

A NOTE ON THE DECOMPOSITION OF POLY-LINKING SYSTEMS AND THE MINORS OF GENERALIZED POLYMATROIDS

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Abstract In this note we shall describe the results obtained by applying the decomposition principle established in [2], [3] to poly-linking systems. In order to state those results in a logically self-consistent way, we shall introduce a new notion of 'minors' of generalized polymatroids.

Firstly, it is observed that the subsystems defined from a poly-linking system through our decomposition method are not in general poly-linking systems any more. This difficulty can be overcome by considering a poly-linking system as a special case of generalized polymatroids. In fact, it can be shown by an easy calculation that a poly-linking system is equivalent to a special case of generalized polymatroids. And when a poly-linking system is considered as a generalized polymatroid, its resultant subsystems through our decomposition method are seen to be the 'minors' of this generalized polymatroid. The notion of a 'minor' of a generalized polymatroid is first introduced in this paper. Hence from the point of view of our decomposition principle, the notion of 'poly-linking system' is not self-consistent, and it should be treated as a special case of generalized polymatroids.

Secondly, as a direct consequence of the results in [2], [3], a direct-sum decomposition of the optimal solutions of the intersection problems on poly-linking systems is induced.

Lastly, we shall investigate the parametrized type of the intersection problem on poly-linking systems where the rank functions are multiplied by positive real parameters.

1. Introduction

This is a succeeding note to the paper [3] by the same author. A part of the results are already reported in an informal publication [4]. The results of this paper have arisen from the application of the decomposition principle for submodular functions established in [2], [3] to poly-linking systems. The concept of poly-linking systems, which is a natural generalization of multi-terminal network flows, is first invented by Schrijver [5].

In case of the polymatroid intersection problems, the resultant submodular functions through our decomposition method are the rank functions of certain minors of the original polymatroids. In contrast with this, the resultant subsystems defined in the decomposition of a poly-linking system are generally not poly-linking systems any more. This kind of difficulty arises from the fact that a poly-linking system is defined from a single bisubmodular function, while its subsystems are characterized by a pair of a submodular and a supermodular functions. By introducing a new notion of 'minors' of generalized polymatroids, this situation can be formulated in a self-contained and self-consistent way.

It is fully described in [2], [3] that our decomposition method induces a direct-sum decomposition of the solutions of the intersection problem of a pair of polymatroids, i.e. the maximum common independent vectors. The present paper describes an analogous result for the intersection problems on poly-linking systems. That is, the direct-sum decomposition of its solutions is shown to arise from the same decomposition principle.

The decomposition method and its associated results which are used in this paper are presented in [3] in detail, so that we refer to [3] for their detailed description. For further information, the reader may refer to the references of [3].

2. Mathematical Preliminaries and A Decomposition of Submodular Functions

Let S denote a nonempty finite set throughout this paper, and 2^S denote the collection of all the subsets of S , which forms a Boolean lattice under inclusion relation. A sub-collection L of 2^S forms a sublattice if it is closed under union and intersection. A real-valued function f on 2^S is said to be submodular if

$$(2.1) \quad f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y) \quad \text{for } X, Y \subseteq S.$$

A function g is said to be supermodular if $-g$ is submodular.

Let L be a sublattice of 2^S , and S^+ and S^M be the minimum and the maximum element of L , respectively. Take any maximal chain

$$S^+ = A_0 \subsetneq A_1 \subsetneq \dots \subsetneq A_p = S^M \quad (\text{where } A_i \in L \text{ for each } 0 < i < p)$$

in L , and define $T = \{A_i - A_{i-1} : i=1, \dots, p\}$. Obviously, T is a partition of $S^M - S^+$ into nonempty subsets. And it is easy to verify that the resultant partition

$$(2.2) \quad (S^+, \{F: F \in T\}, S^-) \quad \text{where } S^- = S - S^M$$

of S is uniquely determined and independent of the choice of the maximal chain $\{A_i\}$. Let f be a submodular function on 2^S , and define the following submodular functions on the subsets of the partition (2.2) based on a maximal chain $\{A_i\}$:

$$(2.3) \quad \begin{aligned} & \text{(i) On } S^+ : f^+(X) = f(X) && \text{for } X \subseteq S^+, \\ & \text{(ii) On each } F = A_i - A_{i-1} \in T \text{ (} i=1, \dots, p \text{):} \\ & \quad f^F(X) = f(X \cup A_{i-1}) - f(A_{i-1}) && \text{for } X \subseteq F, \\ & \text{(iii) On } S^- : f^-(X) = f(X \cup S^M) - f(S^M) && \text{for } X \subseteq S^-. \end{aligned}$$

Combining the above, we have

$$(2.4) \quad (f^+, \{f^F : F \in T\}, f^-).$$

Proposition 1. (Theorem 3.1 of [3]) The decomposition (2.4) of f is uniquely determined and independent of the choice of the maximal chain $\{A_i\}$ if and only if

$$(2.5) \quad f(A) + f(B) = f(A \cup B) + f(A \cap B) \quad \text{for every } A, B \in L.$$

If the equality of (2.5) holds, then L is called an f -skeleton.

For two vectors $u \in R^S$ and $v \in R^{S'}$ (where $S \cap S' = \emptyset$), $u \oplus v$ denotes their direct-sum defined by

$$(u \oplus v)(e) = \begin{cases} u(e) & \text{if } e \in S, \\ v(e) & \text{if } e \in S'. \end{cases}$$

For $u \in R^S$ and $A \subseteq S$, $u(A)$ denotes $\sum_{i \in A} u_i$. Let f be a submodular function on 2^S with $f(\emptyset) = 0$. Then we shall call the polyhedron $P(f)$ defined by

$$(2.6) \quad P(f) = \{u \in R^S : u(X) \leq f(X) \quad \text{for } X \subseteq S\}$$

the *submodular polyhedron* of f , and the polytope defined by

$$(2.7) \quad B(P(f)) = \{u \in P(f) : u(S) = f(S)\}$$

the *base polyhedron* of $P(f)$. If L is an f -skeleton, then such a vector in $P(f)$ that the inequality constraints in (2.6) holds as an equality for every $A \in L$ will be decomposed into a direct-sum corresponding to the decomposition (2.4) of f derived from L . That is, we have

Proposition 2. (Theorem 6.1 of [3])

$$(2.8) \quad \begin{aligned} & \{u \in \mathbb{R}^S : u \in P(f), u(A) = f(A) \text{ for every } A \in L\} \\ & = B(P(f^+)) \oplus \left[\bigoplus_{F \in \mathcal{I}} B(P(f^F)) \right] \oplus P(f^-). \end{aligned}$$

3. Polymatroids, Poly-linking Systems and Generalized Polymatroids

Let f be a real-valued function on 2^S such that

$$(3.1) \quad \begin{aligned} (P1) \quad & f(\emptyset) = 0, \\ (P2) \quad & \text{if } A \subseteq B, \text{ then } f(A) < f(B), \\ (P3) \quad & f \text{ is submodular.} \end{aligned}$$

Then a polytope P defined by

$$(3.2) \quad \begin{aligned} P & = \{u \in \mathbb{R}^S : u > 0, u(X) < f(X) \quad X \subseteq E\} \\ & (= \{u \in P(f) : u > 0\}) \end{aligned}$$

is called a polymatroid, and f its rank function.

A poly-linking system D is such a polytope that

$$(3.3) \quad \begin{aligned} D & = \{(u, v) : u \in \mathbb{R}^{E_1}, v \in \mathbb{R}^{E_2}, u > 0, v > 0, u(E_1) = v(E_2), \\ & \quad u(X) + v(Y) - u(E_1) < h(X, Y) \quad X \subseteq E_1, Y \subseteq E_2\} \end{aligned}$$

where E_1 and E_2 are mutually disjoint nonempty finite sets, and h is a real-valued function on $2^{E_1} \times 2^{E_2}$ satisfying

$$(3.4) \quad \begin{aligned} (S1) \quad & h(X, \emptyset) = h(\emptyset, Y) = 0, \\ (S2) \quad & \text{if } A \subseteq A' \subseteq E_1 \text{ and } B \subseteq B' \subseteq E_2, \text{ then } h(A, B) < h(A', B'), \\ (S3) \quad & h(X, Y) + h(X', Y') > h(X \cup X', Y \cap Y') + h(X \cap X', Y \cup Y'). \end{aligned}$$

[bisubmodularity]

h is called the rank function of D . The concept of a poly-linking system was first invented by Schrijver [5], who defined it in the style of (3.3).

However the manner of the definition of (3.3) is not suitable to our purposes.

So we rewrite its definition in another equivalent way as below.

$$(3.5) \quad \begin{aligned} D & = \{(u, v) : u \in \mathbb{R}^{E_1}, v \in \mathbb{R}^{E_2}, \\ & \quad u(X) - v(E_2 - Y) < h(X, Y), \\ & \quad -u(E_1 - X) + v(Y) < h(X, Y) \quad X \subseteq E_1, Y \subseteq E_2\}. \end{aligned}$$

The equivalence of (3.3) and (3.5) is very easy to check. In fact, if (u, v) belongs to the polytope D defined by (3.3), then it is obvious that (u, v) belongs to that of (3.5). Conversely, take any (u, v) in D of (3.5). Then,

$$(3.6) \quad u(X) - v(E_2 - Y) \leq h(X, Y) \quad \text{for } X \subseteq E_1, Y \subseteq E_2,$$

$$(3.7) \quad -u(E_1 - X) + v(Y) \leq h(X, Y) \quad \text{for } X \subseteq E_1, Y \subseteq E_2.$$

When we put $X = \emptyset$ in (3.6), we have

$$(3.8) \quad -v(E_2 - Y) \leq h(\emptyset, E_2 - Y) = 0 \quad \text{for } Y \subseteq E_2,$$

which is equivalent to

$$(3.9) \quad v(Y') > 0 \quad \text{for } Y' \subseteq E_2.$$

Hence, $v > 0$. $u > 0$ is similarly shown. Setting $X = E_1$ and $Y = \emptyset$ in (3.6), we get

$$(3.10) \quad u(E_1) - v(E_2) \leq h(E_1, \emptyset) = 0$$

Analogously, it follows from (3.7) that

$$(3.11) \quad -u(E_1) + v(E_2) \leq h(\emptyset, E_2) = 0.$$

The combination of (3.10) and (3.11) implies $u(E_1) = v(E_2)$, from which it is readily seen that this (u, v) belongs to the polytope D of (3.3). Accordingly, the definitions (3.3) and (3.5) are shown to be equivalent.

Although the transformation from (3.3) to (3.5) is just an easy routine, this transformation is essential to our arguments, and the formulae of the definition of (3.5) are fully exploited in this paper.

The notion of generalized polymatroids was first presented by Frank [1]. A generalized polymatroid is a polyhedron bounded by a submodular function from upper and bounded by a supermodular function from lower where the pair of the submodular and the supermodular functions are defined on certain intersecting families (not necessarily on the Boolean lattice 2^S) and they are supposed to satisfy a certain relation which will be described later. The original definition of generalized polymatroids is unnecessarily general for our purposes. For instance they are not bounded polytopes in general. So we shall restrict ourselves to such a special case that the pair of a submodular and a supermodular functions are defined on all the subsets of the ground set, and call such a polytope a *simple* generalized polymatroid. A simple generalized polymatroid is necessarily a bounded polytope.

The exact definition is the following. A simple generalized polymatroid is such a polytope, which we shall denote by $W(f, g)$, that

$$W(f, g) = \{x \in \mathbb{R}^S : g(A) < x(A) < f(A) \quad A \subseteq S\}$$

where

(G1) f [resp. g] is a submodular [resp. supermodular] function on 2^S with $f(\emptyset) = 0$ [resp. $g(\emptyset) = 0$],

(G2) f and g meet the following relation

$$(3.12) \quad f(A) - g(B) > f(A \setminus B) - g(B \setminus A) \quad \text{for } A, B \subseteq S.$$

A poly-linking system D can be equivalently transformed to a special case of simple generalized polymatroids. In fact, if we put

$$(3.13) \quad D^* = \{(u, -v) : (u, v) \in D\},$$

then D^* is shown to be a simple generalized polymatroid in \mathbb{R}^S where $S = E_1 \cup E_2$. As for D^* , (3.5) is rewritten as

$$(3.14) \quad D^* = \{(u, w) : u \in \mathbb{R}^{E_1}, w \in \mathbb{R}^{E_2}, u(X) + w(Y) < h(X, E_2 - Y), \\ -u(X) - w(Y) < h(E_1 - X, Y) \quad \text{for } X \subseteq E_1, Y \subseteq E_2\}.$$

We shall define functions f_h and g_h on 2^S by

$$(3.15) \quad f_h(X \cup Y) = h(X, E_2 - Y) \quad X \subseteq E_1, Y \subseteq E_2,$$

$$(3.16) \quad g_h(X \cup Y) = -h(E_1 - X, Y) \quad X \subseteq E_1, Y \subseteq E_2.$$

It is clear by definition that f_h and g_h satisfy the condition (G1). Also it follows from the bisubmodularity of h that f_h and g_h meet the condition (G2). Hence,

$$\begin{aligned} D^* &= \{(u, w) : u(X) + w(Y) < f_h(X \cup Y), \\ &\quad -u(X) - w(Y) < -g_h(X \cup Y)\} \\ &= \{x \in \mathbb{R}^S : g_h(Z) < x(Z) < f_h(Z) \quad Z \subseteq S\} \quad (\text{where } x = (u, w)) \\ &= W(f_h, g_h) \end{aligned}$$

is a simple generalized polymatroid.

4. A Decomposition of Generalized Polymatroids and Poly-linking Systems

Let $W(f, g)$ be a simple generalized polymatroid in \mathbb{R}^S . Suppose a sub-

lattice L of 2^S is given. Take any maximal chain

$$(4.1) \quad S^+ = C_0 \subsetneq C_1 \subsetneq \dots \subsetneq C_q = S^M = S - S^-$$

in L , and let $U = \{C_j - C_{j-1} : j=1, \dots, q\}$. For $A \subseteq S$, \bar{A} denotes its complement, i.e. $\bar{A} = S - A$.

We shall define a decomposition of (f, g) according to the maximal chain (4.1) as follows.

$$(4.2) \quad \begin{aligned} (i) \quad & \text{On } S^+ = C_0 : \text{ For } x \subseteq S^+, \\ & f^+(x) = f(x), \\ & g^-(x) = g(x \cup S^+) - g(S^+), \\ (ii) \quad & \text{On } F = C_j - C_{j-1} \in U : \text{ For } x \subseteq F, \\ & f^F(x) = f(x \cup C_{j-1}) - f(C_{j-1}), \\ & g^F(x) = g(x \cup \bar{C}_j) - g(\bar{C}_j), \\ (iii) \quad & \text{On } S^- = S - S^M = S - C_q : \text{ For } x \subseteq S^-, \\ & f^-(x) = f(x \cup S^-) - f(S^-), \\ & g^+(x) = g(x). \end{aligned}$$

Summarizing the above, we have

$$(4.3) \quad ((f^+, g^-), \{(f^F, g^F) : F \in U\}, (f^-, g^+)).$$

We can easily check that each pair of the functions in (4.3) satisfies both of the conditions (G1) and (G2). Hence each pair of (4.3) defines a certain generalized polymatroid, which is called a *minor* of the original generalized polymatroid.

An analogous result to Proposition 1 holds for generalized polymatroids. That is,

Proposition 3. The decomposition (4.3) of (f, g) is uniquely determined and independent of the choice of the maximal chain in L if and only if for any $A, B \in L$,

$$(4.4) \quad f(A) + f(B) = f(A \cup B) + f(A \cap B)$$

$$(4.5) \quad g(\bar{A}) + g(\bar{B}) = g(\bar{A} \cap \bar{B}) + g(\bar{A} \cup \bar{B}).$$

In case of poly-linking systems, the above result can be stated in the following way.

$2^{E_1} \times (2^{E_2})^*$ denotes the Boolean lattice whose partial order is as

follows: For $(A, B), (A', B') \in 2^{E_1} \times (2^{E_2})^*$,

$$(A, B) < (A', B') \text{ if } A \subseteq A' \text{ and } B \supseteq B'.$$

Let K be a sublattice of $2^{E_1} \times (2^{E_2})^*$, i.e. K be a family such that

$$\text{if } (A, B), (A', B') \in K,$$

$$\text{then } (A \cup A', B \cap B') \in K \text{ and } (A \cap A', B \cup B') \in K.$$

Take any maximal chain $\{(A_i, B_i)\}$ in K such that

$$(4.6) \quad \begin{aligned} (E_1^+, E_2 - E_2^-) = (A_0, B_0) \preceq (A_1, B_1) \preceq \dots \\ \dots \preceq (A_r, B_r) = (E_1 - E_1^-, E_2^+), \end{aligned}$$

which uniquely determines a partition of (E_1, E_2) into

$$(4.7) \quad ((E_1^+, E_2^-), \{(F_1, F_2) : (F_1, F_2) \in V\}, (E_1^-, E_2^+))$$

where $V = \{(A_j - A_{j-1}, B_{j-1} - B_j) : j=1, \dots, r\}$. As is the same with (2.2), the resultant partition (4.7) is independent of the choice of the maximal chain.

Let h be the rank function of a poly-linking system D in $R^{E_1} \times R^{E_2}$. Here we put $S = E_1 \cup E_2$. When we define f_h and g_h by (3.15) and (3.16), then f_h and g_h are obviously a submodular and a supermodular function on 2^S , respectively. For a sublattice K in $2^{E_1} \cup (2^{E_2})^*$, we shall define the following sublattices L_K and L_K^* of 2^S by

$$(4.8) \quad L_K = \{A \cup (E_2 - B) : (A, B) \in K\},$$

$$(4.9) \quad L_K^* = \{(E_1 - A) \cup B : (A, B) \in K\}.$$

Then the following three conditions are equivalent.

$$(4.10) \quad \begin{aligned} (A) \quad h(A, B) + h(A', B') = h(A \cup A', B \cap B') + h(A \cap A', B \cup B') \\ \text{for any } (A, B), (A', B') \in K \end{aligned}$$

$$(B) \quad L_K \text{ is a skeleton for } f_h,$$

$$(C) \quad L_K^* \text{ is a skeleton for } -g_h.$$

When (4.10) is met, K is called a *skeleton* for h .

Suppose K to be a skeleton for h . From (B) and (C) above, L_K and L_K^* are skeleton for f_h and g_h , respectively. Hence L_K and L_K^* determines the decompositions (2.4) of f_h and g_h . Combining the thus obtained functions, we have

$$(4.11) \quad ((f_h^+, g_h^-), \{(f_h^F, g_h^F) : F \in U\}, (f_h^-, g_h^+)).$$

Let $\{(A_j, B_j) : j=0,1,\dots,r\}$ be a maximal chain in K , which gives a unique partition (4.7) of (E_1, E_2) . Let $C_j = A_j \cup (E_2 - B_j)$ for $j=0,1,\dots,r$. Then, $\{C_j\}$ is a maximal chain in L_K . Interpreting the decomposition (4.11) in terms of h , we obtain a decomposition of h into the pairs of bisubmodular functions, each of which is defined on a block of the partition (4.7) derived from K . For instance, take any $F = F_1 \cup F_2 \in U$ ($(F_1, F_2) \in V$). Then the minor on F is

$$\begin{aligned} W_F = \{ & (u, w) : u \in R^{F_1}, w \in R^{F_2}, \\ & u(X) + w(Y) < f_h^F(X, Y) \\ & - u(X) - w(Y) < -g_h^F(X, Y) \quad X \subseteq E_1, Y \subseteq E_2 \} \end{aligned}$$

When we put

$$D_F = \{(u, -w) : (u, w) \in W_F\},$$

then D_F is a subsystem of the original poly-linking system, which is represented as

$$\begin{aligned} D_F = \{ & (u, v) : u \in R^{F_1}, v \in R^{F_2}, \\ & u(X) - v(E_2 - Y) < h_1^F(X, Y) \\ & - u(E_1 - X) + v(Y) < h_2^F(X, Y) \}. \end{aligned}$$

Hence by (3.15) and (4.2), we have

$$\begin{aligned} h_1^F(X, Y) &= f_h^F(X \cup (F_2 - Y)) \\ &= f_h(X \cup (F_2 - Y) \cup C_{j-1}) - f_h(C_{j-1}) \\ &= h(X \cup (C_{j-1} \cap E_1), E_2 - ((F_2 - Y) \cup (C_{j-1} \cap E_2))) \\ &\quad - h(C_{j-1} \cap E_1, E_2 - (E_2 \cap C_{j-1})) \\ &= h(X \cup A_{j-1}, Y \cup B_j) - h(A_{j-1}, B_{j-1}). \end{aligned}$$

In a similar way, we have

$$h_2^F(X, Y) = h(X \cup A_{j-1}, Y \cup B_j) - h(A_j, B_j).$$

The same arguments can be applied to all the pairs in (4.11), which provides the following pairs of bisubmodular functions:

$$(4.12) \quad ((h_1^+, h_2^-), \{(h_1^F, h_2^F) : F = (F_1, F_2) \in V\}, (h_1^-, h_2^+))$$

where

(i) (h_1^+, h_2^-) is defined on (E_1^+, E_2^-) : For $x \subseteq E_1^+, y \subseteq E_2^-$,

$$h_1^+(x, y) = h(x, y \cup (E_2 - E_2^-)),$$

$$h_2^-(x, y) = h(x, y \cup (E_2 - E_2^-)) - h(E_1^+, E_2 - E_2^-).$$

(ii) (h_1^F, h_2^F) is defined on $F = (F_1, F_2) = (A_j - A_{j-1}, B_{j-1} - B_j) \in V$
 $(1 < j < r)$: For $x \subseteq F_1, y \subseteq F_2$,

$$h_1^F(x, y) = h(x \cup A_{j-1}, y \cup B_j) - h(A_{j-1}, B_{j-1}),$$

$$h_2^F(x, y) = h(x \cup A_{j-1}, y \cup B_j) - h(A_j, B_j).$$

(iii) (h_1^-, h_2^+) is defined on (E_1^-, E_2^+) : For $x \subseteq E_1^-, y \subseteq E_2^+$,

$$h_1^-(x, y) = h(x \cup (E_1 - E_1^-), y) - h(E_1 - E_1^-, E_2^+),$$

$$h_2^+(x, y) = h(x \cup (E_1 - E_1^-), y).$$

Since K is supposed to be a skeleton for h , the resultant pairs of bisubmodular functions of (4.12) are uniquely determined and independent of the choice of the maximal chain $\{A_j, B_j\}$ in K . It is very important to remark here that the polytopes defined from the pairs of bisubmodular functions in (4.12) are generally not poly-linking systems any more.

We shall prepare some terminology. For two functions g_1, g_2 on $2^{H_1} \times 2^{H_2}$, we put

$$D(g_1, g_2) = \{(u, v) \in R^{H_1} \times R^{H_2} :$$

$$u(x) - v(H_2 - y) < g_1(x, y),$$

$$- u(H_1 - x) + v(y) < g_2(x, y) \quad x \subseteq H_1, y \subseteq H_2\}$$

In case that h is the rank function of a poly-linking system $D, D = D(h, h)$. Based on this notation, we can restate the above remark as follows. $D(h_1^+, h_2^-), D(h_1^F, h_2^F)$ (for $F \in V$) and $D(h_1^-, h_2^+)$, whose rank functions are defined by (4.13), are not poly-linking systems any more. Furthermore we shall introduce some notions below, which are analogous to the base polyhedra of submodular polyhedra.

$$B^+(D(g_1, g_2)) = \{(u, v) \in D(g_1, g_2) : u(H_1) - v(H_2) = g_1(H_1, \emptyset)\},$$

$$B^-(D(g_1, g_2)) = \{(u, v) \in D(g_1, g_2) : -u(H_1) + v(H_2) = g_2(\emptyset, H_2)\},$$

$$B(D(g_1, g_2)) = B^+(D(g_1, g_2)) \cap B^-(D(g_1, g_2)).$$

Let D be a poly-linking system and h be its rank function. Suppose K to be a skeleton for h . Then K gives the decomposition (4.12) of h . Here we

have a completely analogous result to Proposition 2 for the case of the bi-submodular function h . That is,

Theorem 1.

$$(4.15) \quad \begin{aligned} & \{(u, v) \in D : u(A) - v(E_2 - B) = h(A, B), \\ & \quad - u(E_1 - A) + v(B) = h(A, B) \text{ for every } (A, B) \in K\} \\ & = B^+(D(h_1^+, h_2^-)) \oplus \left[\bigoplus_{F \in V} B(D(h_1^F, h_2^F)) \right] \oplus B^-(D(h_1^-, h_2^+)) \end{aligned}$$

Proof: Making use of the formulae of the definition (3.5), a necessary and sufficient condition for (u, v) to belong to the left-hand side of (4.15) is seen to be the following (a) and (b).

$$(a) \quad \begin{cases} u(X) - v(E_2 - Y) < h(X, Y) & X \subseteq E_1, Y \subseteq E_2, \\ u(A) - v(E_2 - B) = h(A, B) & \text{for } (A, B) \in K, \end{cases}$$

$$(b) \quad \begin{cases} -u(E_1 - X) + v(Y) < h(X, Y) & X \subseteq E_1, Y \subseteq E_2, \\ -u(E_1 - A) + v(B) = h(A, B) & \text{for } (A, B) \in K. \end{cases}$$

In terms of f_h and L_K , (a) is equivalent to

$$(c) \quad \begin{cases} (u \oplus (-v))(Z) < f_h(Z) & Z \subseteq E_1 \cup E_2, \\ (u \oplus (-v))(Q) = f_h(Q) & \text{for } Q \in L_K. \end{cases}$$

By Proposition 2, (c) is equal to

$$(d) \quad u \oplus (-v) \in B(P(f_h^+)) \oplus \left[\bigoplus_{F \in U} B(P(f_h^F)) \right] \oplus P(f_h^-)$$

By using the bisubmodular functions in (4.12), (d) is rewritten as

$$(e1) \quad \text{On } (E_1^+, E_2^-):$$

$$\begin{aligned} u(X) - v(E_2^- - Y) &< h_1^+(X, Y) & X \subseteq E_1^+, Y \subseteq E_2^-, \\ u(E_1^+) - v(E_2^-) &= h_1^+(E_1^+, \emptyset). \end{aligned}$$

$$(e2) \quad \text{On each } F = (F_1, F_2) \in V:$$

$$\begin{aligned} u(X) - v(F_2 - Y) &< h_1^F(X, Y) & X \subseteq F_1, Y \subseteq F_2, \\ u(F_1) - v(F_2) &= h_1^F(F_1, \emptyset). \end{aligned}$$

$$(e3) \quad \text{On } (E_1^-, E_2^+):$$

$$u(X) - v(E_2^+ - Y) < h_1^-(X, Y) \quad X \subseteq E_1^-, Y \subseteq E_2^+.$$

By the same arguments, the condition (b) is seen to be equivalent to the following (f1,2,3).

- (f1) On (E_1^+, E_2^-) :
 - $u(E_1^+ - X) + v(Y) < h_2^-(X, Y) \quad X \subseteq E_1^+, Y \subseteq E_2^-.$
- (f2) On each $F = (F_1, F_2) \in V$:
 - $u(F_1 - X) + v(Y) < h_2^F(X, Y) \quad X \subseteq F_1, Y \subseteq F_2,$
 - $u(F_1) + v(F_2) = h_2^F(\emptyset, F_2).$
- (f3) On (E_1^-, E_2^+) :
 - $u(E_1^- - X) + v(Y) < h_2^+(X, Y) \quad X \subseteq E_1^-, Y \subseteq E_2^-,$
 - $u(E_1^-) + v(E_2^+) = h_2^+(\emptyset, E_2^+).$

Combining (e1,2,3) and (f1,2,3), we have the right-hand side of (4.15).

(End of Proof)

5. Application to the Intersection Problems on Poly-linking Systems

Let D be a poly-linking system in $R_1^{E_1} \times R_2^{E_2}$ and h be its rank function. And let P_1 and P_2 be a polymatroid in $R_1^{E_1}$ and $R_2^{E_2}$, respectively, and f_1 and f_2 denote their rank functions. The associated Intersection Problem is to

$$(5.1) \quad \begin{aligned} &\text{maximize } u(E_1) (= v(E_2)) \\ &\text{subject to } (u, v) \in D, u \in P_1, v \in P_2. \end{aligned}$$

We denote by $M(D, P_1, P_2)$ the collection of all the pairs (u, v) that achieve the maximum of (5.1). The maximal value of (5.1) is known to be

$$(5.2) \quad \begin{aligned} &\max \{u(E_1) : (u, v) \in D, u \in P_1, v \in P_2\} \\ &= \min \{h(X, Y) + f_1(E_1 - X) + f_2(E_2 - Y) : X \subseteq E_1, Y \subseteq E_2\}. \end{aligned}$$

(Schrijver [5])

When we denote by K the collection of all the pairs (X, Y) attaining the minimum in the right-hand side of (5.2), it is easy to prove that K is a sublattice of $2^{E_1} \times (2^{E_2})^*$, and furthermore K is a skeleton for h . At the same time,

$$(5.3) \quad \begin{aligned} K_1 &= \{E_1 - X : (X, Y) \in K \text{ for some } Y\}, \\ K_2 &= \{E_2 - Y : (X, Y) \in K \text{ for some } X\} \end{aligned}$$

are an f_1 -skeleton and an f_2 -skeleton, respectively. The partition (4.7) derived from K shall be denoted by

$$(5.4) \quad ((E_1^+, E_2^-), \{(F_1, F_2) : (F_1, F_2) \in V\}, (E_1^-, E_2^+)).$$

On the other hand, K_1 and K_2 determine the partitions

$$(5.5) \quad (E_1^-, \{F_1 : F_1 \in T_1\}, E_1^+),$$

$$(5.6) \quad (E_2^-, \{F_2 : F_2 \in T_2\}, E_2^+)$$

of E_1 and E_2 , respectively, where

$$\begin{aligned} T_1 &= \{F_1 : (F_1, F_2) \in V \text{ for some } F_2\}, \\ T_2 &= \{F_2 : (F_1, F_2) \in V \text{ for some } F_1\}. \end{aligned}$$

Since K is a skeleton for h , K determines the decomposition (4.12) of h , and K_1 and K_2 give the decomposition (2.4) of f_1 and f_2 , respectively. Then,

Theorem 2. The optimal solutions of the intersection problem (5.1) are decomposed into a direct-sum according to the partition (5.4). More precisely, we have

$$\begin{aligned} M(D, P_1, P_2) &= [B^+(D(h_1^+, h_2^-)) \cap (P(f_1^-) \times B(P(f_2^+)))] \\ &\oplus [\oplus_{F=(F_1, F_2) \in V} \{B(D(h_1^F, h_2^F)) \cap (B(P(f_1^{F_1})) \times B(P(f_2^{F_2})))\}] \\ &\oplus [B^-(D(h_1^-, h_2^+)) \cap (B(P(f_1^+)) \times P(f_2^-))] \end{aligned}$$

Proof: From the min-max equality (5.2), a necessary and sufficient condition for (u, v) to belong to $M(D, P_1, P_2)$ is the following (i), (ii), and (ii').

$$(i) \quad \begin{aligned} (u, v) &\in D, \\ u(A) - v(E_2 - B) &= h(A, B) \quad \text{for any } (A, B) \in K, \\ -u(E_1 - A) + v(B) &= h(A, B) \quad \text{for any } (A, B) \in K, \end{aligned}$$

- (ii) $u \in P_1$,
 $u(A) = f_1(A)$ for any $A \in K_1$,
- (ii') $u \in P_2$,
 $v(B) = f_2(B)$ for any $B \in K_2$.

Applying Theorem 1 to (i) and Proposition 2 to (ii) and (ii'), we have the assertion of the theorem. (End of Proof)

6. Parametrized Intersection Problems and the Dulmage-Mendelsohn Decomposition of Bipartite Graphs

By introducing multiplying parameters for the rank functions, we can consider the parametrized intersection problems. Let a , b and c be positive real numbers. Then the parametrized intersection problem is the following:

$$(6.1) \quad \begin{aligned} & \text{Maximize } u(E_1) (= v(E_2)) \\ & \text{subject to } u \in a P_1, v \in b P_2, (u, v) \in c D. \end{aligned}$$

The min-max equality (5.1) for this parametrized problem is

$$(6.2) \quad \begin{aligned} & \max \{u(E_1) : u \in a P_1, v \in b P_2, (u, v) \in c D\} \\ & = \min \{a f_1(E_1 - A) + b f_2(E_2 - B) + c h(A, B) : A \subseteq E_1, B \subseteq E_2\} \end{aligned}$$

Let $K(a, b, c)$ be the collection of all the pairs (A, B) that achieve the minimum in the right-hand side of (6.2). $K(a, b, c)$ is easily seen to be a skeleton for h . When the parameters a, b, c are changed in a way 'monotonically', then the union of the corresponding skeletons forms a larger skeleton for h . By the word 'monotonically', we mean the following. The 3-tuples (a, b, c) and (a', b', c') of positive real numbers are said to be compatible if either $a/c < a'/c'$, $b/c > b'/c'$ or $a/c < a'/c'$, $b/c > b'/c'$ holds.

Suppose C to be a collection of 3-tuples of positive parameters such that each two in C are compatible. Then

$$(6.3) \quad K_{\text{ext}} = \bigcup K(a, b, c) : (a, b, c) \in C$$

can be shown to be a skeleton for h , and K_{ext} provides the decomposition which is finer than the decomposition corresponding to $K(a, b, c)$ for each $(a, b, c) \in C$.

The Dulmage-Mendelsohn decomposition of bipartite graphs is obtained as a special case of ours as follows. Let $G_B(V_1, V_2)$ be a bipartite graph, whose rank function (as a linking system) is denoted by h_B . The Dulmage-Mendelsohn decomposition corresponds to the case of our parametrized intersection problem (6.1) where the rank functions of polymatroids are cardinality functions, and c is large enough compared with $a = b = 1$. Since c is chosen to be very large enough, the right-hand side of (6.2) turns out to be

$$\begin{aligned}
 & \min \{ |V_1 - A| + |V_2 - B| + c h_B(A, B) : A \subseteq V_1, B \subseteq V_2 \} \\
 (6.4) \quad & = \min \{ |V_1 - A| + |V_2 - B| : A \subseteq V_1, B \subseteq V_2, h_B(A, B) = 0 \} \\
 & = \min \{ |X| + |Y| : (X, Y) \in J \}
 \end{aligned}$$

where

$$(6.5) \quad J = \{ (X, Y) : X \subseteq V_1, Y \subseteq V_2, h_B(V_1 - X, V_2 - Y) = 0 \}.$$

The collection of those (X, Y) which achieve the minimum in (6.4) is just equal to the collection of the minimum-covers of $G_B(V_1, V_2)$. As is well known, the lattice composed of all the minimum-covers defines the Dulmage-Mendelsohn decomposition of the given bipartite graph. Furthermore the parametrized Dulmage-Mendelsohn decomposition is induced from our parametrized problem as follows. Let $J(a, b)$ denote the collection of those pairs (X, Y) which give the minimum of the following:

$$(6.6) \quad \min \quad a|X| + b|Y| \quad \text{subject to} \quad (X, Y) \in J$$

where a and b are positive real numbers. Then,

$$(6.7) \quad J_{\text{par}} = \cup \{ J(a, b) : a > 0, b > 0 \} = \cup \{ J(t, 1 - t) : 0 < t < 1 \}$$

is a skeleton for h_B , which gives the so-called parametrized Dulmage-Mendelsohn decomposition of a bipartite graph $G_B(V_1, V_2)$.

The skeleton J itself clearly contains J_{par} as a sublattice and is far larger than J_{par} . Hence it seems to be quite probable that this J may provide some more useful information for the structure of a bipartite graph.

However this observation is incorrect. In fact, the decomposition and the partial order defined from this distributive lattice J just coincide with the original given bipartite graph, and offers none of new useful information.

Lastly, we shall present an example to illustrate some of the results established in this paper. Let N be a multi-terminal capacitated network shown in Fig. 1. $E_1 = \{x, y, z\}$ and $E_2 = \{m, n\}$ are the set of its sources and sinks, respectively. The collection of the pairs of the inflow at E_1 and the outflow at E_2 forms a poly-linking system on (E_1, E_2) . That is,

$$D_N = \{(u, v) : u \in R_+^{E_1}, v \in R_+^{E_2}, \text{ there is a feasible flow in } N \text{ such that its inflow is } u \text{ and its outflow is } v\}$$

is a typical example of a poly-linking system. And its rank function h_N is determined by

$h_N(X, Y)$ = the maximal value of a flow which can be transported from the sources in X to the sinks in Y .

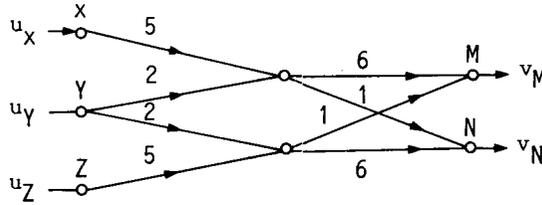


Fig. 1

We shall define two polymatroids P_1, P_2 by

$$P_1 = \{u \in R^{E_1} : 0 < u_x < 4, 0 < u_y < 12, 0 < u_z < 2\},$$

$$P_2 = \{v \in R^{E_2} : 0 < v_m < 4, 0 < v_n < 2\}.$$

And we shall investigate the parametrized problem on $(a P_1, b P_2, c D)$ and the skeleton $K(a, b, c)$ associated with it. Without loss of generality, we may assume $a + b + c = 1$. Then every choice of triples (a, b, c) is represented as a point in the triangle coordinate of Fig. 2.

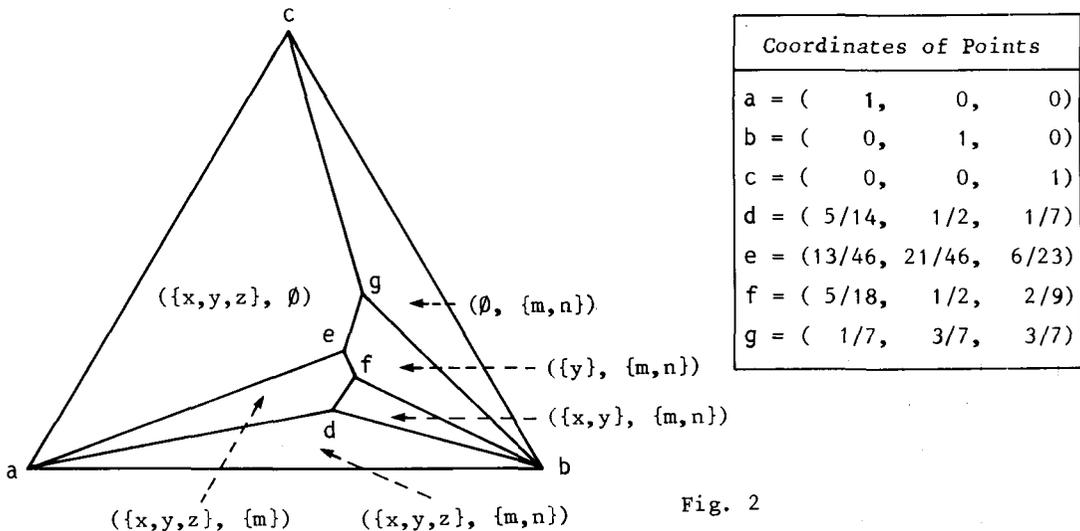


Fig. 2

In our example, the skeleton $K(a, b, c)$ corresponding to a point in a open district consists of a single pair (x, y) , which is filled in there in Fig. 2. On the boundary lines, the corresponding skeleton is the union of the skeletons of the adjacent districts. For instance, on the line segment between e and g (except for the extreme points e and g), the corresponding skeleton is that of Fig. 3.

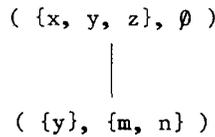


Fig. 3

Also, the skeleton corresponding to a crossing point of boundary lines is the union of the skeletons of the adjacent districts. For example, the skeleton $K(5/18, 1/2, 2/9)$ corresponding to a crossing point $f = (5/18, 1/2, 2/9)$ is presented in Fig. 4. This skeleton provides the partition of the ground set shown in Fig. 5.

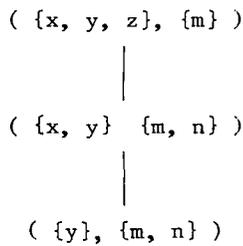


Fig. 4

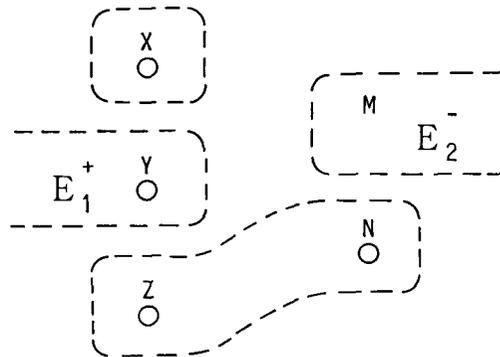


Fig. 5

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