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AN ANALYSIS OF A STOCHASTIC RESOURCE ALLOCATION MODEL WITH VARIOUS UTILITY FUNCTIONS

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Abstract This paper deals with stochastic models of an optimal sequential allocation of resources between consumption and production. We obtain the following results by means of the theory of martingales. For a model of resource allocation we establish the dynamic programming equation and show that a supermartingale characterizes the composition of the model and clear the composition of the optimality, representing the sufficient condition for an allocation to be optimal via a martingale. Further, we show the following three results as the application of the above result. (1) For the Kennedy's model we give another proof for the fact on an optimal allocation. (2) For a model with a convex utility function, we represent an optimal allocation via a supermartingale. (3) For a model with a logarithmic utility function, we obtain an explicit optimal allocation.

1. Introduction

In this paper, we consider stochastic models of resource allocation. Resource is sequentially allocated for consumption and production, and utility and resource are sequentially generated. We want to obtain the maximum sum of the expected utilities.

The model of resource allocation has been investigated by Beckmann [1], Dynkin and Yushkevich [3], Foldes [4], and Kennedy [5]. The model in section 2 is an extension of the Kennedy's model [5] in which the utility function is a special form. Kennedy's model that we treat in section 3 is an extension of the model in [3]. The model described in [3] is the stochastic version of the deterministic model given by Beckmann [1]. Foldes [4] has obtained necessary and sufficient conditions for an allocation to be optimal under the assumption that the utility function is concave and increasing.

In section 2 we analyze a stochastic model of resource allocation via the technique of dynamic programming and the theory of martingales. The analysis proves the sufficient condition for an allocation to be optimal. Section 3 of this paper derives another proof for the fact on an optimal allocation of the Kennedy's model by means of the results in section 2. Section 4 gives the results on optimality for a model with a convex utility function, utilizing the results obtained from section 2. The results in section 4 are similar to those of Kennedy's [5]. Section 5 is devoted to derivation of an explicit optimal allocation for a model with a logarithmic utility function by using the results obtained from section 2. The model is an extension of the deterministic model considered by Beckmann [1]. The results in section 5 are similar to one of the model considered by Beckmann [1]. The model of Dynkin and Yushkevich [3] is another one that gives an explicit optimal allocation.

2. A Model of Resource Allocation

We deal with a discrete time stochastic model of resource allocation with the infinite time horizon. The model behaves in the following way. A resource is allocated for consumption and production at each time. The product of the amount of resource allocated for production and a random parameter is occurred as a resource at next time. The product of the value of the utility function corresponding to the amount of resource allocated for consumption and a random parameter is obtained as a utility at next time.

We formulate the model mathematically. Assume that our argument bases on the probability space (Ω, F, P). Let $U = \{U_n;n\geq 1\}$ and $R = \{R_n;n\geq 1\}$ be two given nonnegative processes. U_n is a random parameter which is a generalization of the discount factor for utility at time n. We assume that only U_n is nonnegative, and independence of $\{U_n\}$ is not assumed. R_n is a nonnegative random parameter for resource at time n. Let $X = \{X_n;n\geq 0\}$ be a nonnegative process and $X_0\equiv 1$. X_n represents the amount of resource at time n. X_0 is the initial resource. Also let $Y = \{Y_n;n\geq 1\}$ and $Z = \{Z_n;n\geq 1\}$ be two nonnegative processes such that $X_{n-1} = Y_n + Z_n$ for all $n\geq 1$. Y_n represents the amount of resource allocated for the consumption at time $n-1$, and Z_n represents the amount of resource allocated for the production at time $n-1$. For convenience, we define $R_0\equiv 1$ and $Z_0\equiv 1$. Let $f : [0, +\infty) \to [-\infty, +\infty)$ be the utility function. Using the description above, the behavior of the model can be described as follows: for all $n\geq 1$, Y_n out of the resource X_{n-1} ia allocated for consumption and $Z_n = X_{n-1} - Y_n$ is allocated for production at time $n-1$. Then the utility $U_n f(Y_n)$ is obtained and the resource $X_n = R_n Z_n$ is occurred at time n.

Also let $\{F_n; n\geq 0\}$ be a sequence of sub-σ-fields of F such that $F_n \subset F_{n+1}$ for any $n\geq 0$. Intuitively, for any $n\geq 0$, F_n represents the information that can be utilized until time n. Concretely, for any $n\geq 0$, F_n represents the information on $\{X_0, U_1, R_1, \ldots, U_n, R_n\}$. Note that $\{F_n; n\geq 0\}$ is independent of our policy. Suppose that Y_n and Z_n are F_{n-1}-measurable for any $n\geq 1$. This assumption means Y_n and Z_n are chosen utilizing the information based on $\{X_0, U_1, R_1, \ldots, U_{n-1}, R_{n-1}\}$. Obviously, U_n and R_n are F_n-measurable for any $n\geq 1$. The objective of our argument is to determine the allocation between Y and Z which maximizes the sum of the expected utilities.

The above model is an extension of the Kennedy's model [5] in which the utility function is $y^{1/p}$ where p is fixed, p>1. That is, we substitute f for $y^{1/p}$. In other words, the model in this section and Kennedy's model are based on the same conditions except for a utility function.

We use the following notations. Let D be the set of processes $A = \{A_n; n\geq 0\}$ such that A_n is F_{n-1}-measurable, $0\leq A_n\leq 1$ for any $n\geq 1$ and $A_0 = 1$. Then we can see that there is a one-to-one relationship among Y, Z and $A\varepsilon D$ given by

$$(2.1) \qquad Y_n = (1-A_n)X_{n-1},\ Z_n = A_n X_{n-1},\ \text{with } X_n = X_{n-1}A_n R_n,\ n\geq 1.$$

A_n represents the proportion of the resource that is allocated for production at time $n-1$. Deservedly $(1-A_n)$ represents one for consumption. Let $D_n = \{A\varepsilon D | A_k=1 \text{ for all } k\ (0\leq k\leq n)\}$ for any $n\geq 0$. Note that $D = D_0$. For any $A\varepsilon D$, let $B = \{B_n; n\geq 0\}$, where

$$(2.2) \qquad B_n = (1-A_n)\prod_{k=0}^{n-1} A_k,\ n\geq 1 \text{ and } B_0 \equiv 0.$$

Also let O, O_n be the sets of such processes corresponding to D, D_n, respectively, through (2.2). That is, let $O = \{B|A\varepsilon D\}$ and $O_n = \{B|A\varepsilon D_n\}$. Note that $O = O_0$. It is easy to see that $O_n = \{B\varepsilon O | B_r\geq 0\ (r\geq 0),\ B_k = 0\ (k\leq n),\ \sum_{r=n+1}^{\infty} B_r \leq 1\}$.

We assume that

$$(2.3) \qquad \sup_{A\varepsilon D} E[\sum_{r=1}^{\infty} U_r f((1-A_r)\prod_{k=0}^{r-1} A_k R_k) | X_0=1] < \infty.$$

Note that $(1-A_r)\prod_{k=0}^{r-1} A_k R_k = Y_r$ represents the amount of the resource consumed at time r. (2.3) guarantees that the supremum of the sum of the expected utilities exists. Formally, the problem is to choose predictable sequences $Y = \{Y_n; n\geq 1\}$ and $Z = \{Z_n; n\geq 1\}$ so as to

$$\begin{aligned}(2.4)\qquad &\text{maximize } E[\sum_{r=1}^{\infty} U_r f(Y_r) \mid X_0=1] \\ &\text{subject to } Y_r+Z_r=X_{r-1},\ Y_r\geq 0,\ Z_r\geq 0\ (r\geq 1);\ Z_0\equiv 1.\end{aligned}$$

We can also describe the problem in the following form,

$$(2.5)\qquad \text{maximize } E[\sum_{r=1}^{\infty} U_r f((1-A_r)\prod_{k=1}^{r-1} A_k R_k)] \text{ over } A\in D.$$

We call $\bar{A}\in D$ an optimal allocation if

$$(2.6)\qquad \sup_{A\in D} E[\sum_{r=1}^{\infty} U_r f((1-A_r)\prod_{k=0}^{r-1} A_k R_k)] = E[\sum_{r=1}^{\infty} U_r f((1-\bar{A}_r)\prod_{k=0}^{r-1} \bar{A}_k R_k)].$$

At this time, we will discuss about the composition of an optimal allocation. Define

$$(2.7)\qquad D^* = \{A\in D \mid E[\sum_{r=1}^{\infty} U_r f((1-A_r)\prod_{k=0}^{r-1} A_k R_k)]>-\infty\}.$$

Then it is evidently in D^*, if the optimal allocation exists. Note that if $f \geq 0$, $D^* = D$. We wish to see an optimal allocation. We accordingly restrict the subject of the investigation to D^*. We show in Theorem 2.1 that the dynamic programming equation holds. (2.9) is similar to the dynamic programming equation established in Striebel [8], and we use the Kennedy's technique [5] in the proof of Theorem 2.1. Though in most systems the dynamic programming equation holds, it must be rigorously proved.

Lemma 2.1 and Definition 2.1 are quoted from [6, p.121]. The proof of Lemma 2.1 is omitted here.

Lemma 2.1. For every family H of random variables X defined on a probability space (Ω, F, P), there exists only one random variable Y such that

(a) $Y \geq X$ a.s. for all $X\in H$

(b) if Z is a random variable such that $Z \geq X$ a.s. for all $X\in H$, then $Z \geq Y$ a.s.

Definition 2.1. For the family H in Lemma 2.1, define

$$\text{ess sup } H\equiv Y,$$

where Y is the random variable in Lemma 2.1.

Definition 2.2. For any $n\geq 0$, $\bar{A} = \{\bar{A}_n; n\geq 0\}\in D^*$, define

$$(2.8)\qquad W_n(\bar{A}_0,\ldots,\bar{A}_n) = \operatorname*{ess\,sup}_{\substack{A\in D^* \\ A_k=\bar{A}_k (0\leq k\leq n)}} E[\sum_{r=1}^{\infty} U_r f((1-A_r)\prod_{k=0}^{r-1} A_k R_k) \mid F_n].$$

Remark: $W_n(\bar{A}_0,\dots,\bar{A}_n)$ represents the maximum conditional expected utility at n if we use $\bar{A}_0,\dots,\bar{A}_n$ until time n.

Theorem 2.1. For $W_n(\bar{A}_0,\dots,\bar{A}_n)$ for any $n\geq 0$ with $\bar{A}=\{\bar{A}_n;n\geq 0\}\varepsilon D^*$ defined by (2.8), we have

$$(2.9)\qquad W_n(\bar{A}_0,\dots,\bar{A}_n) = \operatorname*{ess\,sup}_{\substack{A\varepsilon D^* \\ A_k=\bar{A}_k(0\leq k\leq n)}} E[W_{n+1}(A_0,\dots,A_n,A_{n+1})|F_n].$$

Proof: For any $n\geq 0$, $\bar{A}\varepsilon D^*$;

$$\begin{aligned}
W_n(\bar{A}_0,\dots,\bar{A}_n) &= \operatorname*{ess\,sup}_{\substack{A\varepsilon D^* \\ A_k=\bar{A}_k(0\leq k\leq n)}} E[\sum_{r=1}^{\infty} U_r f((1-A_r)\prod_{k=0}^{r-1} A_k R_k)|F_n] \\
&= \operatorname*{ess\,sup}_{\substack{A\varepsilon D^* \\ A_k=\bar{A}_k(0\leq k\leq n)}} E[E[\sum_{r=1}^{\infty} U_r f((1-A_r)\prod_{k=0}^{r-1} A_k R_k)|F_{n+1}]|F_n] \\
(2.10)\qquad &= \operatorname*{ess\,sup}_{\substack{A\varepsilon D^* \\ A_k=\bar{A}_k(0\leq k\leq n)}} E[\operatorname*{ess\,sup}_{\substack{\bar{\bar{A}}\varepsilon D^* \\ \bar{\bar{A}}_k=\bar{A}_k(0\leq k\leq n) \\ \bar{\bar{A}}_{n+1}=A_{n+1}}} E[\sum_{r=1}^{\infty} U_r f((1-\bar{\bar{A}}_r)\prod_{k=0}^{r-1} \bar{\bar{A}}_k R_k)|F_{n+1}]|F_n] \\
&= \operatorname*{ess\,sup}_{\substack{A\varepsilon D^* \\ A_k=\bar{A}_k(0\leq k\leq n)}} E[W_{n+1}(A_0,\dots,A_n,A_{n+1})|F_n].
\end{aligned}$$

The validity of the third equality can be shown as follows. It is trivial that left hand side is not greater than right hand side. Therefore we show that right hand side is not greater than left hand side. For any $A\varepsilon D^*$ such that $A_k=\bar{A}_k(0\leq k\leq n)$, define

$$(2.11)\qquad F^* = \{E[\sum_{r=1}^{\infty} U_r f((1-\bar{\bar{A}}_r)\prod_{k=0}^{r-1} \bar{\bar{A}}_k R_k)|F_{n+1}];\ \bar{\bar{A}}\varepsilon D^*,\ \bar{\bar{A}}_k=\bar{A}_k(0\leq k\leq n),\ \bar{\bar{A}}_{n+1}=A_{n+1}\}.$$

Then F^* is directed upwards (cf. [6, p.95]) for any $A\varepsilon D^*$ such that $A_k=\bar{A}_k(0\leq k\leq n)$, that is, given any two random variables from the set there is a third dominating the two almost surely. Hence we can choose the sequence $\{f_j;\ j=1,2,\dots\}$ in increasing a.s. and then ess sup $F^*=\lim_{j\to\infty} f_j$ a.s. (cf. [6, p.121]).

Therefore, for any positive integer j, $A\varepsilon D^*$ such that $A_k=\bar{A}_k(0\leq k\leq n)$,

$$(2.12)\qquad \operatorname*{ess\,sup}_{\substack{A\varepsilon D^* \\ A_k=\bar{A}_k(0\leq k\leq n)}} E[E[\sum_{r=1}^{\infty} U_r f((1-A_r)\prod_{k=0}^{r-1} A_k R_k)|F_{n+1}]|F_n]\geq E[f_j|F_n].$$

Letting $j\to\infty$ and using monotone convergence theorem we have

$$(2.13)\qquad \operatorname*{ess\,sup}_{\substack{A\in D^* \\ A_k=\bar{A}_k(0\le k\le n)}} E[E[\sum_{r=1}^{\infty} U_r f((1-A_r)\prod_{k=0}^{r-1} A_kR_k)|F_{n+1}]|F_n]\ge E[\text{ess sup } F^*|F_n],$$

for any $A\in D^*$ such that $A_k=\bar{A}_k$ $(0\le k\le n)$.
From (2.13) we can see that right hand side is not greater than left hand side.

Q.E.D.

The following Lemma 2.2 can be easily proved.

Lemma 2.2. Suppose that the set of random variables $G = \{X(\alpha)|\alpha\varepsilon C;$ $E[X(\alpha)]>-\infty\}$ is directed upwards.
Then $E[\text{ess sup } G] = \sup_{\substack{\alpha\varepsilon C \\ E[X(\alpha)]>-\infty}} E[X(\alpha)]$.

The following Theorem 2.2 corresponds to the results in [5], and represents a characteristic of the model.

Theorem 2.2. Define $W_n(\bar{A}_0,\dots,\bar{A}_n)$, for any $n\ge 0$, $\bar{A} = \{\bar{A}_n;n\ge 0\}\varepsilon D^*$ in (2.8). Then for any $\bar{A}\varepsilon D^*$, $\{W_n(\bar{A}_0,\dots,\bar{A}_n);n\ge 0\}$ is a supermartingale with respect to $\{F_n\}$. That is, $\{W_n(\bar{A}_0,\dots,\bar{A}_n)\}$ satisfies the following condition; (1) for any $n\ge 0$, $W_n(\bar{A}_0,\dots,\bar{A}_n)$ is adapted to $\{F_n\}$, (2) $E[W_n(\bar{A}_0,\dots,\bar{A}_n)^-]>-\infty$, where let $x^-=\min(x,0)$, and (3) $E[W_{n+1}(\bar{A}_0,\dots,\bar{A}_n,\bar{A}_{n+1})|F_n]\le W_n(\bar{A}_0,\dots,\bar{A}_n)$.

Proof: Fix $\bar{A}\varepsilon D^*$ arbitrarily. (1) (2) It is trivial that $\{W_n(\bar{A}_0,\dots,\bar{A}_n)\}$ is an adapted process and $E[W_n(\bar{A}_0,\dots,\bar{A}_n)^-]>-\infty$.
(3) From Theorem 2.1, for any $n\ge 0$, $A\varepsilon D^*$ s.t. $A_k=\bar{A}_k(0\le k\le n)$,

$$(2.14)\qquad \begin{aligned} W_n(\bar{A}_0,\dots,\bar{A}_n) &= \operatorname*{ess\,sup}_{\substack{A\varepsilon D^* \\ A_k=\bar{A}_k(0\le k\le n)}} E[W_{n+1}(A_0,\dots,A_n,A_{n+1})|F_n] \\ &\ge E[W_{n+1}(\bar{A}_0,\dots,\bar{A}_n,A_{n+1})|F_n]. \end{aligned}$$

Consequently, for any $n\ge 0$, $W_n(\bar{A}_0,\dots,\bar{A}_n)\ge E[W_{n+1}(\bar{A}_0,\dots,\bar{A}_n,\bar{A}_{n+1})|F_n]$. Q.E.D.

The following Theorem 2.3 corresponds to the result in [8], and the sufficient condition for an allocation to be optimal is represented via a martingale.

Theorem 2.3. Suppose that the following three conditions (i)-(iii) hold.
(i) There exists the sequence $\{a_n;n\ge 0\}$ such that $\lim_{n\to\infty} a_n=0$ and $\sup_{A\varepsilon D^*} E[\sum_{r=n+1}^{\infty} U_r f((1-A_r)\prod_{k=0}^{r-1} A_kR_k)]\le a_n$ for any $n\ge 0$. (ii) $\{W_n(\bar{A}_0,\dots,\bar{A}_n);n\ge 0\}$ is a martingale with respect to $\{F_n;n\ge 0\}$, that is, for any $n\ge 0$ (1) $\{W_n(\bar{A}_0,\dots,\bar{A}_n)\}$ is an adapted process, (2) $E|W_n(\bar{A}_0,\dots,\bar{A}_n)|<\infty$, (3) $E[W_{n+1}(\bar{A}_0,\dots,\bar{A}_{n+1})|F_n] = W_n(\bar{A}_0,\dots,\bar{A}_n)$. (iii) $\bar{A}$ satisfies

$$(2.15)\qquad \sum_{r=1}^{\infty} \mathrm{E}\left|U_r f((1-\bar{A}_r)\prod_{k=0}^{r-1}\bar{A}_k R_k)\right|<+\infty.$$

Then $\bar{A}$ is an optimal allocation.

Remark 1. If for any $y\geq 0$, there exists a constant $K<\infty$ such that $f(y)\leq K(1+y)$ and $\mathrm{E}[\sum_{r=1}^{\infty} U_r(1+\prod_{k=0}^{r-1} R_k)]<+\infty$, the assumption of the Theorem 2.3 holds.

Remark 2. (2.15) always holds if $f\geq 0$.

Remark 3. The assumption of the Theorem 2.3 holds, if

$$(2.16)\qquad \limsup_{n\to\infty}\ \sup_{A\in D^*} \mathrm{E}[\sum_{r=n+1}^{\infty} U_r f((1-A_r)\prod_{k=0}^{r-1} A_k R_k)]=0.$$

Proof: As $\{W_n(\bar{A}_0,\ldots,\bar{A}_n)\}$ is a martingale, for any $n\geq 0$

$$(2.17)\qquad \mathrm{E}[W_0(\bar{A}_0)] = \mathrm{E}[W_n(\bar{A}_0,\ldots,\bar{A}_n)].$$

Accordingly,

$$(2.18)\qquad \mathrm{E}[W_0(\bar{A}_0)] = \lim_{n\to\infty} \mathrm{E}[W_n(\bar{A}_0,\ldots,\bar{A}_n)].$$

Now, for any $\bar{A}\in D^*$, from Lemma 2.2 and directed upwards property,

$$(2.19)\qquad \mathrm{E}[W_0(\bar{A}_0)] = \sup_{A\in D^*} \mathrm{E}[\sum_{r=1}^{\infty} U_r f((1-A_r)\prod_{k=0}^{r-1} A_k R_k)].$$

That is, the left-hand-side of the equation (2.18) represents the sup of the sum of the expected utilities. On the other side, from Definition 2.2, (2.15), the assumption of Theorem 2.3, and directed upwards property, for any $n\geq 0$,

$$(2.20)\qquad \begin{aligned} \mathrm{E}[W_n(\bar{A}_0,\ldots,\bar{A}_n)] &= \mathrm{E}[\sum_{r=1}^{n} U_r f((1-\bar{A}_r)\prod_{k=0}^{r-1}\bar{A}_k R_k)] \\ &\quad + \sup_{\substack{A\in D^* \\ A_k=\bar{A}_k (0\leq k\leq n)}} \mathrm{E}[\sum_{r=n+1}^{\infty} U_r f((1-A_r)\prod_{k=0}^{r-1} A_k R_k)] \\ &\qquad \text{(from Lemma 2.2)} \\ &\leq \mathrm{E}[\sum_{r=1}^{n} U_r f((1-\bar{A}_r)\prod_{k=0}^{r-1}\bar{A}_k R_k)]+a_n. \end{aligned}$$

Hence, from $\lim_{n\to\infty} a_n=0$, (2.15), (2.18) and (2.19), Theorem 2.3 is proved.

Q.E.D.

3. Kennedy's Model

Kennedy has cleared the composition of an optimal allocation [5] (We call the model Kennedy's model). Here, we give another proof for the fact on an optimal allocation in Kennedy's model by means of Theorem 2.3.

Kennedy's model and the model in section 2 are in fact the same one except for a utility function which is $f(y)=y^{1/p}$ in Kennedy's model, where p>1. (note that it is a nonnegative and concave function) The definitions of D, D_n, O, O_n are the same as the ones in section 2. The decision maker with a concave utility function is called risk averse ([2], [7]).

The assumption in Kennedy's model is as follows:

$$\text{(3.1)} \qquad E[\{\sum_{r=1}^{\infty} U_r^q(\prod_{k=0}^{r-1} R_k)^{q/p}\}^{1/q}]<+\infty, \text{ where } q=p/(p-1).$$

For any $n\geq 0$, define

$$\text{(3.2)} \qquad W_n = \operatorname*{ess\,sup}_{A\in D_n} E[\sum_{r=n+1}^{\infty} U_r((1-A_r)\prod_{k=0}^{r-1} A_k R_k)^{1/p}|F_n],$$

$$\text{(3.3)} \qquad \tau = \min\{n\geq 0|E[W_{n+1}|F_n] = 0\}.$$

$\{W_n\}$ satisfies a dynamic programming equation which is a special form of Theorem 2.1 (we will discuss about it in section 4), and is a supermartingale ([5]). We also note that W_n represents the maximum conditional expected utility at n if no consumption takes place before time $n+1$.

We can find the properties on $\{W_n\}$ in Lemma 3.1. They can easily be proved.

Lemma 3.1. For any $n\geq 0$, $E[W_{n+1}|F_n] = 0$ on $\{\tau\leq n\}$ and $W_n>0$ on $\{\tau>n\}$.

The following Theorem 3.1 is the result given by Kennedy [5]. Here, we can prove it by using Theorem 2.3.

Theorem 3.1. An optimal allocation $\{\bar{A}_n;n\geq 0\}$ is as follows:

$$\bar{A}_0 = 1$$

$$\text{(3.4)} \qquad \bar{A}_{r+1} = (E[W_{r+1}|F_n]/W_r)^q I\{\tau>r\} \text{ for any } r\geq 0$$

$$\bar{A} = \{\bar{A}_n;n\geq 0\}.$$

where W_n and τ are defined by (3.2) and (3.3), respectively.

Remark: The composition of an optimal allocation in Kennedy's model is similar to the one in Beckmann [1] which is solved explicitly.

Proof: It is easy to see the assumption (i) of Theorem 2.3 and (2.15) hold. We show that $\{W_n(\bar{A}_0,\dots,\bar{A}_n)\}$ is a martingale in the following descriptions. (1) It is trivial that $\{W_n(\bar{A}_0,\dots,\bar{A}_n)\}$ is an adapted process. (2) From (3.1) we have $E|W_n(\bar{A}_0,\dots,\bar{A}_n)|<+\infty$ for any $n\geq 0$. (3) For any $n\geq 0$, from Theorem 2.1,

$$\begin{aligned}
W_n(\bar{A}_0,\dots,\bar{A}_n) &= \operatorname*{ess\,sup}_{\substack{A\in D^*\\ A_k=\bar{A}_k(0\leq k\leq n)}} E[W_{n+1}(A_0,\dots,A_n,A_{n+1})|F_n] \\
&= \sum_{r=1}^{n} U_r((1-\bar{A}_r)\prod_{k=0}^{r-1}\bar{A}_kR_k)^{1/p} \\
&\quad + (\prod_{k=0}^{n}\bar{A}_k)^{1/p}\cdot \operatorname*{ess\,sup}_{0\leq A_{n+1}\leq 1}\{E[U_{n+1}|F_n](\prod_{k=0}^{n}R_k)^{1/p}(1-A_{n+1})^{1/p} \\
(3.5)\qquad &\quad + A_{n+1}^{1/p}E[W_{n+1}|F_n]\} \\
&= \sum_{r=1}^{n} U_r((1-\bar{A}_r)\prod_{k=0}^{r-1}\bar{A}_kR_k)^{1/p} \\
&\quad + (\prod_{k=0}^{n}\bar{A}_k)^{1/p}\{E[U_{n+1}|F_n](\prod_{k=0}^{n}R_k)^{1/p}(1-\bar{A}_{n+1})^{1/p} \\
&\quad + \bar{A}_{n+1}^{1/p}E[W_{n+1}|F_n]\} \\
&= E[\sum_{r=1}^{n+1} U_r((1-\bar{A}_r)\prod_{k=0}^{r-1}\bar{A}_kR_k)^{1/p} + (\prod_{k=0}^{n+1}\bar{A}_k)^{1/p}W_{n+1}|F_n].
\end{aligned}$$

The third equality follows from Lemma 3.1 and the fact that if we let $h(x)=a(1-x)^{1/p}+bx^{1/p}$ $(a,b\geq 0;0\leq x\leq 1)$, where $q=p/(p-1)$, $\max_{0<x<1} h(x)=h(b^q/(a^q+b^q))$, and $W_n{}^q=(E[U_{n+1}|F_n](\prod_{k=0}^{n}R_k)^{1/p})^q + (E[W_{n+1}|F_n])^q$ for all $n\geq 0$ (See the remark of Proposition 4.1). On the other hand, for any $n\geq 0$

$$(3.6)\qquad W_{n+1}(\bar{A}_0,\dots,\bar{A}_n,\bar{A}_{n+1}) = \sum_{r=1}^{n+1} U_r((1-\bar{A}_r)\prod_{k=0}^{r-1}\bar{A}_kR_k)^{1/p}+(\prod_{k=0}^{n+1}\bar{A}_k)^{1/p}W_{n+1}.$$

Hence from (3.5) and (3.6), for any $n\geq 0$,
$W_n(\bar{A}_0,\dots,\bar{A}_n)=E[W_{n+1}(\bar{A}_0,\dots,\bar{A}_n,\bar{A}_{n+1})|F_n]$.
Consequently, $\{W_n(\bar{A}_0,\dots,\bar{A}_n)\}$ is a martingale. Therefore $\bar{A}$ is an optimal allocation by Theorem 2.3. Q.E.D.

4. A Model with a Convex Utility Function

In section 2 we explored the composition of an optimal allocation. Hence we can see more definitely the composition of the model having a special convex utility function.

The model in this section and the Kennedy's model are indeed the same one except for a utility function which is $f(y)=y^{\mathrm{p}}$ in this section, where $\mathrm{p}\geq 1$. (Note that it is nonnegative and convex.) A decision maker having a convex utility function is called a risk taker ([2], [7]). The definitions of D, D_n, O, O_n are the same those mentioned in section 2.

Throughout this section, we make the following assumptions:

$$\mathrm{E}[(\sum_{r=1}^{\infty} U_r^{\mathrm{q}}(\prod_{k=0}^{r-1} R_k)^{\mathrm{pq}})^{1/\mathrm{q}}]<\infty, \text{ where } \mathrm{q}=\mathrm{p}/(\mathrm{p}-1) \qquad (\text{if } \mathrm{p}>1) \tag{4.1}$$

$$\mathrm{E}[(\sum_{r=1}^{\infty} U_r^{\beta}(\prod_{k=0}^{r-1} R_k)^{\beta})^{1/\beta}]<\infty, \text{ where } \beta>1 \qquad (\text{if } \mathrm{p}=1). \tag{4.2}$$

We will explain the composition of an optimal allocation.

Though (2.9) is similar to the dynamic programming equation established in Striebel [8], we can transform it into the form of dynamic programming equation established in Kennedy [5] by means of Proposition 4.1.

Proposition 4.1. Define for any $n\geq 0$

$$W_n = \operatorname*{ess\,sup}_{A\in \mathrm{D}_n} \mathrm{E}[\sum_{r=n+1}^{\infty} U_r((1-A_n)\prod_{k=0}^{r-1} A_k R_k)^{\mathrm{p}}|\mathrm{F}_n]. \tag{4.3}$$

Then for any $n\geq 0$ we have

$$\begin{aligned} W_n &= \operatorname*{ess\,sup}_{0\leq A_{n+1}\leq 1} \{\mathrm{E}[U_{n+1}|\mathrm{F}_n](\prod_{k=0}^{n} R_k)^{\mathrm{p}}(1-A_{n+1})^{\mathrm{p}}+\mathrm{E}[W_{n+1}|\mathrm{F}_n]A_{n+1}^{\mathrm{p}}\} \\ &= \max\{\mathrm{E}[U_{n+1}|\mathrm{F}_n](\prod_{k=0}^{n} R_k)^{\mathrm{p}},\ \mathrm{E}[W_{n+1}|\mathrm{F}_n]\}. \end{aligned} \tag{4.4}$$

Remark: As the proof the first equality in (4.4) is independent of the value of p, we see that $\{W_n\}$ in Kennedy's model also satisfies the dynamic programming equation in the same type as the first equality in (4.4).

Proof: From Theorem 2.1, for any $n\geq 0$, $A\in\mathrm{D}$, (2.9) holds. Hence for any $n\geq 0$

$$W_n(1,\ldots,1) = \operatorname*{ess\,sup}_{\substack{A'\varepsilon D \\ A'_k=1(0\le k\le n)}} E[W_{n+1}(1,\ldots,1,A'_{n+1})|F_n] \tag{4.5}$$

$$= \operatorname*{ess\,sup}_{0\le A_{n+1}\le 1} \{E[U_{n+1}|F_n](\prod_{k=0}^{n} R_k)^p(1-A_{n+1})^p+E[W_{n+1}|F_n]A^p_{n+1}\}.$$

Here for any $n\geq 0$

$$W_n(1,\ldots,1) = W_n. \tag{4.6}$$

Hence from (4.5) and (4.6), we obtain the first equality in (4.4). Maximizing the second expression in (4.4) on A_{n+1}, we have the second equality (4.4). Q.E.D.

From Theorem 2.2, for any $A\varepsilon D$, $\{W_n(A_0,\ldots,A_n)\}$ is a supermartingale. But we find that $\{W_n\}$ is also a supermartingale by means of the following Proposition 4.2.

Proposition 4.2. Define W_n in (4.3). Then $\{W_n;n\geq 0\}$ is a supermartingale with respect to $\{F_n;n\geq 0\}$.

Proof: It is trivial that (1) $\{W_n\}$ is an adapted process and (2) $E[W_n^-]>-\infty$ for any $n\geq 0$. (3) From Proposition 4.1, for any $n\geq 0$,

$$W_n = \max\{E[U_{n+1}|F_n](\prod_{k=0}^{n} R_k)^p,\ E[W_{n+1}|F_n]\} \geq E[W_{n+1}|F_n]. \tag{4.7}$$

Q.E.D.

In the following Theorem 4.1 we explain the composition of an optimal allocation, utilizing Theorem 2.3.

Theorem 4.1. Define W_n in (4.3) and

$$\sigma = \min\{n\geq 0|E[U_{n+1}|F_n](\prod_{k=0}^{n} R_k)^p \geq E[W_{n+1}|F_n]\}, \tag{4.8}$$

$$\bar{\bar{A}}_0 = 1.$$

$$\bar{\bar{A}}_{r+1} = I_{\{\sigma>r\}} \text{ for any } r\geq 0 \tag{4.9}$$

$$\bar{\bar{A}} = \{\bar{\bar{A}}_n;n\geq 0\}.$$

Then $\bar{\bar{A}}\varepsilon D$ is an optimal allocation and the optimal value is

$$E[U_{\sigma+1}(\prod_{k=0}^{\sigma} R_k)^p I\{0\leq\sigma<+\infty\}]. \tag{4.10}$$

Proof: It is trivial that $\bar{\bar{A}}\varepsilon D$. And it is obvious that the assumption (i) of Theorem 2.3 and (2.15) hold. We shall show in the following descriptions that $\{W_n(\bar{\bar{A}}_0,\ldots,\bar{\bar{A}}_n)\}$ is a martingale. (1) It is trivial that $\{W_n(\bar{\bar{A}}_0,\ldots,\bar{\bar{A}}_n)\}$ is an adapted process. (2) It follows from (4.1) and (4.2) that $\mathrm{E}|W_n(\bar{\bar{A}}_0,\ldots,\bar{\bar{A}}_n)|<\infty$ for any $n\geq 0$. (3) For any $n\geq 0$, from (2.8), (2.9) and (4.4),

$$
\begin{aligned}
W_n(\bar{\bar{A}}_0,\ldots,\bar{\bar{A}}_n) &= \operatorname*{ess\,sup}_{\substack{A\in\mathrm{D}^* \\ A_k=\bar{\bar{A}}_k (0\leq k\leq n)}} \mathrm{E}[W_{n+1}(A_0,\ldots,A_n,A_{n+1})\,|\,\mathrm{F}_n] \\
&= \sum_{r=1}^{n} U_r((1-\bar{\bar{A}}_r)\prod_{k=0}^{r-1}\bar{\bar{A}}_k R_k)^{\mathrm{p}} \\
&\quad + (\prod_{k=0}^{n}\bar{\bar{A}}_k)^{\mathrm{p}}\max\{\mathrm{E}[U_{n+1}|\mathrm{F}_n](\prod_{k=0}^{n}R_k)^{\mathrm{p}},\mathrm{E}[W_{n+1}|\mathrm{F}_n]\} \qquad (4.11)\\
&= \sum_{r=1}^{n} U_r((1-\bar{\bar{A}}_r)\prod_{k=0}^{r-1}\bar{\bar{A}}_k R_k)^{\mathrm{p}} \\
&\quad + (\prod_{k=0}^{n}\bar{\bar{A}}_k)^{\mathrm{p}}\{\mathrm{E}[U_{n+1}|\mathrm{F}_n](\prod_{k=0}^{n}R_k)^{\mathrm{p}}(1-\bar{\bar{A}}_{n+1})^{\mathrm{p}} \\
&\quad + \bar{\bar{A}}_{n+1}^{\mathrm{p}}\mathrm{E}[W_{n+1}|\mathrm{F}_n]\} \\
&= \mathrm{E}[W_{n+1}(\bar{\bar{A}}_0,\ldots,\bar{\bar{A}}_n,\bar{\bar{A}}_{n+1})|\mathrm{F}_n]\}
\end{aligned}
$$

where the fourth equality derives from the fact $\bar{\bar{A}}_{n+1}=1$, $\mathrm{E}[U_{n+1}|\mathrm{F}_n](\prod_{k=0}^{n}R_k)^{\mathrm{p}} < \mathrm{E}[W_{n+1}|\mathrm{F}_n]$ on $\{\sigma>n\}$ and $\bar{\bar{A}}_{n+1}=0$, $\mathrm{E}[U_{n+1}|\mathrm{F}_n] \geq \mathrm{E}[W_{n+1}|\mathrm{F}_n]$ on $\{\sigma=n\}$ and there exists $1\leq k\leq n$ such that $\bar{\bar{A}}_k=0$ on $\{\sigma<n\}$. Hence $\{W_n(\bar{\bar{A}}_0,\ldots,\bar{\bar{A}}_n)\}$ is a martingale. Therefore $\bar{\bar{A}}$ is an optimal allocation by means of Theorem 2.3. Q.E.D.

Intuitively the composition of the optimal allocation is as follows: no allocation for consumption takes place before time σ and we allocate the whole resources for consumption at the random time σ. The meaning of σ is as follows: the conditional expected utility at n is $\mathrm{E}[U_{n+1}|\mathrm{F}_n](\prod_{k=0}^{n}R_k)^{\mathrm{p}}$, if we allocate the whole resources for production before n and allocate the whole resources for consumption at n. And the maximum conditional expected utility at n is $\mathrm{E}[W_{n+1}|\mathrm{F}_n]$, if we allocate the whole resources for production before $n+1$. That is, σ is the first time when the former exceeds or equals the latter. In other words, Theorem 4.1 tells us that compare the conditional expected utility obtained by allocating the whole resources for consumption with the maximum conditional expected utility obtained by allocating the whole resources for production, and allocate the whole resources for the profitable side. Thus the optimal allocation accurately indicates that the decision maker with a convex utility function is speculative.

Below is an example of this model. Assume that a man bought some shares at time 0. He can sell it gradually and do at a single stroke. Here, $\{R_n;n\geq 0\}$ represents that shares fluctuate in price. That is, $R_n>1$ represents a rise and $R_n<1$ a fall and $R_n=1$ no fluctuation for all $n\geq 0$. The model in this section corresponds to the problem that the decision maker with the utility function y^p ($p\geq 1$) decides the allocation to obtain the maximum expected utilities. The decision maker should sell it by one effort in accordance with Theorem 4.1.

On the other hand, the utility function in the Kennedy's model is concave and the optimal allocation is as follows: resource is fittingly allocated for consumption and production at all time. The optimal allocation exactly indicates the attitude of the decision maker with a concave utility function is steady.

5. A Model with a Logarithmic Utility Function

In this section we explicitly obtain an optimal allocation for the model with a logarithmic utility function by using Theorem 2.3.

The model in this section and Kennedy's model are almost the same. The different respects are as follows: (1) $U_n=\beta^{n-1}$ a.s. for any $n\geq 0$, where β is a discount factor such that $0<\beta<1$. (2) The utility function is $f(y)=\log_e y$ for $y>0$ and $f(y)=-\infty$ for $y=0$ (3) For any $n\geq 0$, $R_n>0$ a.s. (4) $\{R_n;n\geq 1\}$ is an i.i.d. process. The definitions of D, D_n, O, O_n are the same those mentioned in section 2.

Definition 5.1. For any $A\varepsilon D$, define

$$\Phi(A) = E[\sum_{r=1}^{\infty} \beta^{r-1} f((1-A_r) \prod_{k=0}^{r-1} A_k R_k)]. \tag{5.1}$$

And let $c = E[\log R_1]$, $b = E|\log R_1|$.

Remark: $\Phi(A)$ represents the sum of the expected utilities that we have obtained by using $A\varepsilon D$.

The following assumptions will be made throughout this section:

$$b<\infty. \tag{5.2}$$

To obtain an optimal allocation, we first divide D into D^0 and D^1, where

$$D^0 = \{A\varepsilon D | 0<A_{n+1}<1 \text{ a.s. for any } n>0\}, \tag{5.3}$$

$$D^1 = \{A\varepsilon D | \text{for some } n\geq 1,\ P(A_n=0 \text{ or } A_n=1)\geq 0\} = D\backslash D^0. \tag{5.4}$$

Then it is obvious that $\Phi(A)=-\infty$ for any $A\varepsilon\mathrm{D}^1$. Accordingly, in this section D^* in (2.7) is as follows:

$$\mathrm{D}^* = \{A\varepsilon\mathrm{D}^0 | E[\sum_{r=1}^{\infty} \beta^{r-1}\log((1-A_r)\prod_{k=0}^{r-1} A_k R_k)]>-\infty\}. \tag{5.5}$$

It is evidently in D^*, if the optimal allocation exists. In the following Theorem 5.1 we explicitly obtain an optimal allocation by means of Theorem 2.3.

Theorem 5.1. Suppose that

$$\begin{aligned} &A_0^*=1, \\ &A_n^*=\beta \qquad \text{for any } n\geq 1, \\ &A^*=\{A_n^*; n\geq 0\}, \end{aligned} \tag{5.6}$$

where β is the discount factor with $0<\beta<1$.

Then $A^*\varepsilon\mathrm{D}^*$ is an optimal allocation and the optimal value is

$$(\log(1-\beta))/(1-\beta)+(\beta\log\beta)/(1-\beta)^2+c\beta/(1-\beta)^2. \tag{5.7}$$

Proof: It is trivial that $A^*\varepsilon\mathrm{D}^*$. It is easy to see that the assumption (i), (iii) of Theorem 2.3 hold. For any $n\geq 0$, from (2.8) and (2.9)

$$\begin{aligned} W_n(A_0^*,\ldots,A_n^*) &= \operatorname*{ess\,sup}_{\substack{A\varepsilon\mathrm{D}^* \\ A_k=A_k^*(0\leq k\leq n)}} E[W_{n+1}(A_0,\ldots,A_n,A_{n+1})|\mathrm{F}_n] \\ &\leq E[\sum_{r=1}^{n+1} \beta^{r-1}\log((1-\beta)\beta^{r-1}\prod_{k=0}^{r-1} R_k) + \sum_{r=n+2}^{\infty} \beta^{r-1}\log\beta^{n+1} \\ &\quad + \operatorname*{ess\,sup}_{\substack{0<A_{n+2}<1 \\ 0<A_{n+3}<1 \\ \vdots}} E[\sum_{r=n+2}^{\infty} \beta^{r-1}\log((1-\bar{A}_r)\prod_{k=n+2}^{r-1}\bar{A}_k\prod_{k=0}^{r-1} R_k) \\ &\quad |\mathrm{F}_{n+1}]|\mathrm{F}_n] \\ &= E[W_{n+1}(A_0^*,\ldots,A_n^*,A_{n+1}^*)|\mathrm{F}_n]. \end{aligned} \tag{5.8}$$

The inequality follows from the fact that $\max_{0<x<1} g(x)=g(\mathrm{b}/(\mathrm{a}+\mathrm{b}))$, if $g(x)= \mathrm{a}\cdot\log(1-x)+\mathrm{b}\cdot\log(x)$, where $0<x<1$, $\mathrm{a},\mathrm{b}>0$. On the other hand, from Theorem 2.2 $\{W_n(A_0^*,\ldots,A_n^*)\}$ is a supermartingale. Accordingly, we have for any $n\geq 0$

$$W_n(A_0^*,\ldots,A_n^*) = E[W_{n+1}(A_0^*,\ldots,A_n^*,A_{n+1}^*)|\mathrm{F}_n]. \tag{5.9}$$

Consequently, $\{W_n(A_0^*,\ldots,A_n^*)\}$ is a martingale. Therefore, from Theorem 2.3, A^* is an optimal allocation. Q.E.D.

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