\phi(s_i,\underline{x}) = \phi(s'_i,\underline{x})$, which contradicts the condition of coherency.

(2) (Sufficiency) From the condition, for every i of C and every t of Ω_i, there exists $\underline{x}$ of M_t such that $i\in C_t(\underline{x})$ holds. if $t=0$, we have $M_0=\{\underline{0}\}$ from the non-decreasing property of ϕ. Now, we consider two states s and t such that $s<t(\neq 0)$, since in Definition 3.1 the coherent property is treated as a relation of arbitrary two states of each component. For an arbitrarily fixed $\underline{x}$ of $M_t(t\neq 0)$ and every $s(<t)$ of Ω_i, $(s_i,\underline{x}) < (t_i,\underline{x}) = \underline{x}$ holds. Then $\phi(s_i,\underline{x}) < \phi(t_i,\underline{x})$ since ϕ is a non-decreasing function and $\underline{x}$ is a minimal element of $\phi^{-1}(t)$. Q.E.D.

4. Transformation of EBW systems

This section provides a transformation method of the EBW system into binary systems and some relations between those systems. In the following, we assume that $(\Pi_{i\in C}\Omega_i,S,\phi)$ is an EBW system. Let $\tilde{\Omega}_i = \Omega_i\backslash\{0\}$, $\tilde{S} = S\backslash\{0\}$ and

$\Omega = \{0,1\}$, where Ω is the totally ordered set having two elements 0 and 1 such that $0<1$ holds.

Definition 4.1. We define functions ξ_i, ξ, η_s and ϕ_s as follows, where a notation B^A is the set of all the mapping from a set A to a set B.

(1) ξ_i ($i\varepsilon C$) is the mapping from Ω_i to $\Omega^{\tilde{\Omega}_i}$ such that for every s of Ω_i:

$(\xi_i(s))_r = 1$ if $r\leq s$, and $(\xi_i(s))_t = 0$ if $t>s$,

where $(\cdot)_k$ is the k-th element of $(\cdot)$.

(2) ξ is the mapping from S to $\Omega^{\tilde{S}}$ such that for every s of S:

$(\xi(s))_r = 1$ if $r\leq s$, and $(\xi(s))_t = 0$ if $t>s$.

(3) η_s ($s\varepsilon\tilde{S}$) is the mapping from $\Pi_{i\varepsilon C}\Omega^{\tilde{\Omega}_i}$ to Ω^C such that for every $(f_1,f_2,\ldots,f_n)$ of $\Pi_{i\varepsilon C}\Omega^{\tilde{\Omega}_i}$:

$\eta_s(f_1,f_2,\ldots,f_n) = (f_1(s),f_2(s),\ldots,f_n(s))$.

(4) $\phi_s(s\varepsilon\tilde{S})$ is the mapping from Ω^C to Ω such that for every $\underline{y}$ of Ω^C:

$\phi_s(\underline{y}) = (\xi\circ\phi(\underline{x}))_s$ using $\underline{x}\varepsilon\underline{\xi}^{-1}\circ\eta_s{}^{-1}(\underline{y})$,

where $\underline{\xi} = (\xi_1,\xi_2,\ldots,\xi_n)$ is the mapping from $\Pi_{i\varepsilon C}\Omega_i$ to $\Pi_{i\varepsilon C}\Omega^{\tilde{\Omega}_i}$.

Thus we write $(\xi_1(x_1),\xi_2(x_2),\ldots,\xi_n(x_n))$ for the sake of brevity as follows: for every $\underline{x}$ of $\Pi_{i\varepsilon C}\Omega_i$: $\underline{\xi}(\underline{x}) = (\xi_1,\xi_2,\ldots,\xi_n)(\underline{x})$

$= (\xi_1(x_1),\xi_2(x_2),\ldots,\xi_n(x_n))$.

Example 4.1. We give a simple example to demonstrate the functions ξ_i, ξ, η_s and ϕ_s. Consider an EBW system composed of three components with four states, i.e., $C = \{1,2,3\}$ and $\Omega_i(i\varepsilon C) = S = \{0,1,2,3\}$. We assume that at a given moment $\phi(\underline{x}) = \phi(3,2,1) = 2$ holds. Then $\phi(3,2,1) = 2$ is transformed by using the functions in Definition 4.1 as follows:

$\xi_1(x_1) = \xi_1(3) = ((\xi_1(3))_1,(\xi_1(3))_2,(\xi_1(3))_3) = (1,1,1)$, similarly,

$\xi_2(x_2) = \xi_2(2) = (1,1,0)$ and $\xi_3(x_3) = \xi_3(1) = (1,0,0)$.

$\underline{\xi}(\underline{x}) = (\xi_1(x_1),\xi_2(x_2),\xi_3(x_3)) = ((1,1,1),(1,1,0),(1,0,0))$.

$\xi\circ\phi(\underline{x}) = \xi(2) = ((\xi(2))_1,(\xi(2))_2,(\xi(2))_3) = (1,1,0)$.

$\eta_3\circ\underline{\xi}(\underline{x}) = ((\xi_1(3))_3,(\xi_2(2))_3,(\xi_3(1))_3) = (1,0,0)$, similarly,

$\eta_2\circ\underline{\xi}(\underline{x}) = (1,1,0)$ and $\eta_1\circ\underline{\xi}(\underline{x}) = (1,1,1)$.

$\phi_3\circ\eta_3\circ\underline{\xi}(\underline{x}) = \phi_3(1,0,0) = (\xi\circ\phi(\underline{x}))_3 = 0$,

$\phi_2\circ\eta_2\circ\underline{\xi}(\underline{x}) = \phi_2(1,1,0) = (\xi\circ\phi(\underline{x}))_2 = 1$ and

$\phi_1\circ\eta_1\circ\underline{\xi}(\underline{x}) = \phi_1(1,1,1) = (\xi\circ\phi(\underline{x}))_1 = 1$.

In this section, we investigate the property of functions ξ_i, ξ, η_s and ϕ_s ($s\varepsilon S$) for an EBW system $(\Pi_{i\varepsilon C}\Omega_i,S,\phi)$.

Theorem 4.1. The mapping ϕ_s in Definition 4.1.(4) is well-defined.

Proof: In order to prove that ϕ_s is well-defined, it is sufficient to show that the following two conditions hold for every s of $\tilde{S}$.

(i) For every $\underline{y}$ of Ω^C, $\underline{\xi}^{-1}\circ\eta_s^{-1}(\underline{y}) \neq \Phi$ holds, where Φ is the empty set.

(ii) $\eta_s\circ\underline{\xi}(\underline{z}) = \eta_s\circ\underline{\xi}(\underline{w})$ implies $(\xi\circ\phi(\underline{z}))_s = (\xi\circ\phi(\underline{w}))_s$.

(i) For any $\underline{y}$ of Ω^C we determine $\underline{x}$ of $\Pi_{i\varepsilon C}\Omega_i$ as follows:

$$x_i = s \text{ if } y_i = 1, \quad\text{and}\quad x_i = 0 \text{ if } y_i = 0.$$

Then $\eta_s\circ\underline{\xi}(\underline{x}) = ((\xi_1(x_1))_s, (\xi_2(x_2))_s, \ldots, (\xi_n(x_n))_s) = \underline{y}$ using Definition 4.1 (1), i.e.,

$$(\xi_i(x_i))_s = 1 \text{ if } x_i = s, \text{ and } (\xi_i(x_i))_s = 0 \text{ if } x_i = 0.$$

(ii) Let $\eta_s\circ\underline{\xi}(\underline{z}) = \eta_s\circ\underline{\xi}(\underline{w}) = \underline{y}$. From definitions of η_s and $\underline{\xi}$,

$$y_i = 1 \text{ implies } z_i \geq s \text{ and } w_i \geq s,$$

$$y_i = 0 \text{ implies } z_i < s \text{ and } w_i < s.$$

If $(\xi\circ\phi(\underline{z}))_s = 1$, then there exists $\underline{x}$ of $M_{s\leqq}$ such that $\underline{z} \geq \underline{x}$ since $\phi(\underline{z}) \geq s$ holds. Owing to $M_{s\leqq} \subset \{0,s\}^n$ and the relations between z_i and w_i shown above, $\underline{z} \geq \underline{x}$ implies further $\underline{w} \geq \underline{x}$, $\phi(\underline{w}) \geq s$ and $(\xi\circ\phi(\underline{w}))_s = 1$. By the same argument, it is easily obtained that $(\xi\circ\phi(\underline{z}))_s = 0$ if and only if $(\xi\circ\phi(\underline{w}))_s = 0$.

Q.E.D.

We assume that the ordinary product ordered relations are given on $\Omega^{\tilde{\Omega}_i}$ and $\Omega^{\tilde{S}}$. Hence for f and g of $\Omega^{\tilde{S}}$, $f \leq g$ if and only if $f(s) \leq g(s)$ holds for any s of $\tilde{S}$. Since Ω_i and S are totally ordered sets, ξ_i and ξ are non-decreasing functions from their definitions. Moreover, the composite function $\xi\circ\phi$ is non-decreasing by the properties of ϕ. Thus $\underline{w} \leq \underline{z}$ implies

$$(\xi\circ\phi(\underline{w}))_s \leq (\xi\circ\phi(\underline{z}))_s.$$

Next we define binary monotonic systems which are particular multi-state systems whose largest element m is 1, and consider the relationship between EBW systems and the corresponding binary systems constructured by transformations in Definition 4.1.

Definition 4.2. The triplet $(\Pi_{i\varepsilon C}Q_i, Q, \psi)$ is a binary monotonic system if and only if ψ is a non-decreasing surjection from $\Pi_{i\varepsilon C}Q_i$ to Q, where $Q_i = Q = \{0,1\}$.

Theorem 4.2. For each s of $\tilde{S}$, $(\Omega^C, \Omega, \phi_s)$ is a binary monotonic system.

Proof: We prove that the function ϕ_s is a non-decreasing surjection.

(1) (Proof of surjection): From definitions of η_s and $\underline{\xi}$, $\eta_s\circ\underline{\xi}(\underline{s}) = \underline{1}$. Since $(\xi\circ\phi(\underline{s}))_s = (\xi(s))_s = 1$, we have $\phi_s(\underline{1}) = 1$. Similarly, since $\eta_s\circ\underline{\xi}(\underline{0})=0$ $(\xi\circ\phi(\underline{0}))_s = (\xi(0))_s = 0$, we have $\phi_s(\underline{0}) = 0$.

(2) (Proof of non-decreasing property):

We assume that $\underline{u} \leq \underline{v}$ holds for $\underline{u}$ and $\underline{v}$ of Ω^C, and determine $\underline{w}$ and $\underline{z}$ of $\Pi_{i\varepsilon C}\Omega_i$ for $\underline{u}$ and $\underline{v}$ respectively as follows:

$w_i = s$ if $u_i = 1$, $w_i = 0$ if $u_i = 0$,
and $z_i = s$ if $v_i = 1$, $z_i = 0$ if $v_i = 0$.

Then we have $\eta_s \circ \underline{\xi}(\underline{w}) = \underline{u}$ and $\eta_s \circ \underline{\xi}(\underline{z}) = \underline{v}$. Since $\underline{u} \leq \underline{v}$, $\underline{w} \leq \underline{z}$ holds. From the non-decreasing property of the composite function $\xi \circ \phi$, we have

$$\phi_s(\underline{u}) = (\xi \circ \phi(\underline{w}))_s \leq (\xi \circ \phi(\underline{z}))_s = \phi_s(\underline{v}). \qquad \text{Q.E.D.}$$

Theorem 4.3. For each s of $\tilde{S}$, $\phi_s(\underline{y}) = \max_{A \in A_s} \min_{i \in A} y_i$, where $\underline{y} \in \Omega^C$.

Proof: In order to prove that $\phi_s(\underline{y}) = 1$ if and only if $\max_{A \in A_s} \min_{i \in A} y_i = 1$ for each s of $\tilde{S}$ and any $\underline{y}$ of Ω^C, we define a function ζ_s as

$$\zeta_s(\underline{y}) = \max_{A \in A_s} \min_{i \in A} y_i \text{ for any } \underline{y} \text{ of } \Omega^C.$$

Then we show that $\zeta_s(\underline{y}) = 1$ if and only if $\phi_s(\underline{y}) = 1$ for any $\underline{y}$ of Ω^C.

(1) (Necessity): $\zeta_s(\underline{y}) = 1$ implies that there exists an element A of A_s and for all i of A, $y_i = 1$ holds. For this set A, $\underline{x}$ is given by

$$x_i = s \text{ if } i \in A, \; x_i = 0 \text{ if } i \notin A.$$

Since $\eta_s \circ \underline{\xi}(\underline{x}) = \underline{y}$ and $\phi(\underline{x}) = s$ hold, we have

$$\phi_s(\underline{y}) = (\xi \circ \phi(\underline{x}))_s = (\xi(s))_s = 1.$$

(2) (Sufficiency): If $\phi_s(\underline{y}) = 1$, then we have $(\xi \circ \phi(\underline{x}))_s = 1$ using $\underline{x}$ such that $\eta_s \circ \underline{\xi}(\underline{x}) = \underline{y}$ holds. $(\xi \circ \phi(\underline{x}))_s = 1$ implies that there exists A of A_s and for all i of A, $x_i \geq s$ holds since $\phi(\underline{x}) \geq s$. From $\eta_s \circ \underline{\xi}(\underline{x}) = \underline{y}$, $y_i = (\xi_i(x_i))_s = 1$ holds for all i of A. Therefore we have $\zeta_s(\underline{y}) = 1$. Q.E.D.

Theorems 4.2 and 4.3 indicate that the EBW system can be transformed into the class composed of m binary monotonic systems using functions ξ_i, ξ, η_s and ϕ_s ($i \in C$, $s \in S$) in Definition 4.1. In other words, we may regard an EBW system as a class composed of some binary monotonic systems. Specifically, we illustrate the transformation method of an EBW system using Example 2.1 (3) in section 2. In this example, we have

$$A_2 = \{(1,2),(1,3),(2,3)\} \text{ and } A_1 = \{(1),(2,3)\}$$

from

$$M_2 = \{(2,2,0),(2,0,2),(0,2,2)\} \text{ and } M_1 = \{(1,0,0),(0,1,1)\}.$$

By A_2 and A_1, two binary monotonic systems ϕ_2 and ϕ_1 are constructed as follows: $\phi_2(\underline{y}) = \max_{A \in A_2} \min_{i \in A} y_i$ for any $\underline{y}$ of Ω^C,

$$\phi_1(\underline{z}) = \max_{A \in A_1} \min_{i \in A} z_i \text{ for any } \underline{z} \text{ of } \Omega^C.$$

Furthermore, we have

$$\underline{y} = (1,0,0) \Rightarrow \phi_2(1,0,0) = 0 \quad \text{and} \quad \underline{z} = (1,1,0) \Rightarrow \phi_1(1,1,0) = 1$$

from the above construction of ϕ_2 and ϕ_1.

Considering the coherent property of Definition 3.1, the following Corollary holds.

Corollary 4.1. If an EBW system satisfies the coherent condition, binary monotonic systems transformed by ξ_i, ξ, η_s and ϕ_s are coherent for each s of $\tilde{S}$.

Since the proof is easy, it is omitted.

5. Relationship between EBW systems and binary systems

In this section we consider a relationship between EBW systems and the classes of binary monotonic systems satisfying the following condition.

Condition 5.1. (1) (Ω^C,Ω,ψ_s), $s=1,2,\ldots,m$, is a binary monotonic system.
(2) For each s and t of $\tilde{S}$, $s \leq t$ implies $\psi_t \leq \psi_s$, i.e.,

$\psi_t(\underline{y}) \leq \psi_s(\underline{y})$ holds for any $\underline{y}$ of Ω^C.

If we denote the set of all the non-decreasing functions from $\tilde{\Omega}_i$ to Ω by $\Omega_{\geq}^{\tilde{\Omega}_i}$, function ξ_i is the order isomorphism from Ω_i to $\Omega_{\geq}^{\tilde{\Omega}_i}$. Then $\underline{\xi}$ is the order isomorphism from $\Pi_{i\varepsilon C}\Omega_i$ to $\Pi_{i\varepsilon C}\Omega_{\geq}^{\tilde{\Omega}_i}$, and ξ is so from S to $\Omega_{\geq}^{\tilde{S}}$. Now we define a function ψ as follows, where

$$\xi^{-1}((\psi_s\circ\eta_s\circ\underline{\xi}(\underline{x}))_{s\varepsilon\tilde{S}}) = \xi^{-1}(\psi_1\circ\eta_1\circ\underline{\xi}(\underline{x}),\ldots,\psi_m\circ\eta_m\circ\underline{\xi}(\underline{x})).$$

Definition 5.1. ψ is the mapping from $\Pi_{i\varepsilon C}\Omega_i$ to S such that $\psi(\underline{x}) = \xi^{-1}((\psi_s\circ\eta_s\circ\underline{\xi}(\underline{x}))_{s\varepsilon\tilde{S}})$ holds for any $\underline{x}$ of $\Pi_{i\varepsilon C}\Omega_i$, where the function ψ_s satisfies Condition 5.1 and functions η_s, $\underline{\xi}$ and ξ are defined in Definition 4.1.

Theorem 5.1. The triplet $(\Pi_{i\varepsilon C}\Omega_i,S,\psi)$ is an EBW system.

Proof: We prove that function ψ is a non-decreasing surjection, and that the condition of Definition 2.3 is satisfied.

(1) (Non-decreasing): $\underline{\xi}$ and η_s are non-decreasing functions and ψ_s, $s=1,2,\ldots,m$, is also a non-decreasing function from Condition 5.1. Since ξ is the order isomorphism, ξ^{-1} becomes non-decreasing. Thus ψ is non-decreasing from Definition 5.1.

(2) (Surjection): Let B_s be $\{C_1(\underline{x});\underline{x}\varepsilon M_1(\psi_s)\}$, $(s\varepsilon S)$. For an element B of B_s define $\underline{x}$ of $\Pi_{i\varepsilon C}\Omega_i$ by $x_i = s$ if $i\varepsilon B$, $x_i = 0$ if $i\notin B$. Then we have $\underline{\xi}_s\circ\eta_s\circ\underline{\xi}(\underline{x}) = 1$. Since $\underline{\xi}(\underline{x}) = (\xi_1(x_1),\xi_2(x_2),\ldots,\xi_n(x_n))$, we determine $\underline{\xi}(\underline{x})$ as follows:

For $r \leq s$, $(\xi_i(x_i))_r = 1$ if $i\varepsilon B$, $(\xi_i(x_i))_r = 0$ if $i\notin B$,
and for $t > s$ and for all i of C, $(\xi_i(x_i))_t = 0$.

Then since ψ_s satisfies Condition 5.1, $\psi_s\circ\eta_s\circ\underline{\xi}(\underline{x})$, $s=1,\ldots,m$, is represented by $\psi_r\circ\eta_r\circ\underline{\xi}(\underline{x}) = 1$ if $r \leq s$, and $\psi_t\circ\eta_t\circ\underline{\xi}(\underline{x}) = 0$ if $t > s$.

Therefore we have $\psi(\underline{x}) = \xi^{-1}((\psi_k\circ\eta_k\circ\underline{\xi}(\underline{x}))_{k\varepsilon\tilde{S}}) = s$.

(3) Finally, we prove that all the minimal elements of $\psi^{-1}(s\leq)$ belong to $\{0,s\}^n$. From the definition of ξ and Condition 5.1,

$$\psi(\underline{y}) = \xi^{-1}((\psi_k \circ \eta_k \circ \underline{\xi}(\underline{y}))_{k \in \tilde{S}}) \geq s \text{ if and only if } \psi_s \circ \eta_s \circ \underline{\xi}(\underline{y}) = 1.$$

Then for some B of B_s and for each element i of B, $(\xi_i(y_i))_s = 1$ and equivalently $y_i \geq s$. Using this set B, define $\underline{x}$ of $\Pi_{i \in C}\Omega_i$ by

$$x_i = s \text{ if } i \in B, \; x_i = 0 \text{ if } i \notin B.$$

Then from the proof of above (2), we have $\psi(\underline{x}) = s$. Since clearly $\underline{x} \leq \underline{y}$ holds, it follows that $\underline{x}$ is a minimal element of $\psi^{-1}(s\leq)$. Thus all the minimal elements of $\psi^{-1}(s\leq)$ belong to $\{0,s\}^n$. Q.E.D.

Definition 5.2. The function ψ_s is called 's-monotonic' if and only if ψ_s satisfies Condition 5.1 for each s of $\tilde{S}$.

Definition 5.3. The function ψ_s is called 'well-defined' if and only if for each s of $\tilde{S}$ and for any $\underline{z}$ and $\underline{w}$ of $\Pi_{i \in C}\Omega_i$,

$$\eta_s \circ \underline{\xi}(\underline{z}) = \eta_s \circ \underline{\xi}(\underline{w}) \text{ implies } (\xi \circ \psi(\underline{z}))_s = (\xi \circ \psi(\underline{w}))_s,$$

where $\psi_s(\eta_s \circ \underline{\xi}(\underline{x})) = (\xi \circ \psi(\underline{x}))_s$ for any $\underline{x}$ of $\Pi_{i \in C}\Omega_i$, and ψ is a multi-state system.

Theorem 5.2. The function ψ_s is well-defined for each s of $\tilde{S}$ if and only if ψ_s is s-monotonic.

Proof: Suppose that ψ_s is not s-monotonic, i.e., for some s and $t(s<t)$ of $\tilde{S}$, there exists $\underline{y}_0$ of Ω^C such that $\psi_s(\underline{y}_0) < \psi_t(\underline{y}_0)$. Since for every $\underline{y}$ of Ω^C and for each s of $\tilde{S}$, the value of $\psi_s(\underline{y})$ is either 0 or 1, we have

$$\psi_s(\underline{y}_0) = 0 < \psi_t(\underline{y}_0) = 1.$$

Furthermore, we define $\underline{z}$ and $\underline{w}$ of $\Pi_{i \in C}\Omega_i$ for $\underline{y}_0$ as follows:

$$z_i = s, \; w_i = t \text{ if } (\underline{y}_0)_i = 1 \quad \text{and} \quad z_i = 0, \; w_i = 0 \text{ if } (\underline{y}_0)_i = 0.$$

Since $\underline{y}_0 = \eta_s \circ \underline{\xi}(\underline{z}) = \eta_t \circ \underline{\xi}(\underline{w})$, we have $\eta_s \circ \underline{\xi}(\underline{z}) = \eta_s \circ \underline{\xi}(\underline{w})$.
However, $(\xi \circ \psi(\underline{z}))_s = 0$ and $(\xi \circ \psi(\underline{w}))_s = 1$ hold, which contradicts that ψ_s is well-defined. Therefore, if the function ψ_s is well-defined for each s of $\tilde{S}$, then ψ_s is s-monotonic.

On the other hand, it follows from Theorem 5.1 that the triplet $(\Pi_{i \in C}\Omega_i, S, \psi)$, which generated by ψ_s $(s \in \tilde{S})$ using functions η_s, $\underline{\xi}$ and ξ, is an EBW system if ψ_s is s-monotonic. Further, by Theorem 4.1 the function ψ_s is well-defined for each s of $\tilde{S}$ if ψ is an EBW system, hence the proof is complete.

Q.E.D.

Corollary 5.1. If a multi-state system $(\Pi_{i \in C}\Omega_i, S, \psi)$ does not belong to the class of EBW systems, then the function ψ_s is not s-monotonic.

It is obvious by Theorem 5.2.

As for the coherent property, the following two Corollaries hold.

Corollary 5.2. If ψ_s is coherent for each s of $\tilde{S}$, then ψ is coherent.

Corollary 5.3. A multi-state system $(\Pi_{i\epsilon C}\Omega_i, S, \psi)$ satisfying the following two conditions each s of $\tilde{S}$ is a coherent EBW system.

(i) A binary monotonic system $(\Omega^C, \Omega, \psi_s)$ is coherent.

(ii) ψ_s is well-defined.

Since the proofs of Corellaries 5.2 and 5.3 are easy, they are omitted.

As shown in Section 4, an EBW system $(\Pi_{i\epsilon C}\Omega_i, S, \phi)$ generates a class $\{\phi_1, \phi_2, \ldots, \phi_m\}$ of binary monotonic systems using functions η_s $(s\epsilon\tilde{S})$, $\underline{\xi}$ and ξ. On the other hand, from Theorems 5.1 and 5.2, if the structure function ψ_s is well-defined for each s of $\tilde{S}$, then ψ_s becomes s-monotonic, so that the multi-state system $(\Pi_{i\epsilon C}\Omega_i, S, \psi)$ obtained by the class of those binary monotonic systems $(\Omega^C, \Omega, \psi_s)$, $s=1,2,\ldots,m$, is always an EBW system. Conversely, if a multi-state system does not satisfy the condition of EBW systems, the class of binary monotonic systems generated by the system is not s-monotonic. Hence such a multi-state system can not define the class of those binary monotonic systems. Thus EBW systems are the maximum class of the multi-state systems which can be transformed into the class of binary monotonic systems whose number is identical to the number of states of multi-state system. Moreover, it is immediate that the class of EBW systems includes multi-state systems with 'well-defined binary image' proposed by [8].

The following Theorems show the relationship of the two distinct EBW systems ψ and ϕ composed of the classes of binary systems $(\Omega^C, \Omega, \psi_s)$ and $(\Omega^C, \Omega, \phi_s)$, $s=1,2,\ldots,m$, respectively.

Theorem 5.3. Let $(\Pi_{i\epsilon C}\Omega_i, S, \psi)$ and $(\Pi_{i\epsilon C}\Omega_i, S, \phi)$ be EBW systems and let ψ_s and ϕ_s be binary monotonic systems for every s of $\tilde{S}$. We assume that the two EBW systems ψ and ϕ are composed of the classes of binary monotonic systems ψ_s and ϕ_s $(s\epsilon\tilde{S})$, respectively. If $\psi_s = \phi_s$ for every s of $\tilde{S}$, then $\psi = \phi$.

Proof: Suppose that $\psi \neq \phi$, i.e., there exists $\underline{x}$ of $\Pi_{i\epsilon C}\Omega_i$ such that $\psi(\underline{x}) > \phi(\underline{x})$ holds. Then there exists an element k of $\tilde{S}$ satisfying both $\psi(\underline{x}) \geq k$ and $\phi(\underline{x}) < k$. Since for this fixed $\underline{x}$,

$$\psi(\underline{x}) = \xi^{-1}((\psi_s \circ \eta_s \circ \underline{\xi}(\underline{x}))_{s\epsilon\tilde{S}}) \geq k,$$

we have $\psi_k \circ \eta_k \circ \underline{\xi}(\underline{x}) = 1$ by Definition 4.1. On the other hand, since $\phi(\underline{x}) < k$, we have $(\xi \circ \phi(\underline{x}))_k = 0$. Then $\phi_k(\eta_k \circ \underline{\xi}(\underline{x})) = (\xi \circ \phi(\underline{x}))_k = 0$. Therefore we have $\psi_k \neq \phi_k$, which contradicts the assumption that $\psi_s = \phi_s$ for every s. Also the case such that $\psi(\underline{x}) < \phi(\underline{x})$ holds can be similarly negated. Q.E.D.

Theorem 5.4. Under the same assumption of Theorem 5.3, if $\psi_s \neq \phi_s$ for some s of $\tilde{S}$, then $\psi \neq \phi$.

Proof: Since $\psi_s \neq \phi_s$ holds for some s of $\tilde{S}$, there exist $\underline{y}$ of Ω^C and s of $\tilde{S}$ such that $\psi_s(\underline{y}) = 1$ and $\phi_s(\underline{y}) = 0$. For this fixed $\underline{y}$, $\underline{x}$ of $\Pi_{i\epsilon C}\Omega_i$ is given by $x_i = s$ if $i \varepsilon C_1(\underline{y})$, $x_i = 0$ if $i \not\varepsilon C_1(\underline{y})$, noting that $C_1(\underline{y}) = \{i; y_i = 1\}$. Since $\psi_s \circ \eta_s \circ \underline{\xi}(\underline{x}) = \psi_s(\underline{y}) = 1$, we have $\psi(\underline{x}) \geqq s$ by Definition 5.1. From $\phi_s(\underline{y}) = 0$, we have $(\xi \circ \phi(\underline{x}))_s = 0$. Thus there exists $\underline{x}$ of $\Pi_{i \varepsilon C} \Omega_i$ such that $\psi(\underline{x}) > \phi(\underline{x})$ holds. Therefore we have $\psi \neq \phi$.

Similarly, if $\psi_s(\underline{y}) = 0$ and $\phi_s(\underline{y}) = 1$, then we have $\psi \neq \phi$ since there exists $\underline{x}$ such that $\psi(\underline{x}) < \phi(\underline{x})$. Q.E.D.

6. Conclusion

We have propesed an EBW system as an extension of Barlow-Wu (BW) systems. The extension method has been accomplished by relaxing the strict condition that the form of minimal path sets of BW systems should be identical for any states. We have discussed some properties of EBW systems and relationship between EBW systems and the class of binary systems. As the main results, we have shown that an EBW system is equivalent to a class of binary systems satisfying some conditions. Furthermore, we clarified that the class of EBW systems is the maximum class of multi-state systems which can be transformed into binary systems whose number is identical to the number of states of the multi-state systems.

Concerning EBW systems, there remain some problems such as modules, analysis of system reliabilities and applications of the concepts of EBW systems to multiple-valued logic systems and so on.

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