

## BOND PORTFOLIO OPTIMIZATION BY BILINEAR FRACTIONAL PROGRAMMING

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**Abstract** A variety of bond portfolio optimization problems of institutional investors are formulated as linear and/or bilinear fractional programming problems and algorithms to solve this class of problems are discussed. Our objective is to optimize certain index of returns subject to constraints on such factors as the amount of cash flow, average maturity and average risk, etc. The resulting objective functions and constraints are either linear, bilinear or bilinear fractional functions. The authors devised a special purpose algorithm for obtaining a local optimal solution of this nonconvex optimization problem containing more than 200 variables. Though it need not generate a global optimum, it is efficient enough to meet users' requirement.

### 1 Introduction

The bond market of Japan has been rapidly expanding since 1975, when a large amount of national bonds were issued and tight government regulations were substantially relaxed to enable a smooth circulation of national bonds. Numerous brands of bonds are now being circulated and several new types of transactions emerged in accordance with the growth of the market.

Thus, there is a strong demand for information service systems which would facilitate a quick and easy decision making of investors. Several bond operation analyzers have been developed in recent years by leading security companies to meet this demand. Unfortunately, however, none of these systems are accessible through open literature and they are not satisfactory enough regarding its range of applicability and reliability as well as processing speed. They either oversimplify the model to the extent that it is no longer valid in a very complex real transaction environment or can at best simulate small scale transactions. In addition, hours of computation is required to generate a solution which need not even be close to a local optimum. This is primarily because they still depend upon outdated general purpose mathematical tools to compute a solution of typically nonconvex optimization problems. In

fact, they sometimes generate a very awkward solution against which an experienced trader can point out a better solution upon casual inspection.

In the meantime, full bank dealing started in 1980 and major banks joined the bond dealing business. Also, future bond market is expanding rapidly since its birth in 1985. In short, we are on the brink of the revolution in the bond dealing business.

Under such circumstances, we propose a new bond portfolio optimization model which covers a variety of transaction environment of the institutional investors. Also we will develop an efficient algorithm for solving a resulting nonconvex optimization problem by exploiting its special structure. This algorithm can generate a very good, if not a globally optimal solution on a real time basis, namely within one minute on a mainframe computer.

A commercial purpose decision support system based upon our model and algorithm is now under development, which we hope will help bond traders make quick and quality decisions. We will, however restrict ourselves here on the exposition of the model and algorithm. The details of decision support system will be discussed in the forthcoming paper.

## 2 Indices Associated with Bond Portfolio

Let us assume that an investor holds  $u_j$  units of bonds  $B_j$ ,  $j = 1, \dots, N$ . Associated with  $B_j$  are four basic indices:

$c_j$  : coupon to be paid at a fixed rate (yen/bond/year)

$f_j$  : principal value to be refunded at maturity (yen/bond)

$p_j$  : present price in the market (yen/bond)

$t_j$  : maturity (number of years until its principal value is refunded)

Returns from bonds consist of two components. One is the income from coupon and the other is the capital gain due to price increase. Bond portfolio is determined by choosing the expected level of returns and risk from among numerous possible combinations on such factors as the size of transaction, magnitude of profit and/or loss, amount of money needed for additional investment and so on.

### 2.1 Indices to Represent Returns

There are three commonly used indices to represent returns, namely, average direct yield, average yield to maturity and average effective yield. (See e.g. [3], [5] for details about these indices)

Direct yield,  $\gamma_j$  of  $B_j$  is defined by

$$(2.1) \quad \gamma_j = \frac{c_j}{p_j}$$

which represents a very short term index of return.

Yield to maturity  $\mu_j$  is a constant satisfying the following equation:

$$(2.2) \quad p_j(1 + t_j \mu_j) = c_j t_j + f_j$$

The right hand side of (2.2) represents the total amount of cash out of one unit of  $B_j$ , while the left hand side stands for the amount of money we get by saving  $p_j$  units of cash for  $t_j$  years at simple interest rate  $\mu_j$ . Thus  $\mu_j$  is given by:

$$(2.3) \quad \mu_j = \frac{c_j + (f_j - p_j)/t_j}{p_j}$$

Effective yield  $\nu_j$  is a constant satisfying

$$(2.4) \quad p_j(1 + \nu_j)^{t_j} = c_j\{1 + (1 + \alpha) + \dots + (1 + \alpha)^{t_j-1}\} + f_j$$

where  $\alpha$  is the estimated reinvestment rate. Interpretation of  $\nu_j$  is analogous to that of  $\mu_j$  except that the former refers to compound interest rate instead of simple interest rate. Solving (2.4) in terms of  $\nu_j$  gives

$$(2.5) \quad \nu_j = \left\{ \frac{c_j\{(1 + \alpha)^{t_j} - 1\}/\alpha + f_j}{p_j} \right\}^{1/t_j} - 1$$

Average direct yield  $\gamma$ , average yield to maturity  $\mu$  and average effective yield  $\nu$  can now be defined as follows:

$$(2.6) \quad \gamma = \frac{\sum_{j=1}^N \gamma_j p_j u_j}{\sum_{j=1}^N p_j u_j}$$

$$(2.7) \quad \mu = \frac{\sum_{j=1}^N \mu_j p_j t_j u_j}{\sum_{j=1}^N p_j t_j u_j}$$

$$(2.8) \quad \nu = \frac{\sum_{j=1}^N \nu_j p_j t_j u_j}{\sum_{j=1}^N p_j t_j u_j}$$

## 2.2 Index to Represent the Risk of Investment

Associated with the investment is the risk due to variation in the price of bonds (The income from coupon is free from variation). We thus need to have an index to measure the magnitude of this risk. We will adopt here the average price variation index, the most commonly used one among the people in this business.

To explain this, let us first rewrite the equation (2.2) as follows

$$(2.9) \quad p_j = \frac{c_j t_j + f_j}{1 + t_j \mu_j}, \quad j = 1, \dots, N$$

Differentiating  $p_j$  with respect to  $\mu_j$ , we obtain

$$(2.10) \quad \frac{d p_j}{p_j} = \frac{-t_j}{1 + \mu_j t_j} d \mu_j, \quad j = 1, \dots, N$$

Price variation index (by way of simple interest)  $\pi_j$  of  $B_j$  is defined by the coefficient of  $d\mu_j$ , i.e.,

$$(2.11) \quad \pi_j = \frac{t_j}{1 + \mu_j t_j}, \quad j = 1, \dots, N$$

This is an increasing function of  $t_j$ , so that larger risk is associated with a bond with longer maturity.

Average price variation index  $\pi$  is defined as follows :

$$(2.12) \quad \pi = \frac{\sum_{j=1}^N \pi_j u_j}{\sum_{j=1}^N u_j}$$

If we use the expression (2.4) instead of (2.2), we get an alternative price variation index (by way of compound interest)

$$(2.13) \quad \sigma_j = \frac{t_j}{1 + \nu_j}, \quad j = 1, \dots, N$$

and its average :

$$(2.14) \quad \sigma = \frac{\sum_{j=1}^N \sigma_j u_j}{\sum_{j=1}^N u_j}$$

### 3 Objectives and Constraints

A bond trader sells and/or buys bonds to improve portfolio. Objectives of these transactions can be very diverse, i.e., some investor wants to maximize average direct yield by buying available bonds in the market and the other wants to minimize average maturity by selling his bonds in stock. Also another investor may want to improve some other index by selling and buying simultaneously. The model we are going to develop is the one which meets all these diverse requirements of the traders.

There are two schemes called "total optimization" and "partial optimization" to evaluate a transaction. Figure 1 shows the difference of these two schemes.

Total optimization refers to the optimization of certain objective function relative to the resulting portfolio after the transaction (Figure 1(c)). Partial optimization, on the other hand refers to the difference of buying portion and selling portion (Figure 1(d)). Though it seems natural to adopt the former from the systems analyst's point of view, the latter is sometimes preferred by bond traders to check the local goodness of each transaction, particularly when the amount sold or bought are relatively small compared to those which remain untouched.

Some of the possible candidates for the objectives and constraints are :

- (a) average direct yield  $\gamma$  defined by (2.6)
- (b) average yield to maturity  $\mu$  defined by (2.7)
- (c) average effective yield  $\nu$  defined by (2.8)
- (d) average maturity

$$(3.1) \quad t \equiv \frac{\sum_{j=1}^N t_j u_j}{\sum_{j=1}^N u_j}$$

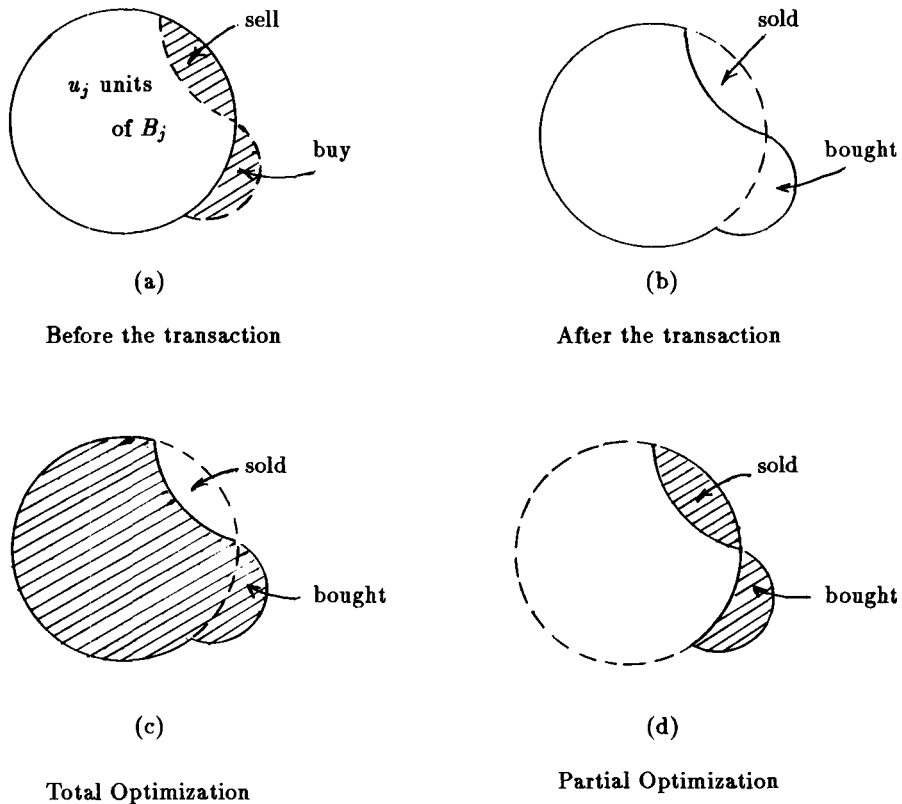


Figure 1

(e) average unit price

$$(3.2) \quad p \equiv \frac{\sum_{j=1}^N p_j u_j}{\sum_{j=1}^N u_j}$$

(f) average risk  $\pi$  (or  $\sigma$ ) defined by (2.12) or (2.14)

Also a trader has to take into account the profit or loss of a transaction in terms of cash. In this regard, there are two possible ways to treat the profit resulting from the transaction. We can either pay tax for the calculated profit or leave it as latent assets, depending upon the state of liquidation. Therefore, the prospect of liquidation affects the choice of portfolio.

An important factor related to this is the so-called unit price adjustment procedure. When a bond trader simultaneously buys and sells bonds through the same agent, he is entitled to choose the actual price of each bond within certain interval provided the agent agrees upon this transaction. The reason why a bond trader agrees to sell certain brands of bonds for the

price lower than the market price is that he wants to reduce the nominal profit out of this transaction, thereby reduce the amount of tax. He may, instead agree to buy certain brands of bonds for the price higher than the market price to compensate the loss of the agent incurred by this transaction.

The actual price of  $B_j$  cannot, however deviate more than a few percent from the market price  $p_j$  due to the transaction regulation. Whereas this option gives more flexibility to a trader, the resulting mathematical model becomes significantly more complicated compared to the one without this procedure.

#### 4 Mathematical Description of the Optimization Model

Let us assume again that an investor holds  $u_j$  units of  $B_j$ ,  $j = 1, \dots, N$  out of which  $n_1$  brands are selected as candidates for sale. In a typical situation,  $N$  is over 500 and  $n_1$  is less than, say 200. This selection process called "filtering" is carried out prior to the optimization process by considering a number of managerial, institutional and market constraints.

Also, let us assume that  $U_k$  units of bond  $B'_k$ ,  $k = 1, \dots, n_2$  are available in the market through an agent where  $n_2$  can be as large as 200.

Let

$$(4.1) \quad x_j = \text{amount of } B_j \text{ to be sold}$$

$$(4.2) \quad X_k = \text{amount of } B'_k \text{ to be purchased}$$

Lower and upper bound constraints are associated with these variables :

$$(4.3) \quad l_j \leq x_j \leq u_j, \quad j = 1, \dots, n_1$$

$$(4.4) \quad L_k \leq X_k \leq U_k, \quad k = 1, \dots, n_2$$

where most of  $l_j$ 's and  $L_k$ 's are zero.

Special case in which  $l_j = u_j = 0$ , for all  $j = 1, \dots, n_1$  is called "buying only" transaction. Alternatively, the case in which  $L_k = U_k = 0$ ,  $k = 1, \dots, n_2$  is called "selling only" transaction.

Unit selling price  $y_j$  of  $B_j$  and unit purchasing price  $Y_k$  of  $B'_k$  must satisfy

$$(4.5) \quad (1 - \lambda_j)p_j \leq y_j \leq (1 + \lambda_j)p_j, \quad j = 1, \dots, n_1$$

$$(4.6) \quad (1 - \lambda'_k)P_k \leq Y_k \leq (1 + \lambda'_k)P_k, \quad k = 1, \dots, n_2$$

where  $p_j$  and  $P_k$  are the reference market price of  $B_j$  and  $B'_k$ , respectively and  $\lambda_j$  is the unit price adjustment coefficient, a positive constant usually less than 0.02.

#### 4.1 Total Optimization Model

Let us introduce twelve indices to be included in our model. For this purpose, let

$$(4.7) \quad S_0 = \sum_{j=1}^N u_j - \sum_{j=1}^{n_1} x_j + \sum_{k=1}^{n_2} X_k$$

$$(4.8) \quad S_1 = \sum_{j=1}^N p_j u_j - \sum_{j=1}^{n_1} p_j x_j + \sum_{k=1}^{n_2} P_k X_k$$

$S_0$  and  $S_1$  stands for the total quantity of bonds and the total value of bonds after the transaction. Also, let us define

$$(4.9) \quad S_2 = \sum_{j=1}^N p_j t_j u_j - \sum_{j=1}^{n_1} p_j t_j x_j + \sum_{k=1}^{n_2} P_k T_k X_k$$

(i) magnitude of sale

$$(4.10) \quad z_1 = \sum_{k=1}^{n_1} x_j$$

(ii) magnitude of purchase

$$(4.11) \quad z_2 = \sum_{k=1}^{n_2} X_k$$

(iii) average coupon

$$(4.12) \quad z_3 = \frac{\sum_{j=1}^N c_j u_j - \sum_{j=1}^{n_1} c_j x_j + \sum_{k=1}^{n_2} C_k X_k}{S_0}$$

(iv) average maturity

$$(4.13) \quad z_4 = \frac{\sum_{j=1}^N t_j u_j - \sum_{j=1}^{n_1} t_j x_j + \sum_{k=1}^{n_2} T_k X_k}{S_0}$$

(v) average unit price

$$(4.14) \quad z_5 = \frac{\sum_{j=1}^N p_j u_j - \sum_{j=1}^{n_1} p_j x_j + \sum_{k=1}^{n_2} P_k X_k}{S_0}$$

(vi) average direct yield

$$(4.15) \quad z_6 = \frac{\sum_{j=1}^N \gamma_j p_j u_j - \sum_{j=1}^{n_1} \gamma_j p_j x_j + \sum_{k=1}^{n_2} \Gamma_k P_k X_k}{S_1}$$

(vii) average yield to maturity

$$(4.16) \quad z_7 = \frac{\sum_{j=1}^N \mu_j p_j t_j u_j - \sum_{j=1}^{n_1} \mu_j p_j t_j x_j + \sum_{k=1}^{n_2} \mu'_k P_k T_k X_k}{S_2}$$

(viii) average effective yield

$$(4.17) \quad z_8 = \frac{\sum_{j=1}^N \nu_j p_j t_j u_j - \sum_{j=1}^{n_1} \nu_j p_j t_j x_j + \sum_{k=1}^{n_2} \nu'_k P_k T_k X_k}{S_2}$$

(ix) average price variation index

$$(4.18) \quad z_9 = \frac{\sum_{j=1}^N \pi_j u_j - \sum_{j=1}^{n_1} \pi_j x_j + \sum_{k=1}^{n_2} \pi'_k X_k}{S_0}$$

(x) total profit

$$(4.19) \quad z_{10} = \sum_{j=1}^{n_1} (y_j - p_{j0}) x_j$$

where  $p_{j0}$  is the book value of  $B_j$

(xi) sum of liquidation

$$(4.20) \quad z_{11} = \sum_{j=1}^{n_1} y_j x_j - \sum_{k=1}^{n_2} Y_k X_k$$

(xii) profit/loss adjustment

$$(4.21) \quad z_{12} = \sum_{j=1}^{n_1} (y_j - p_j) x_j - \sum_{k=1}^{n_2} (Y_k - P_k) X_k$$

Of these indices,  $z_1, z_2$  are linear functions,  $z_3$  through  $z_9$  are linear fractional functions, and the others are bilinear functions.

A bond trader wants to optimize (either maximize or minimize) one of the indices  $z_3$  through  $z_{10}$  subject to constraints on others. Popular candidates for the objective functions are average direct yield  $z_6$  and average maturity  $z_4$ . Constraints on  $z_{10}, z_{11}$  and  $z_{12}$  can be bounded from above and below and constraints on  $z_1$  through  $z_9$  are either bounded from above or below, so that the most general mathematical form of our total optimization model can be written as follows:

$$(4.22) \quad \left\{ \begin{array}{ll} \text{maximize} & \frac{\sigma_0 - \sum_{j=1}^{n_1} (q_j + q'_j y_j) x_j + \sum_{k=1}^{n_2} (Q_k + Q'_k Y_k) X_k}{\pi_0 - \sum_{j=1}^{n_1} r_j x_j + \sum_{k=1}^{n_2} R_k X_k} \\ \text{subject to} & \frac{\phi_0 - \sum_{j=1}^{n_1} f_{ij} x_j + \sum_{k=1}^{n_2} F_{ik} X_k}{\delta_0 - \sum_{j=1}^{n_1} d_{ij} x_j + \sum_{k=1}^{n_2} D_{ik} X_k} \geq \alpha_i, \quad i = 1, \dots, m_1 \\ & \sum_{j=1}^{n_1} (h_{lj} + h'_{lj} y_j) x_j - \sum_{k=1}^{n_2} (H_{lk} + H'_{lk} Y_k) X_k \geq \beta_l, \quad l = 1, \dots, m_2 \\ & l_j \leq x_j \leq u_j, \quad y_j^0 \leq y_j \leq y_j^1, \quad j = 1, \dots, n_1 \\ & L_k \leq X_k \leq U_k, \quad Y_k^0 \leq Y_k \leq Y_k^1, \quad k = 1, \dots, n_2 \end{array} \right.$$

It should be noted that dividends of the expressions in (4.22) are positive for whatever value of the variables provided they satisfy the constraints, so that the first  $m_1$  inequalities can be reduced to linear inequalities. Also, standard normalization technique can be applied to all variables and we get the following normalized form of the total optimization model :

$$(4.23) \quad \left\{ \begin{array}{ll} \text{maximize} & \frac{\sum_{j=1}^n (q_j + q'_j y_j) x_j + q_0}{\sum_{j=1}^n p_j x_j + p_0} \\ \text{subject to} & \sum_{j=1}^n a_{ij} x_j \geq \alpha_{i0}, \quad i = 1, \dots, m_1; \\ & \sum_{j=1}^n (h_{lj} + h'_{lj} y_j) x_j \geq \beta_l, \quad l = 1, \dots, m_2 \\ & 0 \leq x_j \leq 1, \quad 0 \leq y_j \leq 1, \quad j = 1, \dots, n \end{array} \right.$$



where  $n = n_1 + n_2$ ,  $x_{n_1+j}$  and  $y_{n_1+j}$  corresponds to  $X_k$  and  $Y_k$  ( $k = 1, \dots, n_2$ ) respectively. This problem will be called a “bilinear fractional programming problem”. In this problem,  $n$  is usually 100~300 while  $m_1 \leq 9$  and  $m_2 \leq 6$ .

*Remark.* In this normalized formulation,  $y_j = 1/2$  corresponds to the reference market price  $p_j$ . Also,  $|h'_{lj}|$  is much smaller than  $|h_{lj}|$ . In a typical situation,  $|h'_{lj}| \leq |h_{lj}|/20$  for all  $l$  and  $j$ . In particular, if all unit price adjustment coefficients  $\lambda_j$  are zero, then all coefficients of  $y'_j$ s are zero and the problem reduces to a linear fractional program which can be solved by the standard technique ([1, 6]).

## 4.2 Mathematical Analysis of the Total Optimization Model

Let us prove a basic property of the optimal solution of the bilinear fractional programming problem (4.23).

Let us denote

$$(4.24) \quad f(x, y) = \frac{\sum_{j=1}^n (q_j + q'_j y_j) x_j + q_0}{\sum_{j=1}^n p_j x_j + p_0}$$

$$(4.25) \quad g_i(x) = \sum_{j=1}^n a_{ij} x_j, \quad i = 1, \dots, m_1$$

$$(4.26) \quad h_l(x, y) = \sum_{j=1}^n (h_{lj} + h'_{lj} y_j) x_j, \quad l = 1, \dots, m_2$$

and rewrite the problem (4.23) in a compact form as follows :

$$(4.27) \quad \left\{ \begin{array}{ll} \text{maximize} & f(x, y) \\ \text{subject to} & g_i(x) \geq \alpha_{i_0}, \quad i = 1, \dots, m_1; \\ & h_l(x, y) \geq \beta_l, \quad l = 1, \dots, m_2; \\ & 0 \leq x \leq e, \quad 0 \leq y \leq e \end{array} \right.$$

where  $e = (1, 1, \dots, 1)^t$ .

**Theorem 4.1** *If the problem (4.27) has a feasible solution, then it has an optimal solution  $(x^*, y^*)$ , where at least  $n - (m_1 + m_2)$  components of  $x^*$  are 0 or 1. Also, at least  $n - m_2$  components of  $y^*$  are 0 or 1.*

*Proof.*  $f(x, y)$  is continuous on the bounded feasible region, whence there exists an optimal solution  $(\hat{x}, \hat{y})$  if (4.27) is feasible. Consider the linear fractional program :

$$\begin{aligned} & \text{maximize } \{f(x, \hat{y}) \mid g_i(x) \geq \alpha_{i_0}, \quad i = 1, \dots, m_1; \\ & \quad \quad \quad h_l(x, \hat{y}) \geq \beta_l, \quad l = 1, \dots, m_2; 0 \leq x \leq e\} \end{aligned}$$

which has an optimal basic solution  $x^*$ . Since  $\hat{x}$  is a feasible solution of this problem, we have  $f(x^*, \hat{y}) \geq f(\hat{x}, \hat{y})$ . Also, at least  $n - (m_1 + m_2)$  components of  $x^*$  are at their lower or upper bounds.

Let us consider a linear program :

$$\text{maximize } \{f(x^*, y) \mid h_l(x^*, y) \geq \beta_l, \quad l = 1, \dots, m_2; 0 \leq y \leq e\}$$

which has an optimal solution  $y^*$  satisfying  $f(x^*, y^*) \geq f(x^*, \hat{y})$ . Also, at least  $n - m_2$  components of  $y^*$  are 0 or 1. It follows immediately from the above inequalities that  $f(x^*, y^*) \geq f(\hat{x}, \hat{y})$  which means that  $(x^*, y^*)$  is another optimal solution of (4.27).  $\square$

Note that a great majorities of the components of  $x^*$  and  $y^*$  are either 0 or 1 when  $n$  is over 100. This implies that almost all brands are either sold (purchased) to the limit or not sold (purchased) at all. This is very desirable from the practical point of view since there usually exists a minimal transaction unit associated with each bond and it has to be purchased or sold at an integral multiple of this minimal unit.

## 5 A Practical Algorithm for Solving the Total Optimization Problem

This section is devoted to the algorithm to obtain a good local optimal solution of (4.27).

### 5.1 Ascent Procedure by Solving a Sequence of Linear and Linear Fractional Programs

Given a feasible solution  $(x^k, y^k)$  of (4.27), let us solve a linear fractional program:

$$(5.1) \quad \begin{cases} \text{maximize} & f(x, y^k) \\ \text{subject to} & g_i(x) \geq \alpha_{i_0}, \quad i = q, \dots, m_1; \\ & h_l(x, y^k) \geq \beta_l, \quad l = 1, \dots, m_2; \\ & 0 \leq x \leq e \end{cases}$$

and let  $x^{k+1}$  be the resulting optimal basic solution. Obviously  $f(x^{k+1}, y^k) \geq f(x^k, y^k)$  since  $x^k$  is a feasible solution of (5.1). Also, let  $y^{k+1}$  be an optimal basic solution of a linear programming problem ;

$$(5.2) \quad \begin{cases} \text{maximize} & f(x^{k+1}, y) \\ \text{subject to} & h_l(x^{k+1}, y) \geq \beta_l, \quad l = 1, \dots, m_2 \\ & 0 \leq y \leq e \end{cases}$$

Again it is easy to see that  $f(x^{k+1}, y^{k+1}) \geq f(x^{k+1}, y^k)$ . We thus obtain a sequence of feasible solutions  $(x^k, y^k)$  of (4.27) satisfying

$$f(x^{k+1}, y^{k+1}) \geq f(x^k, y^k)$$

We continue this process until the condition

$$f(x^{k+1}, y^{k+1}) = f(x^k, y^k)$$

is satisfied. Since  $f$  is continuous on the bounded feasible region of (4.27),  $f(x^k, y^k)$  converges to the limit, which we denote by  $f^*$ . Also let  $(x^*, y^*)$  be an accumulation point of  $(x^k, y^k)$ .

*Remark.* Both (5.1) and (5.2) can be solved very cheaply. Also our computational experience shows that the sequence  $(x^k, y^k)$  converges very quickly, typically within 3 or 4 iterations.

## 5.2 Further Improvement by Simultaneous Change of Quantity and Price

Once  $(x^*, y^*)$  is reached, we can no longer improve  $f$  by fixing either  $x$  or  $y$ . We thus search for a feasible ascent direction of  $f$  by allowing simultaneous change of  $x$  and  $y$ .

Let

$$(5.3) \quad I_0(x^*) = \{j \mid x_1^* = 0\}, \quad I_1(x^*) = \{j \mid x_j^* = 1\}$$

$$(5.4) \quad I_0(y^*) = \{j \mid y_1^* = 0\}, \quad I_1(y^*) = \{j \mid y_j^* = 1\}$$

Also let

$$(5.5) \quad \begin{cases} \sum_{j=1}^n a_{ij} x_j^* = \alpha_{i_0}, & i \in I \\ \sum_{j=1}^n (h_{lj} + h'_{lj} y_j^*) x_j^* = \beta_l, & l \in L \end{cases}$$

A feasible direction vector

$$(5.6) \quad d = (\phi_1, \phi_2, \dots, \phi_n, \psi_1, \psi_2, \dots, \psi_n)$$

must satisfy

$$(5.7) \quad \begin{cases} \sum_{j=1}^n a_{ij} \phi_j \geq 0, & i \in I \\ \sum_{j=1}^n h'_{lj} x_j^* \psi_j + \sum_{j=1}^n (h_{lj} + h'_{lj} y_j^*) \phi_j \geq 0, & l \in L \\ \phi_j \geq 0, & j \in I_0(x^*); \quad \phi_j \leq 0, \quad j \in I_1(x^*) \\ \psi_j \geq 0, & j \in I_0(y^*); \quad \psi_j \leq 0, \quad j \in I_1(y^*) \end{cases}$$

In addition,  $\phi_j$ 's and  $\psi_j$ 's have to satisfy

$$(5.8) \quad \sum_{j=1}^n \frac{\partial f(x^*, y^*)}{\partial x_j} \phi_j + \sum_{j=1}^n \frac{\partial f(x^*, y^*)}{\partial y_j} \psi_j \geq 0$$

The existence of the vector  $d \in R^{2n}$  satisfying (5.7) and (5.8) can be checked by solving a linear program

$$(5.9) \quad \begin{cases} \text{maximize} & z = \sum_{j=1}^n \frac{\partial f(x^*, y^*)}{\partial x_j} \phi_j + \sum_{j=1}^n \frac{\partial f(x^*, y^*)}{\partial y_j} \psi_j \\ \text{subject} & (5.7) \end{cases}$$

*Case 1.* (5.9) generates an unbounded ray with direction  $(\phi_1^*, \dots, \phi_n^*, \psi_1^*, \dots, \psi_n^*)$

Let  $\alpha^*$  be the largest  $\alpha$  for which  $(x^* + \alpha \phi^*, y^* + \alpha \psi^*)$  is feasible for all  $\alpha \in [0, \alpha^*]$ . This can be obtained by solving a set of linear and quadratic equations.

*Case 1.1*  $\alpha^* > 0$

We execute a line search on the interval  $[(x^*, y^*), (x^* + \alpha^* \phi^*, y^* + \alpha^* \psi^*)]$  and obtain a new feasible solution  $(\hat{x}, \hat{y})$  such that

$$f(\hat{x}, \hat{y}) > f(x^*, y^*)$$

In this case we will return to the procedure explained in Section 5.1 by taking  $(\hat{x}, \hat{y})$  as the starting feasible solution.

*Case 1.2*  $\alpha^* = 0$

In this case, we failed to identify a feasible ascent direction. Thus we will either try to generate another unbounded ray by further pivoting or stop computation.

*Case 2.* (5.9) generates an optimal solution for which the objective function  $z$  is nonpositive.

**Theorem 5.1** *If (5.9) generates an optimal solution for which  $z \leq 0$ , then  $(x^*, y^*)$  is a Karush-Kuhn-Tucker point of (4.27).*

*Proof.* Since (5.9) has an optimal solution, its dual has a feasible solution. Hence there exists  $\xi_i, i \in I, \zeta_l, l \in L$  such that

$$(5.10) \quad \begin{aligned} & \leq 0, \quad j \in I_0(x^*) \\ \sum_{i \in I} a_{ij} \xi_i + \sum_{l \in L} h'_{lj} x_j^* \zeta_l + \frac{\partial f(x^*, y^*)}{\partial x_j} & \geq 0, \quad j \in I_1(x^*) \\ & = 0, \quad j \notin I_0(x^*) \cup I_1(x^*) \end{aligned}$$

$$(5.11) \quad \begin{aligned} & \leq 0, \quad j \in I_0(y^*) \\ \sum_{l \in L} (h_{lj} + h'_{lj} y_j^*) \zeta_l + \frac{\partial f(x^*, y^*)}{\partial y_j} & \geq 0, \quad j \in I_1(y^*) \\ & = 0, \quad j \notin I_0(y^*) \cup I_1(y^*) \end{aligned}$$

$$(5.12) \quad \xi_i \geq 0, \quad i \in I, \quad \zeta_l \geq 0, \quad l \in L$$

which is exactly the Karush-Kuhn-Tucker condition for (4.27) at  $(x^*, y^*)$ .  $\square$

### 5.3 Procedure to Obtain a Starting Feasible Solution

Let us define the Phase-I problem :

$$(5.13) \quad \left| \begin{array}{ll} \text{minimize} & z = \sum_{i=1}^{m_1} v_i + \sum_{l=1}^{m_2} (w_l + w'_l) \\ \text{subject to} & \sum_{j=1}^n a_{ij} x_j - x_{n+i} + v_i = \alpha_{i_0}, \quad i = 1, \dots, m_1 \\ & \sum_{j=1}^n (h_{lj} + h'_{lj} y_j) x_j - x'_{n+l} + w_l = \beta_l, \quad l = 1, \dots, m_2 \\ & 0 \leq x_j \leq 1, \quad j = 1, \dots, n \\ & x_{n+i} \geq 0, \quad v_i \geq 0, \quad i = 1, \dots, m_1 \\ & x'_{n+l} \geq 0, \quad w_l \geq 0, \quad w'_l \geq 0, \quad l = 1, \dots, m_2 \end{array} \right.$$

for fixed  $y_j$  with  $0 \leq y_j \leq 1$ , we will apply algorithm explained in Section 5.1 and 5.2 by choosing

$$y_j^0 = 1/2, \quad j = 1, \dots, n$$

as the starting value of  $y_j, j = 1, \dots, n$ . Note that this choice is equivalent to setting the price of each bond equal to its market price. We solve a sequence of linear programs until the sum of infeasibility  $z$  reduces to zero.

In case  $z$  cannot be reduced to zero, we will choose a randomly generated vector  $y$  and try the same procedure again. If several trials turn out to be failures, we stop calculation and suggest bond traders to modify parameters to make the problem more loosely constrained. (The detail of this will be discussed in the forthcoming paper).

Figure 2 shows the flowchart of our procedure to solve the total optimization problem.

## 6 Partial Optimization Model

Partial optimization refers to the difference of buying portion and selling portion (See Figure 1(d)). Thus all the indices  $z_1$  through  $z_{12}$  are redefined as the difference of each index associated with buying portion and selling portion, so that our model can be rewritten (after suitable normalization) as follows :

$$(6.1) \quad \left\{ \begin{array}{l} \text{maximize} \quad -\frac{\sum_{j=1}^{n_1} (q_j + q'_j y_j) x_j + q_0}{\sum_{j=1}^{n_1} r_j x_j + r_0} + \frac{\sum_{k=1}^{n_2} (Q_k + Q'_k Y_k) X_k + Q_0}{\sum_{k=1}^{n_2} R_k X_k + R_0} \\ \text{subject to} \quad -\frac{\sum_{j=1}^{n_1} f_{ij} x_j + f_{i0}}{\sum_{j=1}^{n_1} d_{ij} x_j + d_{i0}} + \frac{\sum_{k=1}^{n_2} F_{ik} X_k + F_{i0}}{\sum_{k=1}^{n_2} D_{ik} X_k + D_{i0}} \geq \alpha_i, \quad i = 1, \dots, m_1; \\ \quad -\sum_{j=1}^{n_1} (h_{lj} + h'_{lj} y_j) x_j + \sum_{k=1}^{n_2} (H_{lk} + H'_{lk} Y_k) X_k \geq \beta_l, \quad l = 1, \dots, m_1; \\ \quad 0 \leq x_j \leq 1, \quad 0 \leq y_j \leq 1, \quad j = 1, \dots, n_1 \\ \quad 0 \leq X_k \leq 1, \quad 0 \leq Y_k \leq 1, \quad k = 1, \dots, n_2 \end{array} \right.$$

This problem is much more difficult than its counterpart (4.27). In particular, even when  $q_j$ 's,  $Q_k$ 's,  $h_{lj}$ 's and  $H_{lk}$ 's are all zero, it is not solvable by standard linear fractional programming algorithm since the objective function is no longer quasi-convex and the feasible region may not even be a connected region. We thus employ a heuristic algorithm based upon the algorithm developed in Section 5.

Given a feasible solution  $(x^p, y^p, X^p, Y^p)$  of (6.1), we fix the pair of variable  $(x, y)$  at their current level  $(x^p, y^p)$  and solve the resulting bilinear fractional programming problem :

$$(6.2) \quad \left\{ \begin{array}{l} \text{maximize} \quad \frac{\sum_{k=1}^{n_2} (Q_k + Q'_k Y_k) X_k + Q_0}{\sum_{k=1}^{n_2} R_k Y_k + R_0} \\ \text{subject to} \quad \frac{\sum_{k=1}^{n_2} F_{ik} X_k + F_{i0}}{\sum_{k=1}^{n_2} D_{ik} Y_k + D_{i0}} \geq \alpha_i^p, \quad i = 1, \dots, m_1; \\ \quad \sum_{k=1}^{n_2} (H_{lk} + H'_{lk} Y_k) X_k \geq \beta_l^p, \quad l = 1, \dots, m_1; \\ \quad 0 \leq X_k \leq 1, \quad 0 \leq Y_k \leq 1, \quad k = 1, \dots, n_2 \end{array} \right.$$

by the algorithm of the preceding section by using  $(X^p, Y^p)$  as the starting solution. Let  $(X^{p+1}, Y^{p+1})$  be the resulting local optimal solution of (6.1).

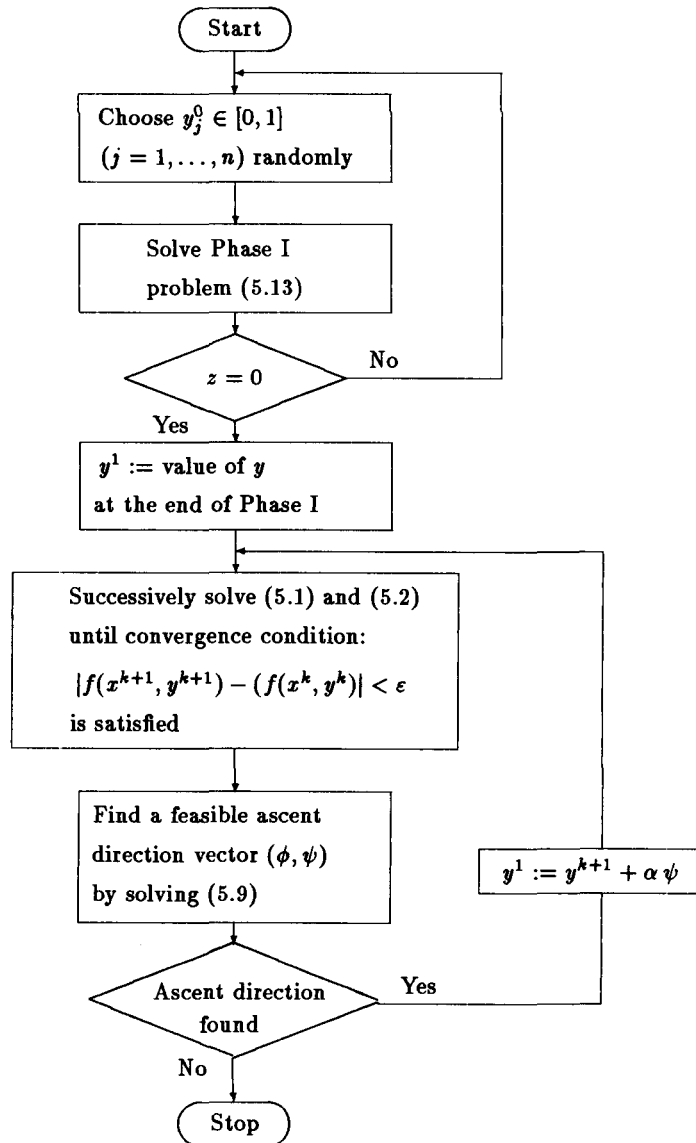


Figure 2: Flow Chart of Total Optimization Algorithm

We next fix the value of  $(X, Y)$  at  $(X^{p+1}, Y^{p+1})$  and solve another bilinear fractional programming problem :

$$(6.3) \quad \left| \begin{array}{l} \text{maximize} \quad - \frac{\sum_{j=1}^{n_1} (q_j + q'_j y_j) x_j + q_0}{\sum_{j=1}^{n_1} r_j x_j + r_0} \\ \text{subject to} \quad - \frac{\sum_{j=1}^{n_1} f_{ij} x_j + f_{i0}}{\sum_{j=1}^{n_1} d_{ij} x_j + d_{i0}} \geq \tilde{\alpha}_i^p \quad i = 1, \dots, m_1; \\ \quad \quad \quad - \sum_{j=1}^{n_1} (h_{lj} + h'_{lj} y_j) x_j \geq \tilde{\beta}_l^p \quad l = 1, \dots, m_2; \\ \quad \quad \quad 0 \leq x_j \leq 1, \quad 0 \leq y_j \leq 1, \quad j = 1, \dots, n_1 \end{array} \right.$$

Denote the optimal solution of this problem by  $(x^{p+1}, y^{p+1})$ .

We will continue this process until appropriate convergence condition is satisfied. This is admittedly only a heuristic algorithm but it turned out to generate solutions much better than the ones predicted by professional bond traders prior to our calculation.

## 7 Computational Results and Conclusions

We have implemented the algorithm for total optimization as well as partial optimization in Fortran IV and tested them on Burroughs 7900 computer.

The essential part of the routine is the procedure to solve upper bounded linear fractional program :

$$(7.1) \quad \left| \begin{array}{l} \text{maximize} \quad \frac{q^t x + q_0}{p^t x + p_0} \\ \text{subject to} \quad Ax \geq b \\ \quad \quad \quad 0 \leq x \leq e \end{array} \right.$$

where  $A$  is almost dense.

Table 1 and 2 show the statistics for a few dozens of test problems for the total and partial optimization model. Objective functions were chosen among average direct yield (2.6), average maturity (3.1) and average risk (2.14). These test problems very well simulate the practical transaction. All the problems successfully generated a good locally optimal solution against

Problem No.	$m$	$n$	Average CPU sec
1 - 30	9	10	2.0 (0.8 - 3.1)
31	6	30	2.8
32	6	50	3.0
33 - 35	11	165	10.0 (8.9 - 10.9)

Table 1: Total Optimization Model

Problem No.	$m$	$n_1$	$n_2$	Average CPU sec
36	9	10	10	15.8
37	6	10	10	14.5

Table 2: Partial Optimization Model

which no better alternative solution could be identified by professional bond traders. Some of the solutions were much better than those expected prior to the computation. Also, Table 1 indicates that the amount of computation for total optimization model depends at most linearly on the number of variables. Thus we believe that our algorithm will work for the problems of the size referred to in Section 4.

The software package based upon the algorithm we developed here will be used as the core of the decision support system currently under development at the Nihon Keizai Data Development Center. It will provide the bond trader with the information regarding the optimal investment strategy within one minute after he identifies objectives and constraints. Thus it will enable him to figure out his optimal investment on a real time basis.

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