Journal of the Operations Research
Society of Japan
Vol. 31, No. 4, December 1988

THEORY AND ALGORITHMS OF THE LAGUERRE TRANSFORM, PART I : THEORY

Ushio Sumita
University of Rochester

Masaaki Kijima
Tokyo Institute of Technology

(Received June 8, 1987; Revised March 3, 1988)

Abstract The Laguerre transform, introduced by Keilson and Nunn (1979), Keilson, Nunn and Sumita (1981) and further studied by Sumita (1981), provides an algorithmic framework for the computer evaluation of repeated combinations of continuum operations such as convolution, integration, differentiation and multiplication by polynomials. The procedure enables one to numerically evaluate many distribution results of interest, which have been available only formally behind the 'Laplacian curtain'. Since the initial development, the formalism has been extended to incorporate matrix and bivariate functions and finite signed measures. The purpose of this paper is to summarize theoretical results on the Laguerre transform obtained up to date. In a sequel to this paper, a summary is given focusing on algorithmic aspects. The two summary papers will enable the reader to use the Laguerre transform with ease.

§0. Introduction

In applied probability and statistics, one often encounters expressions involving repeated combinations of continuum operations such as multiple convolutions, differentiation, integration and multiplication by polynomials. The server busy period density for M/G/1 queueing systems (Takács [29]), the effective service time density of M/G/1 queueing systems with service interruptions (Gaver [2] and Keilson [6]) and the test of uniformity on the sphere (Beran [1] and Giné [3]) are typical examples. Numerical evaluation of such continuum operations is, in general, quite tedious. The brute force approach via discretization often fails due to problems of expense and uncharacterized accumulated errors. These numerical barriers have limited the use of many theoretical findings and have substantially impeded the growth of applied probability as an engineering and management resource.

The Laguerre transform, introduced by Keilson and Nunn [7], Keilson, Nunn and Sumita [8], and further studied by Sumita [18], provides an algorithmic framework for the computer evaluation of multiple convolutions and other continuum operations as described above. The transform based on generalized Fourier series employs the Laguerre functions as a basis, and maps functions $f(x)$ in L_2 into discrete sequences $(f_n^{\#})_{-\infty}^{\infty}$. Correspondingly, various continuum operations are mapped into lattice operations, thereby providing the desired algorithmic basis. The procedure enables one to numerically evaluate many distribution results of interest, which have been available only formally behind the 'Laplacian curtain', with speed and accuracy. The power of the Laguerre transform has been demonstrated through a variety of applications in applied probability and statistics [7,8,9,10,11,12,18,19].

The formalism of the Laguerre transform requires further extensions to deal with more complicated models. The matrix Laguerre transform has been developed by Sumita [18,21] for the study of semi-Markov processes and associated semi-Markov renewal processes. Convolutions and other continuum operations involving matrix functions can be evaluated through the matrix version of the Laguerre transform. Consequently this extension provides modeling flexibility in the study of reliability systems and in the performance evaluation of computer and communication systems, enabling one to conduct dynamic analysis of such systems. Successful applications of the matrix Laguerre transform have been reported in [16,18,21,26].

An extension of the formalism for bivariate functions is also useful and important. In the study of bivariate processes, expressions of many theoretical results involve repeated combinations of bivariate continuum operations such as multiple bivariate convolutions, marginal convolutions, double integration, partial differentiation and multiplication by bivariate polynomials. A cumulative shock model associated with correlated pairs of random variables by Sumita and Shanthikumar [27] and the waiting time structure of split queueing systems analyzed by Sumita and Kijima [23] are such examples. Numerical evaluation of these bivariate continuum operations is substantially harder than that for the univariate case. The bivariate Laguerre transform has been developed by Sumita and Kijima [22] and further studied by Kijima [14] for mechanizing various bivariate continuum operations, thereby enhancing numerical exploration of applied probability models with bivariate distributions and bivariate processes. The bivariate formalism is an extension of the univariate Laguerre transform, using the product orthonormal basis generated from the Laguerre functions. This extension is nontrivial since the bivariate transform has many peculiar problems of its own.

In Figure 0.1, the development of the Laguerre transform to date is described. The figure consists of 12 basic components which are classified into two categories: Theory and Applications. Logical relations among these components are indicated by arrows. The purpose of this paper is to provide a concise summary of theoretical aspects of the Laguerre transform. In Section 1, the Laguerre transform is formally introduced. Some useful identities for the Laguerre sharp coefficients and basic operational properties are then summarized. Extensions of the formalism to the matrix form and finite signed measures are also described. Section 2 is devoted to the discussion of the bivariate Laguerre transform. Through a product orthonormal basis generated by the Laguerre functions, the bivariate Laguerre transform is formally introduced. Some useful identities for bivariate sharp coefficients and various operational properties are given next. Of particular interest is the discussion of the minimum and maximum of a pair of correlated random variables through the bivariate Laguerre transform. A discussion of the bivariate Laguerre transform for finite bivariate signed measures is also given. The paper is intended to provide an overview of the development of the Laguerre transform and a collection of the main results with proofs and details, for the most part, omitted.

In a sequel to this paper, we will focus on algorithmic and applicational aspects of the Laguerre transform. In particular, computational procedures will be summarized for finding Laguerre coefficients of many probability density fucntions of interest in applied probability and stochastic processes. Operational properties will be described in an algorithmic form and many successful applications of the Laguerre transform will be reported. Combining these papers, the reader will be able to use the Laguerre transform with ease.

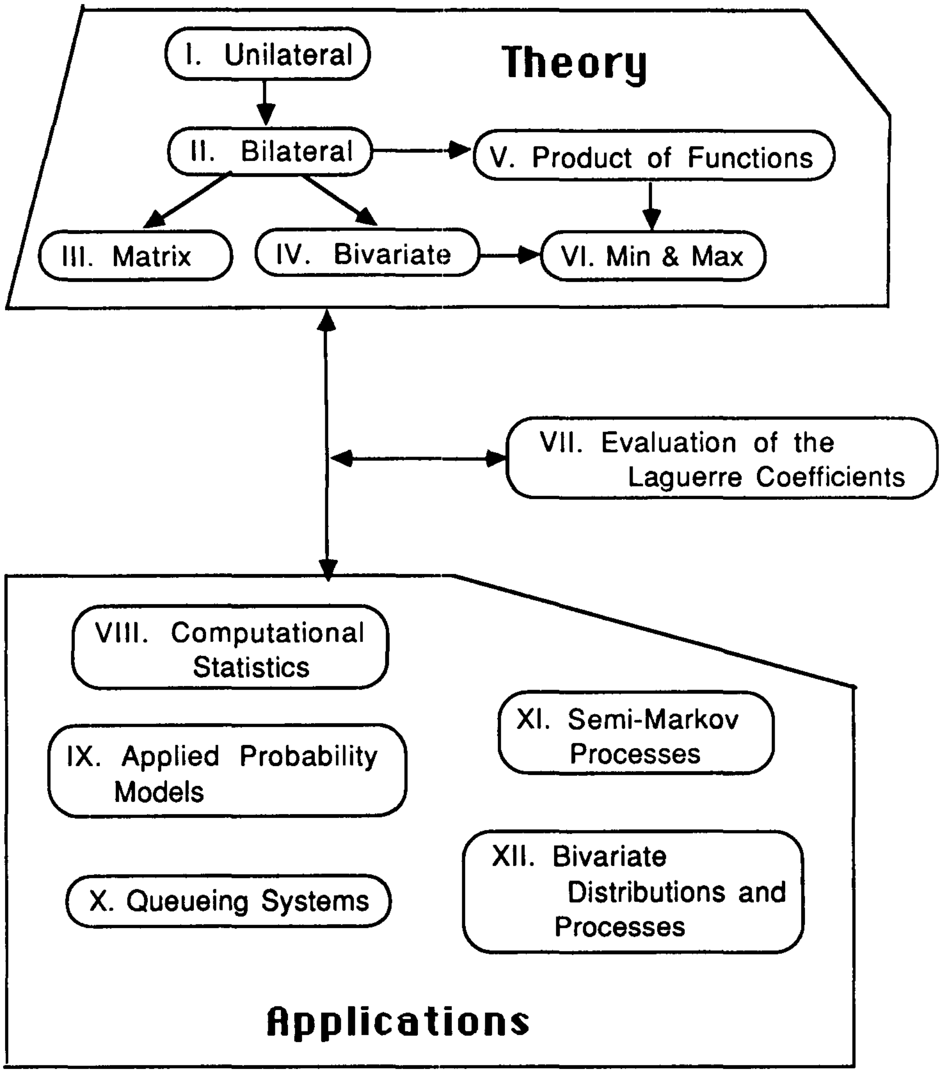

Figure 0.1 Development of the Laguerre Transform Method

§1. Theory of the Univariate Laguerre Transform

§1.1 The Laguerre Transform

The Laguerre polynomials $L_n(x)$ defined by the Rodrigues formula

$$L_n(x) = \left(\frac{e^x}{n!} \frac{d}{dx}\right)^n (x^n e^{-x}) \tag{1.1.1}$$

form a set of orthonormal polynomials with weighting function $w(x) = e^{-x}$ on $(0, \infty)$, see e.g. Szegö [27]. The associated Laguerre functions $\ell_n(x) = e^{-x/2} L_n(x)$ then constitute an orthonormal basis of $L_2(0, \infty) = \{f : \int_0^\infty f^2(x)dx < \infty\}$. To incorporate functions on full continuum, we introduce the extended Laguerre function $h_n(x)$ of order n by

$$h_n(x) = \begin{cases} \ell_n(x)U(x), & \text{n} \geq 0, \\ -\ell_{-n-1}(-x)U(-x), & n < 0, \end{cases} \tag{1.1.2}$$

where $U(x) = 1$ for $x \geq 0$ and $U(x) = 0$ for $x < 0$. One then easily sees that the extended Laguerre functions constitute an orthonormal basis of $L_2(-\infty, \infty) = \{f : \int_{-\infty}^{\infty} f^2(x)dx < \infty\}$. The function space $L_2(-\infty, \infty)$ is a Hilbert space with an inner product $< f, g >= \int_{-\infty}^{\infty} f(x)g(x)dx$. For any $f \in L_2(-\infty, \infty)$, one has the Fourier-Laguerre expansion

$$f(x) = \sum_{n=-\infty}^{\infty} f_n^\dagger h_n(x); \quad f_n^\dagger =< f, h_n >= \int_{-\infty}^{\infty} f(x)h_n(x)dx. \tag{1.1.3}$$

Equality holds in the sense of the limit in the mean, i.e. if we define

$$S_{MN}(x) = \sum_{n=-M}^{N} f_n^\dagger h_n(x), \tag{1.1.4}$$

one has $\lim_{M,N\to\infty} \int_{-\infty}^{\infty} \{f(x) - S_{MN}(x)\}^2 dx = 0$. The speed of convergence of the Laguerre dagger coefficients $(f_n^\dagger)_{-\infty}^{\infty}$ to zero depends on the smoothness and the boundedness of $f(x)$. Let

$$f_+(x) = f(x)U(x); \quad f_-(x) = f(x)U(-x), \tag{1.1.5}$$

so that $f(x) = f_+(x) + f_-(x)$. It has been shown in [7,8,18] that if $f_+(x)$ and $f_-(-x)$ belong to the class of rapidly decreasing functions denoted by $C_\downarrow^\infty(0,\infty)$, then the sequence of the dagger coefficients $(f_n^\dagger)_{-\infty}^\infty$ is also rapidly decreasing in the sense that $|\, n\, |^k |\, f_n^\dagger \,| \to 0$ as $|\, n\, | \to \infty$ for any positive integer k. We define a class of such functions f by

$$C_\downarrow^{\infty *}(-\infty,\infty) = \{f : \ f_+, f_- \in C_\downarrow^\infty(0,\infty)\}. \tag{1.1.6}$$

We note that $f \in C_\downarrow^{\infty *}(-\infty,\infty)$ may have discrepancy at the origin, i.e. $f(x) \,|_{x=0-}^{0+} = f(0+) - f(0-) \neq 0$. It is clear that for any $f \in C_\downarrow^{\infty *}(-\infty,\infty)$, the ordinary pointwise convergence of $S_{MN}(x)$ to $f(x)$ is assured almost everywhere. In what follows, we restrict ourselves to this class for theoretical simplicity. The reader is referred to [7,8,18] for further detailed discussions.

The extended Laguerre functions possess two important properties which play an essential role in developing the Laguerre transform. The first property is that the values of $h_n(x)$ can be generated efficiently through the recursion formula. The Laguerre functions satisfy

$$\ell_{n+1}(x) = \frac{1}{n+1}[(2n+1-x)\ell_n(x) - n\ell_{n-1}(x)], \quad n \geq 1, \tag{1.1.7}$$

starting with $\ell_0(x) = e^{-x/2}$ (see e.g. Rainville [17]). This recursion formula is numerically stable and enables one to generate values of $\ell_n(x)$ with speed and accuracy. Using (1.1.2), the values of $\ell_n(x)$ can be easily converted to those of $h_n(x)$. Hence once $(f_n^\dagger)_{-\infty}^\infty$ is obtained, values of $f(x)$ can be calculated through (1.1.3) efficiently.

The second important property of $h_n(x)$ as a tool for mechanizing continuum operations is the form of the Laplace transform given by

$$\int_{-\infty}^\infty e^{-sx} h_n(x)dx = \frac{1}{s+1/2}\left(\frac{s-1/2}{s+1/2}\right)^n, \quad |\, Re(s)\,| < 1/2, \quad -\infty < n < \infty. \tag{1.1.8}$$

The fact that the index n of $h_n(x)$ appears as a geometric power in its Laplace transform enables one to relate the generating function of the Laguerre coefficients $(f_n^\dagger)_{-\infty}^{\infty}$ of $f(x)$ with the Laplace transform $\phi_f(s) = \int_{-\infty}^{\infty} e^{-sx} f(x)dx$ in a simple manner. Let

$$T_f^\dagger(u) = \sum_{n=-\infty}^{\infty} f_n^\dagger u^n; \quad T_f^\#(u) = (1-u)T_f^\dagger(u) = \sum_{n=-\infty}^{\infty} f_n^\# u^n. \tag{1.1.9}$$

We note that

$$f_n^\dagger = \sum_{m=-\infty}^{n} f_m^\#; \quad f_n^\# = f_n^\dagger - f_{n-1}^\dagger, \quad -\infty < n < \infty. \tag{1.1.10}$$

For $f \in C_\downarrow^{\infty *}(-\infty, \infty)$, one sees from (1.1.3) and (1.1.8) that

$$\phi_f(s) = \sum_{n=-\infty}^{\infty} f_n^\dagger \frac{1}{s+1/2} \left(\frac{s-1/2}{s+1/2}\right)^n. \tag{1.1.11}$$

By letting $u = \frac{s-1/2}{s+1/2}$, Equation (1.1.11) then leads to

$$T_f^\#(u) = \phi_f\left(\frac{1}{2}\frac{1+u}{1-u}\right). \tag{1.1.12}$$

Equation (1.1.11) is valid at least for $s \in Im = \{s : \ s = it, t \in R\}$. Correspondingly, Equation (1.1.12) is valid at least for $u \in A = \{u : \ |u| = 1, u \neq 1\}$.

Equation (1.1.12) is the key formula that provides a bridge between continuum operations and lattice operations. For example, consider the convolution $(f * g)(x) = \int_{-\infty}^{\infty} f(x-t)g(t)dt$ for $f, g \in C_\downarrow^{\infty *}(-\infty, \infty)$. Since $\phi_{f*g}(s) = \phi_f(s)\phi_g(s)$, it can be readily seen from (1.1.12) that

$$T_{f*g}^\#(u) = T_f^\#(u) T_g^\#(u) \tag{1.1.13}$$

or equivalently

$$(f * g)_n^\# = \sum_{m=-\infty}^{\infty} f_{n-m}^\# g_m^\#. \tag{1.1.14}$$

The Laguerre transform maps functions f, g into sequences $(f_n^\#)_{-\infty}^{\infty}$ and $(g_n^\#)_{-\infty}^{\infty}$. Correspondingly, the continuum convolution $(f * g)$ is mapped into a lattice convolution via

(1.1.14). The resulting Laguerre sharp coefficients $((f*g)_n^{\#})_{-\infty}^{\infty}$ can then be inverted back onto the continuum using (1.1.10) and the series representation (1.1.3). As we will see, other basic continuum operations are also mapped into lattice operations conveniently.

§1.2. Some Useful Identities for the Laguerre Sharp Coefficients

The key to accuracy of the Laguerre transform method is the ability to represent a function $f(x)$ with a sequence of Laguerre sharp coefficients of reasonable length. Quantification of truncation error is quite hard, as its counterpart in Fourier series theory where error bounding is known to be quite difficult, and broad and useful results are hard to come by. The key formula (1.1.12) enables one to develop various identities for Laguerre sharp coefficients. Such identities can be useful for heuristically determining how many terms are needed for a given accuracy.

Theorem 1.2A. (Moment Formula)

$$M(i) = \int_{\infty}^{\infty} x^i f(x)dx = 4^i \sum_{n=-\infty}^{\infty} (-1)^n n^i f_n^{\#}, \quad 0 \le i \le 2.$$

Extensive numerical experiments suggest that if truncation points M and N are determined, for a given $\epsilon > 0$, by

$$|M(i) - 4^i \sum_{n=-M}^{N} (-1)^n n^i f_n^{\#}| < \epsilon, \quad 0 \le i \le 2, \tag{1.2.1}$$

it is likely that

$$|f(x) - \sum_{n=-M}^{N} f_n^{\dagger} h_n(x)\,| < \epsilon, \quad x \in R. \tag{1.2.2}$$

For all practical purposes, this criterion for determining the truncation points M and N is satisfactory. Other identities of interest are:

Theorem 1.2B.

(a) $\sum_{n=-\infty}^{\infty} f_n^{\#} = 0.$

(b) $\sum_{n=-\infty}^{\infty} n f_n^{\#} = f(x)|_{x=0-}^{0+}.$

(c) $2\sum_{n=0}^{\infty} f_{2n+1}^{\#} = -\int_0^{\infty} f(x)dx.$

(d) $\sum_{n=-\infty}^{\infty} |n| f_n^{\#2} = \int_{-\infty}^{\infty} |x| f^2(x)dx.$

§1.3. Operational Properties

Using the key formula (1.1.12), the operational properties of Laplace transforms can be converted to those of the Laguerre sharp generating functions, thereby mapping continuum operations into lattice operations. Accordingly, the utility of the Laguerre transform method is enhanced by a large number of simple rules, permitting one to generate Laguerre sharp coefficients needed from other known Laguerre transforms. In this section, we summarize key operational properties of the Laguerre transform. For a given sequence $(a_n)_{-\infty}^{\infty}$, we define the first difference by $\Delta[a_n] = a_n - a_{n-1}$. Higher differences $\Delta^k[a_n]$ are defined similarly.

Theorem 1.3A.

(a) $r(x) = f(-x) \iff r_n^{\#} = f_{-n}^{\#};$

(b) $r(x) = \int_{-\infty}^{\infty} f(x-y)h_m(y)dy \iff r_n^{\#} = \Delta[f_{n-m}^{\#}].$

Theorem 1.3B. (Convolution and Integration)

(a) $r(x) = \int_{-\infty}^{\infty} f(x-y)g(y)dy \iff r_n^{\#} = \sum_{m=-\infty}^{\infty} f_{n-m}^{\#} g_m^{\#};$

(b) $r(x) = \int_{-\infty}^{\infty} f(x+y)g(y)dy \iff r_n^{\#} = \sum_{m=-\infty}^{\infty} f_{n+m}^{\#} g_m^{\#};$

(c) $r(x) = \int_x^{\infty} f(y)dy$, f on $[0,\infty) \iff r_n^{\#} = -2f_n^{\#} + 4\sum_{m=0}^{\infty}(-1)^m f_{m+1+n}^{\#}.$

Remark 1.3C.

The tail integral $\int_x^{\infty} f(y)dy$ for f on $(-\infty,\infty)$ can be evaluated from Theorem 1.3B(c) in the following manner. We recall from (1.1.6) that $f(x) = f_+(x) + f_-(x)$. When $x \geq 0$, one has

$$\int_x^\infty f(y)dy = \int_x^\infty f_+(y)dy. \tag{1.3.1}$$

Since $f_+(x)$ has non-negative support, Theorem 1.3B(c) is directly applicable. For $x < 0$, we note that

$$\int_x^\infty f(y)dy = \int_{-\infty}^\infty f(y)dy - \int_x^\infty f_-(-y)dy. \tag{1.3.2}$$

The first term can be found from Theorem 1.2A with $i = 0$. The second term can be evaluated using Theorem 1.3B(c) since $f_-(-y)$ has support on $[0,\infty)$.

<u>Theorem 1.3D.</u> (Differentiation)

(a) $r(x) = \frac{d}{dx}f(x),\ x \neq 0 \iff r_n^\# = \frac{1}{2}(f_n^\dagger + f_{n-1}^\dagger) - \delta_{no}f(x)|_{x=0-}^{0+}$.

(b) $r(x) = \frac{d}{dx}f(x) + \frac{1}{2}f(x),\ x \neq 0 \iff r_n^\# = f_n^\dagger - f(x)|_{x=0-}^{0+}$.

Here $\delta_{ij} = 1,\ i = j$ and $\delta_{ij} = 0,\ i \neq j$.

<u>Theorem 1.3E.</u> (Multiplication by Polynomial and Exponential Function)

(a) $r(x) = xf(x) \iff r_n^\# = -\Delta^2[(n+1)f_{n+1}^\#]$.

(b) $r(x) = e^{-\theta x}f(x),\ f$ on$[0,\infty) \iff r_n^\# = \sum_{m=0}^\infty f_m^\dagger(p_{nm}(\theta) - p_{n-1,m}(\theta))$,

where $p_{mn}(\theta) = \int_0^\infty e^{-\theta x}\ell_m(x)\ell_n(x)dx$.

<u>Remark 1.3F.</u>

In a series of papers [4,5], Karlin and McGregor study the spectral structure of birth-death processes from an analytic and an applied point of view. In particular, they find in [5] that a linear growth birth-death process $N(t)$ with birth rates $\lambda_n = n+1,\ n \geq 0$ and death rates $\mu_n = n,\ n > 0$ has a spectral representation in terms of the Laguerre polynomials $L_n(x)$ associated with spectral measure e^{-x} on $(0,\infty)$. The matrix $\underline{\underline{P}}(\theta) =$

$[p_{mn}(\theta)]$ appearing in Theorem 1.3E(b) is the transition probability matrix of this birth-death process. A computational procedure for evaluating $p_{mn}(\theta)$ can be found in [20,25].

Theorem 1.3G. (Shifting and Scaling)

(a) $r(x) = f(x-T) \iff r_n^{\#} = \sum_{m=-\infty}^{\infty} f_{n-m}^{\#} \Delta[h_m(T)]$.

(b) $r(x) = e^{-\frac{1}{2}(1-c)x} f(cx),\ f$ on $(0,\infty),\ 0 < c < 1 \iff$

$r_n^{\#} = \Delta[\sum_{m=n}^{\infty} f_m^{\dagger} \binom{m}{n} c^n (1-c)^{m-n}]$.

Remark 1.3H.

It has been shown in [13] that the sequence $(\Delta[h_n(T)])_{-\infty}^{\infty}$ is square summable but is not absolutely summable. Hence the sequence has a rather long tail. Shifting operation can be performed with accuracy only when the tails of $(f_n^{\#})$ disappear rapidly.

It is well known that the Laplace transform does not have any operational property for product of functions. Hence for developing the Laguerre transform of a product of functions, a different approach is necessary.

Theorem 1.3I (Product of Functions)

$$r(x) = f(x)g(x),\ f, g \text{ on } [0,\infty] \iff r_n^{\dagger} = \underline{f}^{\dagger T} \underline{\underline{A}}(n) \underline{g}^{\dagger}, \quad n \geq 0,$$

where $\underline{f}^{\dagger} = [f_0^{\dagger}, f_1^{\dagger}, \ldots]$, $\underline{g}^{\dagger} = [g_0^{\dagger}, g_1^{\dagger}, \ldots]$, and $\underline{\underline{A}}(n) = [a_{ij}(n)]$ with $a_{ij}(n) = \int_0^{\infty} l_i(x) l_j(x) l_n(x) dx$, $i, j, n \geq 0$.

Matrices $\underline{\underline{A}}(n)$ can be computed efficiently based on a recursion formula, see [14,25].

Remark 1.3J

Let $f(x) = f_+(x) + f_-(x)$ and let $g(x) = g_+(x) + g_-(x)$. One then easily sees that $f(x)g(x) = f_+(x)g_+(x) + f_-(x)g_-(x)$. Hence a product of functions on $(-\infty, +\infty)$ can easily be treated based on Theorem 1.3I.

§1.4. The Matrix Laguerre Transform

A $K \times K$ matrix function $\underline{\underline{a}}(x) = [a_{ij}(x)]$ is called a matrix p.d.f. if $a_{ij}(x) \geq 0$ for all $i,\ j,\ -\infty < x < \infty$, and if $\underline{\underline{a}} = \int_{-\infty}^{\infty} \underline{\underline{a}}(x)dx$ is a stochastic matrix. In the study of additive processes on finite Markov chains and related Markov renewal processes (see Part II), one often encounters matrix convolution of two such matrix p.d.f.'s

$$\underline{\underline{g}}(x) = \underline{\underline{a}}(x) * \underline{\underline{b}}(x) = \int_{-\infty}^{\infty} \underline{\underline{a}}(x-y)\underline{\underline{b}}(y)dy. \tag{1.4.1}$$

Hence it is of interest to extend the Laguerre transform procedure to this matrix setting.

Let $\underline{\underline{L}}_2(-\infty,\infty) = \{\underline{\underline{a}}(x) = [a_{ij}(x)] : a_{ij}(x) \in L_2(-\infty,\infty)\}$. Then the space $\underline{\underline{L}}_2(-\infty,\infty)$ is a Hilbert space with an inner product

$$< \underline{\underline{a}}(x),\ \underline{\underline{b}}(x) >= \frac{1}{K}\int_{-\infty}^{\infty} tr\{\underline{\underline{a}}(x)\underline{\underline{b}}(x)^T\}dx. \tag{1.4.2}$$

The set of matrix functions $\{\underline{\underline{h}}_n(x)\}$ with $\underline{\underline{h}}_n(x) = h_n(x)\underline{\underline{I}}$ becomes an orthonormal basis of $\underline{\underline{L}}_2(-\infty,\infty)$ in the following matrix sense. For any $\underline{\underline{a}}(x) \in \underline{\underline{L}}_2(-\infty,\infty)$ there exists a unique sequence of the Laguerre dagger coefficient matrices $(\underline{\underline{a}}_n^\dagger)_{-\infty}^{\infty}$ such that

$$\underline{\underline{a}}(x) = \sum_{n=-\infty}^{\infty} \underline{\underline{a}}_n^\dagger \underline{\underline{h}}_n(x); \quad \underline{\underline{a}}_n^\dagger = \int_{-\infty}^{\infty} \underline{\underline{a}}(x)\underline{\underline{h}}_n(x)dx. \tag{1.4.3}$$

The key formula (1.1.12) for the scalar Laguerre transform now extends to the matrix form naturally. Let $\underline{\underline{\alpha}}(s) = \int_{-\infty}^{\infty} e^{-sx}\underline{\underline{a}}(x)dx$. From (1.4.3), one then has

$$\underline{\underline{\alpha}}(s) = \sum_{n=-\infty}^{\infty} \underline{\underline{a}}_n^\dagger \frac{1}{s+1/2}\left(\frac{s-1/2}{s+1/2}\right)^n. \tag{1.4.4}$$

The matrix generating functions $\underline{\underline{T}}_a^\dagger(u) = \sum_{n=-\infty}^{\infty} \underline{\underline{a}}_n^\dagger u^n$ and $\underline{\underline{T}}_a^\#(u) = \sum_{n=-\infty}^{\infty} \underline{\underline{a}}_n^\# u^n$ with $\underline{\underline{a}}_n^\# = \underline{\underline{a}}_n^\dagger - \underline{\underline{a}}_{n-1}^\dagger$ then lead to the matrix version of the key formula

$$\underline{\underline{T}}_a^\#(u) = (1-u)\underline{\underline{T}}_a^\dagger(u) = \underline{\underline{\alpha}}\left(\frac{1}{2}\frac{1+u}{1-u}\right). \tag{1.4.5}$$

For matrix convolution $\underline{\underline{g}}(x) = \underline{\underline{a}}(x) * \underline{\underline{b}}(x)$, one has $\underline{\underline{\gamma}}(s) = \underline{\underline{\alpha}}(s)\underline{\underline{\beta}}(s)$ so that, as for the scalar case,

$$\underline{\underline{T}}_g^{\#}(u) = \underline{\underline{T}}_a^{\#}(u)\underline{\underline{T}}_b^{\#}(u). \tag{1.4.6}$$

Equation (1.4.6) implies that

$$\underline{\underline{g}}_n^{\#} = \sum_{m=-\infty}^{\infty} \underline{\underline{a}}_{n-m}^{\#} \underline{\underline{b}}_m^{\#}, \tag{1.4.7}$$

providing again an algorithmic basis for evaluating matrix convolution. Other operational properties can be developed similarly.

§1.5. Extension to Finite Signed Measures

It has been shown in [13,18] that the Laguerre sharp transform maps every finite measure μ on $(-\infty, \infty)$ into a square summable sequence $(\mu_n^{\#})$ and there is a one to one correspondence between the probability measures and their sharp transforms. The Laguerre sharp norm in this context can then be used as a distance between any two probability measures. The norm is a practical tool for measuring convergence of iterative procedures and provides a stopping criterion for such procedures. This section gives a brief summary of these interesting results.

For any finite signed measure μ on $(-\infty, \infty)$, let $F_\mu(x) = \int_{-\infty}^{x} \mu(dx')$. Since $\Delta h_n(x)$ are uniformly bounded by one, the Laguerre sharp coefficients

$$\mu_n^{\#} = \int_{-\infty}^{\infty} \Delta[h_n(x)]dF_\mu(x). \tag{1.5.1}$$

are well defined. One then has:

Theorem 1.5A. Every finite signed measure μ is mapped into a square summable sequence $(\mu_n^{\#})_{-\infty}^{\infty}$ by (1.5.1). This Laguerre sharp transform $\mu \leftrightarrow (\mu_n^{\#})_{-\infty}^{\infty}$ is a one to one mapping.

Theorem 1.5B. For a finite signed measure μ, let $\phi_\mu(s) = \int_{-\infty}^{\infty} e^{-sx} dF_\mu(x)$. Define $T_\mu^\#(u) = \sum_{n=-\infty}^{\infty} \mu_n^\# u^n$. Then

$$T_\mu^\#(u) = \phi_\mu\left(\frac{1}{2}\frac{1+u}{1-u}\right). \tag{1.5.2}$$

Theorem 1.5C. For a finite signed measure μ, let $\|\mu\|_2^\# = \sqrt{\sum_{n=-\infty}^{\infty}(\mu_n^\#)^2}$. Then a sequence of probability measures μ_j converges weakly to μ if and only if $\|\mu_j - \mu\|_2^\# \to 0$ as $j \to \infty$.

Theorem 1.5D. Let μ and ν be probability measures. Then $\|\mu * \nu\|_2^\# \leq \|\mu\|_2^\# \|\nu\|_2^\#$.

§2. Theory of the Bivariate Laguerre Transform

§2.1. The Bivariate Laguerre Transform

Let $L_2(R^2) = \{f : \int\int_{R^2} f^2(x,y)dx\, dy < \infty\}$ and define

$$h_{mn}(x,y) = h_m(x)h_n(y), \quad -\infty < x, y < \infty. \tag{2.1.1}$$

It is then easy to see that $\{h_{mn}(x,y)\}$ is an orthonormal basis of $L_2(R^2)$. For any $f \in L_2(R^2)$, the Laguerre dagger coefficients $f_{mn}^\dagger$ are given by

$$f_{mn}^\dagger = \langle f, h_{mn}\rangle = \int\int_{R^2} f(x,y)h_{mn}(x,y)dx\, dy, (m,n) \in Z^2. \tag{2.1.2}$$

One has the Fourier-Laguerre series expansion

$$f(x,y) = \lim_{k\to\infty} S_{I(k)}(x,y); \quad S_{I(k)}(x,y) = \sum_{(m,n)\in I(k)}\sum f_{mn}^\dagger h_{mn}(x,y) \tag{2.1.3}$$

where $I(k)$ is a sequence of nested compact sets in Z^2 such that $\cup_k I(k) = Z^2$.

For any $f, g \in L_2(R^2)$ with $(f_{mn}^\dagger)$, $(g_{mn}^\dagger)$, we have

$$\langle f, g\rangle = \int\int_{R^2} f(x,y)g(x,y)dxdy = \sum_{(m,n)\in Z^2}\sum f_{mn}^\dagger g_{mn}^\dagger \tag{2.1.4}$$

and

$$\|f\|_2^2 = \int\int_{R^2} f^2(x,y)dx\ dy = \sum_{(m,n)\in Z^2}\sum f_{mn}^{\dagger 2}. \tag{2.1.5}$$

Let $f_{++}(x,y) = f(x,y)U(x)U(y)$, $f_{+-}(x,y) = f(x,y)U(x)U(-y)$, $f_{-+}(x,y)$ =f(x,y)U(-x)U(y) and $f_{--}(x,y) = f(x,y)U(-x)U(-y)$ so that $f(x,y) = \sum_{a,b\in\{1,-1\}} f_{ab}(x,y)$, where $f_{ab}(x,y) = f_{+-}(x,y)$ if $a = 1$ and $b = -1$, etc. If each $f_{ab}(ax, by)$ belongs to the class of rapidly decreasing bivariate functions, we say that $f \in C_{\downarrow}^{\infty *}(R^2)$, the bivariate counterpart of (1.1.6). Note that $f \in C_{\downarrow}^{\infty *}(R^2)$ possibly has discrepancies on the set $\{(x,y):\ xy = 0\}$. If $f \in C_{\downarrow}^{\infty *}(R^2)$ then the ordinary pointwise convergence of $S_{I(k)}(x,y)$ to $f(x,y)$ is guaranteed almost everywhere. In what follows, we treat functions belonging to $C_{\downarrow}^{\infty *}(R^2)$ only. The reader is referred to [14,22] for detailed discussions.

We note from (1.18) and (2.1.1) that for all $(m,n) \in Z^2$

$$\int\int_{R^2} e^{-sx-wy}h_{mn}(x,y)dxdy = \frac{1}{s+\frac{1}{2}}\left(\frac{s-\frac{1}{2}}{s+\frac{1}{2}}\right)^m \frac{1}{w+\frac{1}{2}}\left(\frac{w-\frac{1}{2}}{w+\frac{1}{2}}\right)^n,\quad \max\{|Re(s)|,|Re(w)|\} < \frac{1}{2}. \tag{2.1.6}$$

For $f \in C_{\downarrow}^{\infty *}(R^2) \subset L_1(R^2)$, its bivariate Laplace transform

$$\phi_f(s,w) = \int\int_{R^2} e^{-sx-wy}f(x,y)dxdy$$

is well defined at least for $(s,w) \in Im \times Im = \{(s,w)|s = ia,\ w = ib,\ a,b \in R\}$. Hence, for such (s,w), $\phi_f(s,w)$ can be formally obtained from (2.13) by interchanging the summation and the integration, and then by using (2.1.6). Namely, one has

$$\phi_f(s,w) = \sum_{m=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} f_{mn}^{\dagger}\frac{1}{s+\frac{1}{2}}\left(\frac{s-\frac{1}{2}}{s+\frac{1}{2}}\right)^m \frac{1}{w+\frac{1}{2}}\left(\frac{w-\frac{1}{2}}{w+\frac{1}{2}}\right)^n,\quad (s,w) \in Im \times Im. \tag{2.1.7}$$

We now define the bivariate generating functions $T_f^{\dagger}(u,v)$ and $T_f^{\#}(u,v)$ by

$$T_f^{\dagger}(u,v) \stackrel{\text{def}}{=} \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} f_{mn}^{\dagger}u^m v^n \tag{2.1.8}$$

and

$$T_f^{\#}(u,v) \stackrel{\text{def}}{=} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f_{mn}^{\#} u^m v^n \stackrel{\text{def}}{=} (1-u)(1-v)T_f^{\dagger}(u,v). \tag{2.1.9}$$

It should be noted that the dagger coefficients $(f_{mn}^{\dagger})$ and the sharp coefficients $(f_{mn}^{\#})$ are related in the following manner. For a bivariate sequence (a_{mn}), we define the three types of the first differences:

$$\Delta_1(a_{mn}) = a_{mn} - a_{m-1,n}; \quad \Delta_2(a_{mn}) = a_{mn} - a_{m,n-1}; \tag{2.1.10}$$

$$\underline{\Delta}(a_{mn}) = \Delta_1\Delta_2(a_{mn}) = \Delta_2\Delta_1(a_{mn}). \tag{2.1.11}$$

The higher differences are denoted by Δ_1^k, Δ_2^k, and $\underline{\Delta}^k$. One then has

$$f_{mn}^{\#} = \underline{\Delta}(f_{mn}^{\dagger}); \quad f_{mn}^{\dagger} = \sum_{i=-\infty}^{m} \sum_{j=-\infty}^{n} f_{ij}^{\#}. \tag{2.1.12}$$

By letting $u = (s-\frac{1}{2})/(s+\frac{1}{2})$ and $v = (w-\frac{1}{2})/(w+\frac{1}{2})$, one sees from (2.1.7) through (2.1.9) that

$$T_f^{\#}(u,v) = \phi_f\left(\frac{1}{2}\frac{1+u}{1-u}, \frac{1}{2}\frac{1+v}{1-v}\right). \tag{2.1.13}$$

We note that Equation (2.1.13) is valid for (u,v) being in the set A where

$$A = \{(u,v): \ |u| = |v| = 1, \ u \neq 1, \ v \neq 1\}. \tag{2.1.14}$$

Equation (2.1.13) is the key formula, providing a bridge between bivariate continuum operations and bivariate lattice operations. For example, consider the bivariate convolution

$$(f * g)(x,y) = \int\int_{R^2} f(x-x', y-y')g(x',y')dx'dy'$$

for $f, g \in C_{\downarrow}^{\infty *}(R^2)$. Since $\phi_{f*g}(s,w) = \phi_f(s,w)\phi_g(s,w)$, one has from (1.13) that

$$T_{f*g}^{\#}(u,v) = T_f^{\#}(u,v)T_g^{\#}(u,v), \tag{2.1.15}$$

or equivalently

$$(f * g)^{\#}_{mn} = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} f^{\#}_{m-i,n-j} g^{\#}_{ij}. \tag{2.1.16}$$

The bivariate Laguerre transform maps functions f, g into bivariate sequences $(f^{\#}_{mn}), (g^{\#}_{mn})$ by (2.1.13). Correspondingly, their bivariate continuum convolution is mapped into a bivariate lattice convolution via (2.1.15). The resulting sharp transform $(f * g)^{\#}_{mn}$ can then be inverted back onto the continuum using (2.1.12) and the series representation (2.1.3).

§2.2. Some Useful Identities for the Bivariate Sharp Coefficients

As for sharp coefficients $(f^{\#}_n)$ of the univariate case, there also exist various identities for bivariate sharp coefficients $(f^{\#}_{mn})$. The first two moments of $f(x,y)$, for example, are conveniently obtained from $(f^{\#}_{mn})$ as we will see. These identities are not only of theoretical interest but also of practical importance. Since storage capacity of computers is limited, the series expansion (2.1.3) must be truncated. Because of the bivariate nature of the sequence $(f^{\#}_{mn})$, developing a means that heuristically determines truncation points in (2.1.3) for a given accuracy is of great importance.

Theorem 2.2A. (Moment Formulas)

$$M(i,j) = \int\int_{R^2} x^i y^j f(x,y)dxdy = 4^{i+j} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} (-1)^{m+n} m^i n^j f^{\#}_{mn}, \quad 0 \le i,\ j \le 2.$$

For many functions the values of moments $M(i,j)$, $0 \le i,\ j \le 2$ are known. Extensive numerical experiments suggest that if truncation points M_1, M_2, N_1 and N_2 are determined, for a given $\epsilon > 0$, by

$$|M(i,j) - 4^{i+j} \sum_{m=-M_1}^{M_2} \sum_{n=-N_1}^{N_2} (-1)^{m+n} m^i n^j f^{\#}_{mn}| < \epsilon, \quad 0 \le i,\ j \le 2, \tag{2.2.1}$$

it is likely that

$$|f(x,y) - \sum_{m=-M_1}^{M_2} \sum_{n=-N_1}^{N_2} f^{\dagger}_{mn} h_{mn}(x,y)| < C\epsilon, \quad -\infty < x,y < \infty.$$

The constant C depends on a function f but is typically between 1 and 1000. Other identities of interest are:

Theorem 2.2B.

(a) $\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} mn f^{\#}_{mn} = f(x,y)|_{x=0-}^{0+}|_{y=0-}^{0+}$

(b) $4\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} f^{\#}_{2m+1,2n+1} = \int_0^{\infty}\int_0^{\infty} f(x,y)dx\ dy$

(c) $4\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} f^{\#}_{2m+1,-2n-1} = \int_0^{\infty}\int_{-\infty}^{0} f(x,y)dx\ dy$

(d) $4\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} f^{\#}_{-2m-1,2n+1} = \int_{-\infty}^{0}\int_0^{\infty} f(x,y)dx\ dy$

(e) $4\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} f^{\#}_{-2m-1,-2n-1} = \int_{-\infty}^{0}\int_{-\infty}^{0} f(x,y)dx\ dy$

§2.3. Bivariate Operational Properties

In this section, various operational properties of the bivariate Laguerre transform are provided via the key formula (2.1.13). As for the univariate case, the bivariate Laguerre transform bypasses numerical integration, multiple convolutions and differentiation. All of these basic continuum operations are conveniently mapped into lattice operations, thereby providing powerful numerical tools.

Theorem 2.3A. (Symmetry)

(a) $r(x,y) = f(y,x) \iff r^{\#}_{mn} = f^{\#}_{nm}.$

(b) $r(x,y) = f(-x,y) \iff r^{\#}_{mn} = f^{\#}_{-m,n}.$

(c) $r(x,y) = f(x,-y) \iff r^{\#}_{mn} = f^{\#}_{m,-n}.$

Theorem 2.3B. (Convolution)

(a) $r(x,y) = \int\int_{R^2} f(x-x',y-y')g(x',y')dx'dy'$

$\iff r^{\#}_{mn} = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} f^{\#}_{m-i,n-j} g^{\#}_{ij}$.

(b) $r(x) = \int_{-\infty}^{\infty} f(x-y,y)dy \iff r^{\#}_m = \sum_{n=-\infty}^{\infty} f^{\#}_{m-n,n}$.

(c) $r(x) = \int_{-\infty}^{\infty} f(x+y,y)dy \iff r^{\#}_m = \sum_{n=-\infty}^{\infty} f^{\#}_{m+n,n}$.

Let $g(x) \in C_{\downarrow}^{\infty *}(R)$ with $(g^{\#}_n)_{-\infty}^{\infty}$.

(d) $r(x,y) = \int_{-\infty}^{\infty} g(x-x')f(x',y)dx' \iff r^{\#}_{mn} = \sum_{k=-\infty}^{\infty} g^{\#}_{m-k} f^{\#}_{kn}$.

(e) $r(x,y) = \int_{-\infty}^{\infty} g(x-x')f(x,y')dy' \iff r^{\#}_{mn} = \sum_{k=-\infty}^{\infty} g^{\#}_{n-k} f^{\#}_{mk}$.

Theorem 2.3C. (Marginal Functions)

(a) $r(x) = \int_{-\infty}^{\infty} f(x,y)dy \iff r^{\#}_m = \sum_{n=-\infty}^{\infty} (-1)^n f^{\#}_{mn}$.

(b) $r(y) = \int_{-\infty}^{\infty} f(x,y)dx \iff r^{\#}_n = \sum_{m=-\infty}^{\infty} (-1)^m f^{\#}_{mn}$.

Theorem 2.3D. (Integration)

Let $f: R^2_+ \to R$ where $R^2_+ = [0,\infty) \times [0,\infty)$.

(a) $r(x,y) = \int_x^{\infty} f(x',y)dx' \iff r^{\dagger}_{mn} = 2\sum_{i=m+1}^{\infty} \sum_{j=0}^{n} (-1)^{m+i} f^{\#}_{ij}$.

(b) $r(x,y) = \int_y^{\infty} f(x,y')dy' \iff r^{\dagger}_{mn} = 2\sum_{i=0}^{m} \sum_{=n+1}^{\infty} (-1)^{n+j} f^{\#}_{ij}$.

(c) $r(x,y) = \int_y^{\infty} \int_x^{\infty} f(x',y')dx'dy' \iff r^{\dagger}_{mn} = 4\sum_{i=m+1}^{\infty} \sum_{j=n+1}^{\infty} (-1)^{m+n+i+j} f^{\#}_{ij}$.

Remark 2.3E. For $f: R^2_+ \to R$, values of $r(x,y) = \int_0^y \int_0^x (x',y')dx'dy'$ can be found easily since

$$r(x,y) = \int_0^{\infty}\int_0^{\infty} f(x,y)dx\, dy + \int_y^{\infty}\int_x^{\infty} f(x',y')dx'dy' - \int_x^{\infty} f_X(x')dx' - \int_y^{\infty} f_Y(y')dy' \tag{2.3.1}$$

where $f_X(x) = \int_0^{\infty} f(x,y)dy$ and $f_Y(y) = \int_0^{\infty} f(x,y)dx$. The first term in (2.3.1) is evaluated from Theorem 2.2A with $i = j = 0$ and the second term is obtained from Theorem 2.3D(c). $f_X(x)$ and $f_Y(y)$ are the marginal functions (see Theorem 2.3C) so that the third and fourth terms are calculated using Theorem 1.3B(c).

Remark 2.3F. For functions $f: R^2 \to R$, the integration procedure is slightly more complicated. As in §2.1, we decompose f into four functions defined on each orthant, i.e. $f(x,y) = \sum_{a,b\in\{1,-1\}} f_{ab}(x,y)$. Clearly $f_{ab}(ax,by)$ is defined on R_+^2. Values of $\int_x^\infty f(x',y)dx', \int_{-\infty}^x f(x',y)dx', \int_y^\infty \int_x^\infty f(x',y')dx'\,dy'$, etc. for any x and y are then evaluated by applying Theorem 2.3D and Remark 2.3E appropriately to $f_{ab}(ax,by)$ and the univariate Laguerre transform. For example, for $x,y<0$,

$$\int_y^\infty \int_x^\infty f(x',y')dx'\,dy' = \int_x^\infty f_X(x')dx' - \int_{-\infty}^{y} f_Y(y')dy' + \int_{-y}^\infty \int_{-x}^\infty f_{--}(-x',-y')dx'\,dy'.$$

<u>Theorem 2.3G.</u> (Partial Differentiation)

Let f be differentiable everywhere.

(a) $r(x,y) = \frac{\partial}{\partial x} f(x,y) \iff r_{mn}^{\#} = \frac{1}{2}\Delta_2[f_{mn}^{\dagger} + f_{m-1,n}^{\dagger}]$.

(b) $r(x,y) = \frac{\partial}{\partial y} f(x,y) \iff r_{mn}^{\#} = \frac{1}{2}\Delta_1[f_{mn}^{\dagger} + f_{m,n-1}^{\dagger}]$.

(c) $r(x,y) = \frac{\partial^2}{\partial x \partial y} f(x,y) \iff r_{mn}^{\#} = \frac{1}{4}(f_{mn}^{\dagger} + f_{m-1,n}^{\dagger} + f_{m,n-1}^{\dagger} + f_{m-1,n-1}^{\dagger})$.

<u>Theorem 2.3H.</u> (Multiplication by Polynomial and Exponential Function)

(a) $r(x,y) = xf(x,y) \iff r_{mn}^{\#} = -\Delta_1^2[(m+1)f_{m+1,n}^{\#}]$.

(b) $r(x,y) = yf(x,y) \iff r_{mn}^{\#} = -\Delta_2^2[(n+1)f_{m,n+1}^{\#}]$.

(c) $r(x,y) = xyf(x,y) \iff r_{mn}^{\#} = \underline{\Delta}^2[(m+1)(n+1)f_{m+1,n+1}^{\#}]$.

(d) $r(x,y) = e^{-\theta_1 x - \theta_2 y} f(x,y)$, f on R_+^2 $\iff$

$r_{mn}^{\#} = \sum_{i=0}^\infty \sum_{j=0}^\infty f_{ij}^{\dagger}(p_{im}(\theta_1) - p_{i,m-1}(\theta_1))(p_{jn}(\theta_2) - p_{j,n-1}(\theta_2))$, where $p_{mn}(\theta)$ is as in (1.4.1).

<u>Theorem 2.3I.</u> (Shifting and Scaling)

(a) $r(x,y) = f(x-a,y-b) \iff r_{mn}^{\#} = \sum_{i=-\infty}^\infty \sum_{j=-\infty}^\infty f_{m-i,n-j}^{\#} \underline{\Delta}[h_{ij}(a,b)]$.

(b) $r(x,y) = e^{-\frac{1}{2}(1-a)x - \frac{1}{2}(1-b)y} f(ax,by)$, f on R_+^2 $\iff$

$r_{mn}^{\#} = \underline{\Delta}[\sum_{m'=m}^\infty \sum_{n'=n}^\infty f_{m'n'}^{\dagger} \binom{m'}{m}\binom{n'}{n} a^m b^n (1-a)^{m'-m}(1-b)^{n'-n}]$.

§2.4. Minimum and Maximum of a Pair of Correlated Random Variables

Let (X, Y) be a pair of correlated nonnegative random variables. In certain reliability models, one often encounters the lifetime distributions involving the minimum $V = min\{X, Y\}$ and the maximum $W = max\{X, Y\}$. If stochastic analysis of the underlying model requires multiple convolutions and other continuum operations involving the probability density functions (p.d.f.'s) of V and W, numerical evaluation of the model becomes quite tedious. The purpose of this section is to develop an algorithmic procedure for calculating the Laguerre coefficients corresponding to the p.d.f.'s of V and W in terms of those of the joint p.d.f. of (X, Y). Detailed discussions are found in [14,24].

Let $F_{XY}(x, y)$ be the joint cumulative distribution function (c.d.f.) of (X, Y) and let $F_X(x) = F_{XY}(x, \infty)$ and $F_Y(y) = F_{XY}(\infty, y)$. The c.d.f.'s of V and W are denoted by $F_V(x)$ and $F_W(x)$ respectively. We assume that the c.d.f. $F_{XY}(x, y)$ is absolutely continuous with a joint p.d.f. $f_{XY}(x, y) = \frac{\partial^2}{\partial x \partial y} F_{XY}(x, y)$. For notational convenience, we define $\overline{F}_{XY}(x, y) = P[X > x, Y > y]$. The survival function $\overline{F}(x)$ of a c.d.f. $F(x)$ is defined in the ordinary manner. It is further assumed that $f_{XY}(x, y)$ is a rapidly decreasing function.

It can be readily seen that

$$\overline{F}_V(x) = \overline{F}_{XY}(x, x); \ \overline{F}_W(x) = \overline{F}_{XY}(x, 0) + \overline{F}_{XY}(0, x) - \overline{F}_V(x). \tag{2.4.1}$$

Using (2.1.12) and Theorem 2.3D(c), the dagger coefficients $(f^{\dagger}_{XY;mn})$ of $f_{XY}(x, y)$ can be converted to the dagger coefficients $(\overline{F}^{\dagger}_{XY;mn})$ of $\overline{F}_{XY}(x, y)$. Hence both $\overline{F}_V(x)$ and $\overline{F}_W(x)$ can be evaluated straightforwardly using $(\overline{F}^{\dagger}_{XY;mn})$ and the series representation

(2.1.3), i.e.

$$\overline{F}_V(x) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\overline{F}^{\dagger}_{XY;mn}\ell_m(x)\ell_n(x), \tag{2.4.2}$$

and

$$\overline{F}_W(x) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\overline{F}^{\dagger}_{XY;mn}(\ell_m(x) + \ell_n(x) - \ell_m(x)\ell_n(x)). \tag{2.4.3}$$

The dagger coefficients $(\overline{F}^{\dagger}_{Vn})$ and $(\overline{F}^{\dagger}_{Wn})$ can now be obtained from (2.4.2) and (2.4.3). For two matrices $\underline{\underline{b}} = [b_{mn}]$ and $\underline{\underline{c}} = [c_{mn}], m, n \geq 0$, we define

$$+/+/\underline{\underline{b}} \otimes \underline{\underline{c}} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} b_{mn}c_{mn}. \tag{2.4.4}$$

<u>Theorem 2.4A.</u> Let $\underline{\underline{\overline{F}}}^{\dagger}_{XY} = [\overline{F}^{\dagger}_{XY;mn}], m, n \geq 0$. Then

$$\overline{F}^{\dagger}_{Vk} = +/+/\underline{\underline{\overline{F}}}^{\dagger}_{XY} \otimes \underline{\underline{A}}(k), \quad k \geq 0.$$

$$\overline{F}^{\dagger}_{Wk} = \sum_{n=0}^{\infty}\overline{F}^{\dagger}_{XY;kn} + \sum_{m=0}^{\infty}\overline{F}^{\dagger}_{XY;mk} - \overline{F}^{\dagger}_{Vk}, \quad k \geq 0.$$

The matrix $\underline{\underline{A}}(k), k \geq 0$ has been introduced in Theorem 1.3I.

The Laguerre dagger coefficients $(\overline{F}^{\dagger}_{Vn})$ and $(\overline{F}^{\dagger}_{Wn})$ should be obtained if analysis of the underlying stochastic model requires multiple convolutions and other continuum operations involving V and W. When only the moments of V and W are needed, however, one can bypass the computation of the Laguerre coefficients.

<u>Theorem 2.4B</u> Let $\underline{\underline{V}} = \lim_{M\to\infty}(-\underline{\underline{Q}}(M))$ where $\underline{\underline{Q}}(M)$ is defined in (1.4.6). Then

(a) $E[V^k] = k(+/+/\underline{\underline{\overline{F}}}^{\dagger}_{XY} \otimes \underline{\underline{V}}^{k-1}), \quad k \geq 1.$

(b) $E[W^k] = E[X^k] + E[Y^k] - E[V^k], \quad k \geq 1.$

We note from Theorem 2.3C that the Laguerre sharp coefficients $(f^{\#}_{Xn})$ and $(f^{\#}_{Yn})$ for the marginal densities $f_X(x)$ and $f_Y(y)$ respectively can be generated easily from $(f^{\#}_{XY\,mn})$. By applying Theorem 1.2A repeatedly, $E[X^k]$ and $E[Y^k]$ are computed.

Of related interest is the probability

$$P[X \le Y] = \int_0^\infty dx \int_x^\infty f_{XY}(x,y)dy.$$

Let $r(x,y) = \int_y^\infty f_{XY}(x,y')dy'$. The Laguerre coefficients $(r^\dagger_{mn})$ of $r(x,y)$ can be obtained from Theorem 2.3D(b). Then

$$P[X \le Y] = \int_0^\infty r(x,x)dx = \sum_{m=0}^\infty \sum_{n=0}^\infty r^\dagger_{mn}\delta_{mn}. \tag{2.4.5}$$

The probability $P[X > Y]$ is found by $1 - P[X \le Y]$.

§2.5. Extension to Finite Bivariate Signed Measures

An extension of the univariate Laguerre sharp transform to finite signed measures has been discussed in §1.6. In this section, we describe the bivariate counterpart of this extension (see also [14]). For any finite bivariate signed measures μ on R^2, let $F_\mu(x,y) = \int_{-\infty}^y \int_{-\infty}^x \mu(dx', dy')$. Since $\underline{\Delta}[h_{mn}(x,y)]$ is uniformly bounded, one has the Laguerre sharp coefficients

$$\mu^\#_{mn} = \int\int_{R^2} \underline{\Delta}[h_{mn}(x,y)]dF_\mu(x,y). \tag{2.5.1}$$

In parallel with Theorems 1.6A through 1.6D, the following theorems hold.

Theorem 2.5A. Every finite bivariate signed measure μ is mapped into a square summable sequence $(\mu^\#_{mn})_{m,n=-\infty}^\infty$ by (2.5.1). This Laguerre sharp transform $\mu \leftrightarrow (\mu^\#_{mn})$ is a one to one mapping.

Theorem 2.5B. For a finite bivariate signed measure μ, let $\phi_\mu(s,w) = \int\int_{R^2} e^{-sx-wy}dF_\mu(x,y)$. Define $T^\#_\mu(u,v) = \sum_{m=-\infty}^\infty \sum_{n=-\infty}^\infty \mu^\#_{mn}u^m v^n$. Then

$$T^\#_\mu(u,v) = \phi_\mu\left(\frac{1}{2}\frac{1+u}{1-u}, \frac{1}{2}\frac{1+v}{1-v}\right). \tag{2.5.2}$$

Theorem 2.5C. For a finite bivariate signed measure μ, let $\|\mu\|_2^\# = \sqrt{\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} (\mu_{mn}^\#)^2}$. Then a sequence of bivariate probability measures μ_j converges weakly to μ if and only if $\|\mu_j - \mu\|_2^\# \to 0$ as $j \to \infty$.

Theorem 2.5D. Let μ and ν be bivariate probability measures. Then $\|\mu * \nu\|_2^\# \leq \|\mu\|_2^\# \|\nu\|_2^\#$.

ACKNOWLEDGMENT

The authors wish to thank Sue North and Paulo Goes for their extensive technical contributions.

REFERENCES

[1] Beran, R.J. (1968), "Testing for Uniformity on Compact Homogeneous Space," *Journal of Applied Probability*, Vol. 5, pp. 177-195.

[2] Gaver, D.P. (1962), "A Waiting Line with Interruption Service, Including Priorities," *Journal of the Royal Statistical Society, Series B*, Vol. 24, pp. 73-90.

[3] Giné, M. (1975), "Invariant Tests for Uniformity on Compact Riemannian Manifolds based on Sobolev Norms," *The Annals of Statistics*, Vol. 3, pp. 1243-1266.

[4] Karlin, S. and McGregor, J. (1957), "The Differential Equations of Birth and Death Processes and the Stieltjes Moment Problem," *Transactions of the American Mathematical Society*, Vol. 85, pp. 489-546.

[5] Karlin, S. and McGregor, J. (1958), "Linear Growth Birth and Death Processes," *Journal of Mathematics and Mechanics*, Vol. 7, No. 4, pp. 643-662.

[6] Keilson, J. (1962), "Queues Subject to Service Interruption," *The Annals of Mathematical Statistics*, Vol. 33, No. 4, pp. 1314-1322.

[7] Keilson, J. and Nunn, W.R. (1979), "Laguerre Transformation as a Tool for the Numerical Solution of Integral Equations of Convolution Type," *Applied Mathematics and Computation*, Vol. 5, pp. 313-359.

[8] Keilson, J., Nunn, W.R. and Sumita, U. (1981), "The Bilateral Laguerre Transform," *Applied Mathematics and Computation*, Vol. 8, No. 2, pp. 137-174.

[9] Keilson, J., Petrondas, D., Sumita, U. and Wellner, J. (1983), "Significant Points for Some Tests of Uniformity on the Sphere," *Journal of Statistical Computation and Simulation*, Vol. 17, No. 3, pp. 195-218.

[10] Keilson, J. and Sumita, U. (1981), "Waiting Time Distribution Response to Traffic Surges via the Laguerre Transform," *The Proceedings of the Conference on Applied Probability-Computer Science: The Interface*, Boca Raton, Florida.

[11] Keilson, J. and Sumita, U. (1983), "The Depletion Time for M/G/1 Systems and a Related Limit Theorem," *Advances in Applied Probability*, Vol. 15, pp. 420-443.

[12] Keilson, J. and Sumita, U. (1983), "Evaluation of the Total Time in System in a Preempt/Resume Priority Queue via a Modified Lindley Process," *Advances in Applied Probability*, Vol. 15, pp. 840-856.

[13] Keilson, J. and Sumita, U. (1983), "A General Laguerre Transform and Related Distance Between Probability Measures," to appear in *Journal of Mathematical Analysis and Applications*.

[14] Kijima, M. (1986), "Development of the Bivariate Laguerre Transform for Numerical Study of Bivariate Distributions and Bivariate Processes," Ph.D. Thesis, William E. Simon Graduate School of Business Administration, University of Rochester.

[15] Ledermann, W. and Reuter, G.E.H. (1954), "Spectral Theory for the Differential Equations of Simple Birth and Death Processes," *Philosophical Transactions of the Royal Society of London. Series A*, Vol. 246, pp. 321-369.

[16] Masuda, Y., Shanthikumar, J.G. and Sumita, U. (1986), "A General Software Availability/Reliability Model: Numerical Exploration via the Matrix Laguerre Transform," *Stochastic Models*, Vol. 2, pp. 203-236.

[17] Rainville, E.D. (1971), *Special Functions*, Chelsea Publishing Co., New York, Revised Ed.

[18] Sumita, U. (1981), "Development of the Laguerre Transform Method for Numerical Exploration of Applied Probability Models," Ph.D. Thesis, William E. Simon Graduate School of Business Administration, University of Rochester.

[19] Sumita, U. (1983), "On Sums of Independent Logistic and Folded Logistic Variants-Structual Tables and Graphs," *Journal of Statistical Computation and Simulation*, Vol. 17, pp. 251-274.

[20] Sumita, U. (1984), "The Laguerre Transform and a Family of Functions with Nonnegative Laguerre Coefficients," *Mathematics of Operations Research*, Vol. 9, pp. 510-521.

[21] Sumita, U. (1984), "The Matrix Laguerre Transform," *Applied Mathematics and Computation*, Vol. 15, pp. 1-28.

[22] Sumita, U. and Kijima, M. (1985), "The Bivariate Laguerre Transform and Its Applications–Numerical Exploration of Bivariate Processes," *Advances in Applied Probability*, Vol. 17, pp. 683-708.

[23] Sumita, U. and Kijima, M. (1985), "Numerical Exploration of a Bivariate Lindley Process via the Bivariate Laguerre Transform," to appear in *The Proceeding of the Conference on Statistical and Computational Problems in Probability Modeling*, Williamburg, New York.

[24] Sumita, U. and Kijima, M. (1985) "Evaluation of Minimum and Maximum of a Correlated Pair of Random Variables via the Bivariate Laguerre Transform," *Stochastic Models*, Vol. 2, pp. 123-149.

[25] Sumita, U. and Kijima, M. (1987), "The Laguerre Transform of Product of Functions", submitted for publication.

[26] Sumita, U. and Masuda, Y. (1986), "Analysis of Software Availability/Reliability under the Influence of Hardware Failures," *IEEE Transactions on Software Engineering: the Special Issue on Software Reliability - PART II*, Vol. SE-12, pp. 32-41.

[27] Sumita, U. and Shanthikumar, J.G. (1985), "A Class of Correlated Cumulative Shock Models," *Advances in Applied Probability*, Vol. 17, pp. 133-147.

[28] Szegö, G. (1975), *Orthogonal Polynomials*, 4th Ed. American Mathematical Society Colloquim Publications, Vol. 23.

[29] Takács, L. (1962), "The Time Dependence of a Single-Server Queues with Poisson Input and General Service Times," *The Annals of Mathematical Statistics*, Vol. 33, pp. 1340-1348.

Ushio Sumita: William E. Simon
Graduate School of Business
Administration, University of
Rochester, Rochester NY 14627
U.S.A.