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EQUIPMENT REPLACEMENT WITH CONSIDERATION OF TECHNOLOGICAL ADVANCES — DETERMINATION OF REPLACEMENT TIMES BY CONTROL LIMIT POLICY —

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Abstract In recent years, equipment with lower operating costs has been appearing successively as a result of technological advances, and the need has increased for an effective replacement decision system in management. In order to design a practical decision system applicable to a dynamic situation with technological advances, we treat a replacement problem with a finite planning horizon. In such a situation, it is important to devise a policy using forecast and adaptive re-estimation for taking into consideration changes in circumstances.

In this paper, we propose a method of determining an economical replacement times of equipment for a finite planning horizon using the Control Limit Policy, and clarify the sufficient condition that the control limit policy is optimal. We analyze a special case where cash flow functions are exponential, and discuss two case studies related to the investigation of a certain bolt manufacturing firm. Further, by comparing with Kusaka's results [4], we show that this study can be interpreted as an essential extension of the former. Finally, we present a method for dealing with changes in circumstances. The method can easily give not only the ordinary optimal policy, but also a new revised optimal policy even when the replacement time is altered for strategic or other reasons.

1. Introduction

Recently, equipment with low operating costs has been appearing one after another as a result of advances in technology. One of the most important subjects for management now is to determine the timing for economical equipment replacement. In dealing with this problem, it is necessary to realize that "there exists an infinite chain of replacements", that is, "the present decision is affected by the subsequent sets of decisions". Since it is impossible to forecast technological advances over an infinite time horizon, it is consequently difficult to determine rationally a sequence for replacements.

Therein lies the fundamental problem in equipment replacement with consideration of technological advances, and it has been an important aspect in conventional research.

Terborgh [10] studied a replacement problem with technological advances and developed the MAPI system. His study was epoch-making in introducing the consideration of technological advances into the replacement decision. However, there were some problems: he assumed the adverse minimum of future competing equipment (the minimum value of annual average expenses) to be equal to that of the present equipment, and did not consider the change in acquisition cost of equipment, and so on [8]. Since his study, various attempts were made at replacement models with consideration of technological advances. Bellman [1] formulated a replacement problem in an infinite time horizon by Dynamic Programming (DP). Dreyfus [3] treated multiple replacement alternatives at each decision time in a finite time horizon and gave numerical examples. Nakamura [6] determined a sequence of replacement times for a problem in which the technological advances affect the equipment acquisition cost and annual profit, and clarified some problems in MAPI. His research is characterized by a new model different from MAPI, though the numerical solution is rather tedious.

Sethi and Chand [9] have shown there is such a planning horizon T that the first replacement time in a finite planning horizon becomes optimal for a longer horizon (including an infinite planning horizon) and have presented a procedure for obtaining the optimal replacement time of the first equipment using DP. They have supposed a case where a single replacement alternative is available in each decision period and extended it to a problem with multiple possible alternatives, available at each decision time [2]. Their research is characterized by the possibility of obtaining the first replacement time in an infinite planning horizon based on the optimal policy for a finite planning horizon T, for which a forecast of technological advances is possible. However, the optimality of the subsequent replacement times remains a subject for future study.

Lin et. al. [5] have given an approximate solution method for a problem with an infinite time horizon, in which technical progress affects both equipment acquisition costs and annual operating costs, by making use of "two equal life models (like for like replacement models)" in which technical improvements stop at a certain time in the future. Their research is characterized by its use of an approximate solution, but the numerical solution is rather laborious.

The dynamic nature of the equipment replacement problem and the difficulty of its treatment are described above. It is important in the future to consider the following points in discussing equipment replacement with consideration of technological advances:

(1) It is realistic to treat the problem with an infinite chain of replacements as a problem with a foreseeable finite planning horizon and reestimate it according to changes in circumstances. Especially, we can positively consider a finite planning horizon T in the following cases: the overall replacement is expected at future time T, due to an occurrence of product change based on a new product development, or due to an appearance of novel equipment which can not be considered as the extension of conventional technological advances, based on a new technology development.

(2) As equipment is always exposed to the danger of obsolescence due to technological advances, the decision to replace now or not becomes urgent in many cases. At the same time, we must consider the fact that the present decision will have an affect on subsequent decisions.

(3) In considering the flexibility of equipment replacement, it is necessary to constantly ascertain its state of obsolescence due to technological advances by a simple method, and to reflect the result in future plans accordingly.

From this standpoint, Kusaka [4] has derived a criterion representing the state of obsolescence. He has shown a method for determining whether to keep or to replace the existing equipment at the present time without determining the sequence of subsequent replacement times, and given the upper bound of replacement times for a finite planning horizon. This system has a practical characteristic that the evaluation method is given in the form of an explicit function which enables us to quickly monitor the state of obsolescence, though it is restrictive in its application.

This paper clarifies that:

(1) the above problem is resolved under certain circumstances by introducing the concept of control limit policy [7], formerly treated in the Markovian decision processes;

(2) this policy can be interpreted as an essential extension of the above evaluation system; and

(3) the control limit policy plays a practical role in replacement decision.

Now we shall state a noteworthy difference in control limit policy between the Markovian decision processes and our study. The former treats the problem of replacing a part which deteriorates stochastically in a Markov chain with a new one of the same type, whereas the latter has a deterministic approach and treats the problem of replacing deteriorated or obsolete equipment with a new one having technological advances.

Equipment is replaced for various reasons, and this study discusses the replacement problem with regard to an economical and stable production volume. Therefore, it is difficult to apply the proposed method to replacement for increased capacity due to rapid growth of market or for strategic factors as seen in the semiconductor industry.

2. Model Description

2.1 Determination of replacement times for a finite planning horizon

Equipment purchased in the x-th period (x=0,1,...,t-1) is operating at the present period t. Here, we shall briefly denote these by "time x" and "time t" instead of "the x-th period" and "the t-th period" and express every period in terms of the beginning of the period. At each time of t, t+1, ... and T-1, the existing equipment can be replaced by new equipment having technological advances during a planning horizon $[t,T]$ and is disposed of at time T. The impact of technological advances appears in a decrease in initial operating costs and an increase in purchase price of new equipment. If the existing equipment is kept operating, the operating cost will increase and the disposal value will decrease. Moreover, the tendency of these changes is predictable for the planning horizon $[t,T]$. It is supposed that the newest equipment is purchased in every replacement. The decision maker will determine the sequence of replacement times so as to minimize the present value of total cost for the planning horizon.

2.2 Notation

We introduce the following notations:

$H(x,n)$: operating cost at time n for the equipment purchased at time x.

$I(n)$: purchase price of new equipment at time n.

$V(x,n)$: salvage value at time n for the equipment purchased at time x.

$P(x,n)$: present value at time n of total cost for $[n,T]$, starting at time n with the equipment purchased at time x and following the optimal policy since time n.

α : discount rate per period in discrete compounding interest factor ($0<\alpha<1$).

In order to simplify the notation, we denote the functions by $g(x)\uparrow x$, $g(x)\downarrow x$ and $g(x)$: convex(x), respectively when $g(x)$ is non-decreasing, non-increasing and convex with respect to x.

As stated in the section 2.1, it is considered that the initial operating cost $H(n,n)$ of new equipment decreases, and its purchase price $I(n)$ increases with respect to n due to technological advances. Also it is considered that operating cost $H(x,n)$ increases and salvage value $V(x,n)$ decreases with respect to n due to deterioration, wear, etc. There is no practical case where the equipment is disposed of immediately after its purchase, but, in such a case, the salvage value $V(n,n)$ will be smaller than purchase price $I(n)$, that is, $V(n,n)<I(n)$. From the above reasons, $H(x,n)$, $I(x)$ and $V(x,n)$ may be represented as shown in Fig. 1.

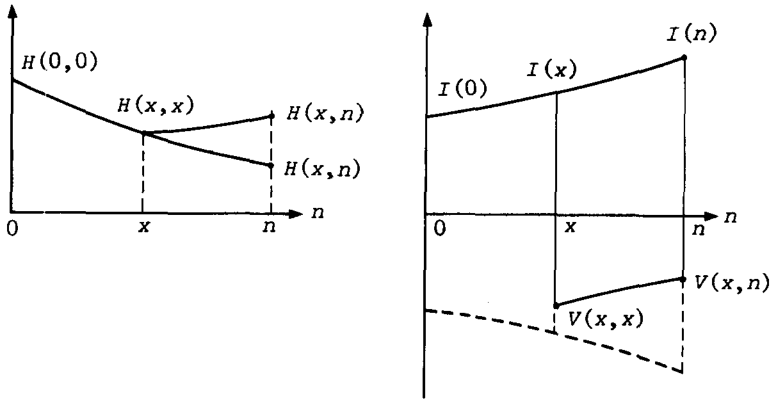


Fig. 1 Relations between the change in x, n and $H(x,n)$, $I(x)$, $V(x,n)$

2.3 Formulation

The present value $P(x,n)$ for the remaining periods $[n,T]$ of the total cost of equipment which was purchased at time x, started at time n and followed the optimal policy for $[n,T]$, is given by

$$P(x,n) = \min \begin{cases} I(n) - V(x,n) + H(n,n) + \alpha P(n,n+1) & \text{if } A_n=R \\ H(x,n) + \alpha P(x,n+1) & \text{if } A_n=K \end{cases} \tag{2.1}$$

$$\text{for } \forall n=1,2,\ldots,T-1; \forall x=0,1,\ldots,n$$

$$P(x,T) = -V(x,T) \quad \text{for } \forall x=0,1,\ldots,T-1$$

where A_n represents the decision at time n, and R and K are "replace" and "keep" actions respectively. The optimal decision A_n at time n ($n=1,2,\ldots,T-1$) can be determined by solving the recurrence equation (2.1).
A decision using (2.1) is equivalent to

$$(2.2)\qquad H(x,n) + V(x,n) + \alpha P(x,n+1) \begin{matrix} > \\ \\ < \end{matrix} H(n,n) + I(n) + \alpha P(n,n+1) \begin{matrix} \to A_n=R \\ \\ \to A_n=K . \end{matrix}$$

Hereinafter to simplify the description, let the left and right hand sides of (2.2) denote respectively as follows:

$$(2.3)\qquad \begin{aligned} L(x,n) &\equiv H(x,n) + V(x,n) + \alpha P(x,n+1) \\ L_0(n) &\equiv H(n,n) + I(n) + \alpha P(n,n+1) . \end{aligned}$$

3. Control Limit Policy

3.1 Definition of the control limit policy and its rôle in replacement decision

Generally in equipment replacement, the older the existing equipment is, the more economical R action is and, on the contrary, the newer it is, the more economical K action is. Therefore for economical replacement decisions, it is natural to suppose a rule to replace the equipment if it were purchased at time x before a certain critical time and to keep it otherwise. For this reason, we define the control limit x_n as this critical time at time n as follows:

Definition 1: When we adopt, with respect to the purchase time x of equipment, such a rule as A_n=R in the case of $0 \leq x \leq x_n$ and A_n=K in the case of $x_n < x \leq n$, we call x_n the "control limit at time n". Furthermore, we call $\{x_n\}$ $(n=1,2,\ldots,T-1)$ as the "control limit policy".

In general, when the optimal replacement decision from (2.1) or (2.2) is given by control limit x_n^* of the above Definition 1, we call x_n^* the "optimal control limit". Fig. 2 represents some typical relations between $L(x,n)$'s and $L_0(n)$. In each case of Fig. 2 (i), (ii), (iii) and (iv), there exists control limit x_n^*. In the case of Fig. 2 (ii), where $L(x,n)=L_0(n)$ holds for x such that x is greater than or equal to a certain value x', we can consider x' as x_n^*. We can also consider the two cases of Fig. 2 (iii) and Fig. 2 (iv) as special cases where there exist control limits x_n^* by letting $x_n^*=T$ and $x_n^*=0$ respectively. On the other hand, Fig. 2 (v) and (vi) show cases where there exists no control limit x_n^* from its definition. In these cases, the optimal replacement decision exists, but it lies out of control limit policy.

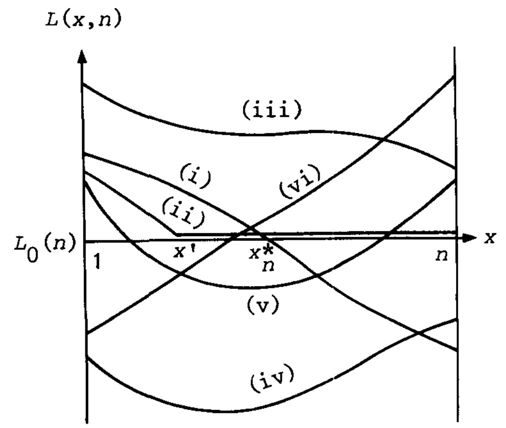


Fig. 2 Example of existing and non-existing patterns of control limit policy

Definition 2: When the optimal replacement decision from (2.2) can be made at every time n (n=1,2,...,T-1) based on the optimal control limit x_n^*, we call $\{x_n^*\}$ (n=1,2,...,T-1) the "optimal control limit policy".

Hereinafter we shall abbreviate "optimal control limit" and "optimal control limit policy" as "control limit" and "control limit policy", respectively.

Supposing the existence of $\{x_n^*\}$, it follows that A_n=R for $x \leq x_n^*$ and A_n=K for $x > x_n^*$. If we vary the value of n, the region of A_n=R is given by the shadowed portion of Fig. 3 (i), in which the vertical axis indicates the state of equipment. Therefore, potting the purchase time x of existing equipment on the vertical axis, A_n=R is selected when the purchase time x reaches the shadowed portion as the time n passes. Noticing this fact, we can easily determine the sequence of optimal replacement times n_1 and n_2, starting from an arbitrary time x, as illustrated in Fig. 3 (ii).

Since it is realistic to consider that equipment replacement is determined with respect to not only economical but also many other factors, especially strategic factors, there may be cases where replacement is postponed even though it reaches the economically optimal time. In this case, the state of equipment remains at x as long as "keep" action has been taken. Therefore, as shown in Fig. 3 (iii), if $x_n^* \uparrow n$, then the height of state x remains within the region of R, i.e., the existing equipment remains in the state of replacement at every time beyond time n. Intuitively, we expect the case of $x_n^* \uparrow n$, but some numerical examples reveal that there is a $\{x_n^*\}$ that possesses a portion of $x_n^* \downarrow n$ as shown in Fig. 3 (iv). Considering the equation (3.1) in the next section, this means that the reduction of operating cost by introducing new equipment becomes small compared with the increase of net invested cost. In these situations, it is economical to postpone replacement until time n'' when more innovative equipment appears, that is, when the reduction of operating cost is relatively large compared with the increase of net invested cost. This circumstance is understood as representing an extremely symbolic aspect in equipment replacement with consideration of technological advances. If

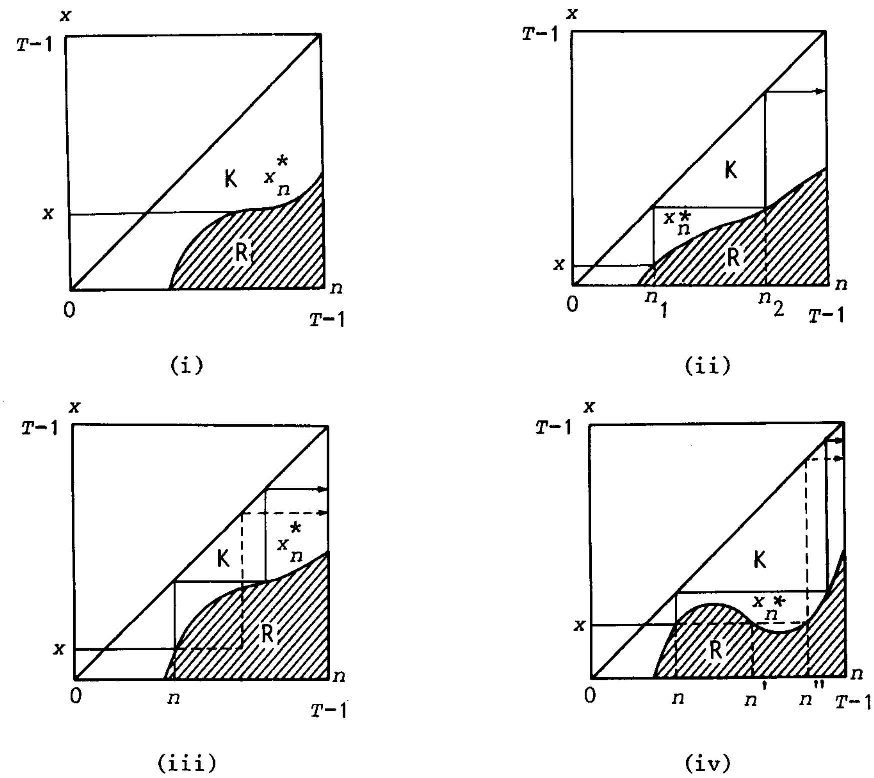


Fig. 3 Control limit policy and replacement decision

an arbitrary time n and the equipment state x at this time are given, the control limit can determine the optimal policy at all times since time n. The advantage of the control limit is that we can predict what policy is to be taken according to changes in the situation, compared with the method of determining either K or R action for a specific time n and state x.

In the next section, we shall give the sufficient conditions for the existence of the control limit policy $\{x_n^*\}$.

3.2 Sufficient conditions for the control limit policy

In this section, we derive the sufficient conditions for the optimal policy constituting control limit policy from (2.2). For convenience, we define an important function:

$$(3.1)\qquad C(x,n) \equiv H(x,n) + V(x,n) - \alpha V(x,n+1) \text{ for } \forall n=1,2,..,T-1;\ \forall x=0,1,..,n.$$

Here, the first term $H(x,n)$ is the operating cost at time n of equipment purchased at time x, and the second and third terms $V(x,n)-\alpha V(x,n+1)$ represent the amount of decrease in salvage value when the above equipment is put into operation at time n. The function $C(x,n)$ can be interpreted as the cost at time n for operating the existing equipment purchased at time x.

Using the function $C(x,n)$, the following lemma holds with respect to $L(x,n)$ of (2.3).

Lemma: For $\forall n=1,2,...,T-1$, $L(x,n) \downarrow x$ on $[0,n]$ if $C(x,n) \downarrow x$.

Proof: We define the function $R(x,n)$ as follows:

$$(3.2)\qquad R(x,n) \equiv P(x,n) + V(x,n) \quad \text{for } \forall\ n=1,2,...,T;\ \forall x=0,1,...,n.$$

The function $R(x,n)$ can be considered as the present value at time n of the total cost, assuming that at time n we purchase used equipment with age $(n-x)$ at price $V(x,n)$ and follow the optimal policy from that time. Rewriting (2.1) using (3.1) and (3.2), the decision using (2.1) is equivalent to

$$(3.3)\qquad R(x,n) = \min\begin{cases} C(n,n) + \alpha R(n,n+1) + I(n) - V(n,n) \\ C(x,n) + \alpha R(x,n+1) \end{cases}$$
$$\text{for } \forall n=1,2,...,T-1;\ \forall x=0,1,...,n$$

$$(3.4)\qquad R(x,T) = 0 \quad \text{for } \forall x=0,1,...,T-1\ .$$

The function $L(x,n)$, using the relations of (3.1) and (3.2), is rewritten as

$$(3.5)\qquad \begin{aligned} L(x,n) &= H(x,n) + V(x,n) + \alpha P(x,n+1) \\ &= H(x,n) + V(x,n) - \alpha V(x,n+1) + \alpha\{P(x,n+1)+V(x,n+1)\} \\ &= C(x,n) + \alpha R(x,n+1) \quad \text{for } \forall\ n=1,2,...,T-1;\ \forall\ x=0,1,...,n. \end{aligned}$$

Under the assumption of $C(x,n) \downarrow x$ and using (3.3)-(3.5), we next show the proof of $L(x,n) \downarrow x$ by induction.

From (3.4), $R(x,n) \downarrow x$ holds as $n=T$. In general, assume $R(x,n+1) \downarrow x$ $(n=1,2,...,T-1)$. Then the lower part of the right hand side of (3.3) becomes non-increasing in x, i.e., $C(x,n)+\alpha R(x,n+1) \downarrow x$ from $0<\alpha<1$ and $C(x,n) \downarrow x$. The upper part of the right hand side of (3.3), $C(n,n)+\alpha R(n,n+1)+I(n)-V(n,n)$ is constant with respect to x. Thus $R(x,n) \downarrow x$ $(n=1,2,...,T-1)$ holds whether the left hand side of (3.3) takes the value of the upper or lower part in the right hand side of (3.3).

Applying this result and $C(x,n) \downarrow x$ to (3.5) yields $L(x,n) \downarrow x$. □

Combining (2.2) and (2.3) with the lemma, the following corollary holds.

Corollary: If $C(x,n) \downarrow x$ on $[0,n]$ for $\forall n=1,2,\dots,T-1$, then there exists an optimal control limit policy $\{x_n^*\}$.

4. Proper Functions in Equipment Replacement

4.1 The case of bi-nonlinear functions

It is natural to consider that both operating cost function $H(x,n)$ and salvage value function $V(x,n)$ are determined by the cost at time x when the equipment is purchased and by the cost of deterioration which depends on the use periods $(n-x)$. Therefore we suppose that both $H(x,n)$ and $V(x,n)$ are bi-nonlinear functions which are represented by the product of a function of x and a function of $(n-x)$. For convenience of analysis, we treat them as continuous variables, though x and n are discrete variables. The functions $H(x,n)$, $I(x)$ and $V(x,n)$ are assumed to be continuous and differentiable with respect to x and n. Under more practical assumptions, these functions are given as follows (see also Fig. 1):

(1) Operating cost $H(x,n)$ is determined by both the initial operating cost $h(x)$ at purchase time x which depends on technological advancements, and the deterioration rate $\psi_h(n-x)$ which depends on the use periods $(n-x)$ in the following way:

  (i) $H(x,n)=h(x)\psi_h(n-x)$ for $\forall n\geq 0$; $\forall n\geq x$.

  (ii) The initial operating cost $h(x)$ decreases year by year due to technological advances, and the decreasing rate diminishes successively. That is,

  (a) $h(x)>0$, (b) $h(x) \downarrow x$, (c) $h(x)$: convex(x).

  (iii) $\psi_h(y)$ becomes higher at an increasing rate due to deterioration, and so forth as the use periods $y=n-x$ becomes longer. That is,

  (a) $\psi_h(y)>0$, (b) $\psi_h(0)=1$, (c) $\psi_h(y) \uparrow y$, (d) $\psi_h(y)$: convex(y).

(2) Equipment purchase price $I(x)$ becomes higher year by year at an increasing rate. That is, we let

  (a) $I(x)>0$, (b) $I(x) \uparrow x$, (c) $I(x)$: convex(x).

(3) Salvage value $V(x,n)$ is determined by the value $v(x)$ at time x, i.e., the salvage value immediately after the equipment is purchased, and the rate $\psi_v(n-x)$ which depends on the use periods $(n-x)$ as follows:

(i) $V(x,n)=v(x)\psi_V(n-x)$ for $\forall\ x\geqq 0;\ \forall\ n\geqq x$.

(ii) $v(x)$ is considered less than the purchase price. That is, we let
$v(x)=\beta I(x)$,
where β $(0\leqq\beta<1)$ is the decreasing rate of equipment salvage value immediately after the equipment is purchased at time x.

(iii) $\psi_V(y)$ is the rate which represents the decrease in salvage value as the use periods $y=n-x$ become longer and the decreasing rate diminishes successively. Further, the second derivative function is decreasing. That is, we let

(a) $\psi_V(y)>0$, (b) $\psi_V(0)=1$, (c) $\psi_V(y)\ \downarrow y$, (d) $\psi_V(y)$: convex(y),

(e) $\dfrac{d^2\psi_V(y)}{dy^2}\ \downarrow y$.

In discussing the properties of $C(x,n)$ under the above assumptions, we shall examine some properties of the functions $H(x,n)$ and $V(x,n)-\alpha V(x,n+1)$ which are terms of $C(x,n)$.

Under assumption (1), it is easily shown that the following properties hold with respect to $H(x,n)$:

(4.1)
(i) $H(x,n)>0$, and is continuous and differentiable for $\forall\ x\geqq 0;\ \forall\ n\geqq x$.
(ii) $H(x,n)\ \downarrow x$.
(iii) $H(x,n)$: convex(x).
(iv) $H(x,n)\ \uparrow n$.

Similarly under assumptions (2) and (3), it is easily shown that the following properties hold with respect to the function $V(x,n)-\alpha V(x,n+1)$:

(4.2)
(i) $V(x,n)-\alpha V(x,n+1)>0$, and is continuous and differentiable for $\forall\ x\geqq 0;\ \forall\ n\geqq x$.
(ii) $V(x,n)-\alpha V(x,n+1)\ \uparrow x$.
(iii) $V(x,n)-\alpha V(x,n+1)$: convex(x).
(iv) $V(x,n)-\alpha V(x,n+1)\ \downarrow n$.

The behavior of $H(x,n)$ and $V(x,n)$ with respect to n is shown in Fig. 4.

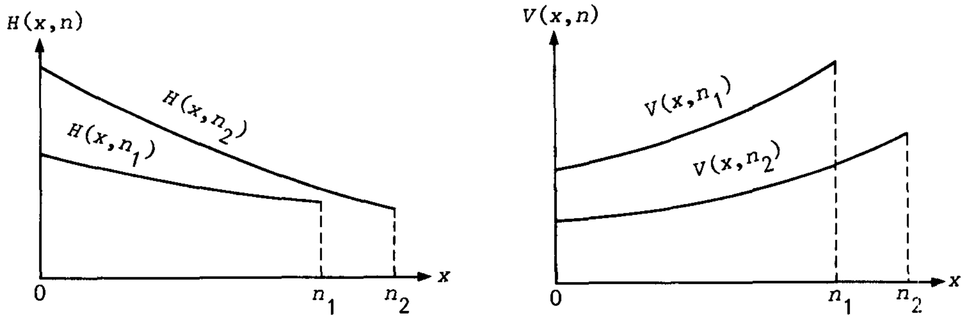


Fig. 4 Change in n and behaviour of $H(x,n)$ and $V(x,n)$

From (i) and (iii) of (4.1), and (i) and (iii) of (4.2), $C(x,n)$ of (3.1) is convex and continuous for x. Therefore, the function $R(x,n)$ is piecewise convex with respect to x from (3.3) and (3.4), so that the function $L(x,n)$ is also piecewise convex in x from (3.5).

When $L(x,n)$ is piecewise convex in x, there is a possibility of an optimal policy in the case of $I(x)>V(x,x)=v(x)$ ($\beta<1$). However, even in this case we can also give a simple numerical example where there is no control limit policy (when $H(x,x)$, $I(x)$ and $V(x,n)$ are given by exponential functions and β is close to 1). In general, if $L(x,n)$ is piecewise convex with respect to x and $L(x,n) \downarrow x$ does not necessarily hold, it is not easy to derive the necessary and sufficient conditions for the existence of the control limit. Hence, we will give a sufficient condition for the existence of the control limit policy if the functions $H(x,n)$, $I(x)$ and $V(x,n)$ satisfy assumptions (1)-(3).

Considering $C(x,n)$: convex(x) under assumptions (1)-(3), the sufficient condition of the corollary in the section 3.2 is equivalent to

$$(4.3) \qquad C(n-1,n) \geq C(n,n) \quad \text{for} \quad \forall\, n=1,2,\ldots,T-1.$$

Here, $C(n-1,n)$ and $C(n,n)$ have been defined in (3.1). Equation (4.3) implies that the cost per period of the newest equipment at an arbitrary time n is less than or equal to that of the newest at time $n-1$.

4.2 The case of exponential functions

In economic phenomena, there are many variables which change at a constant rate according to the law of diminishing returns or the law of increasing returns. Moreover, there will be cases where the decision maker can estimate their rates from past experience and make a rough forecast of future technological advances. Here, as a special case of the section 4.1, we will discuss the case where the functions $H(x,n)$, $I(x)$ and $V(x,n)$ are given in the form of exponential functions, that is,

$$\begin{aligned} H(x,n) &= h_0\tau^x\rho^{n-x} && (h_0\equiv H(0,0)>0,\ \rho>1,\ \tau<1) \\ I(x) &= v_0\delta^x && (v_0\equiv I(0),\ \delta>1) \\ V(x,n) &= \beta I(x)\phi^{n-x} && (0\leqq\beta<1,\ 0<\phi<1) \end{aligned} \tag{4.4}$$

where τ represents the decreasing rate of initial operating cost due to technological advances, ρ the increasing rate of operating cost due to deterioration and so forth, δ the increasing rate of purchase price of equipment, ϕ the decreasing rate of salvage value, and β the falling rate of salvage value of new equipment immediately after its purchase. Equation (4.4) satisfies all the assumptions (1) through (3) with respect to the functions $H(x,n)$, $I(x)$ and $V(x,n)$, and it is easily shown that the function $C(x,n)$ of (3.1) has the form of

$$C(x,n) = h_0\rho^n(\tau/\rho)^x + \beta v_0(1-\phi\alpha)\phi^n(\delta/\phi)^x \tag{4.5}$$

and is convex with respect to x. For β=0, there holds the sufficient condition of the corollary in the section 3.2 from (4.5). On the other hand, for $0<\beta<1$, the value x_n^o, which minimizes $C(x,n)$ with respect to x, is formally obtained as the root of equation $\partial C(x,n)/\partial x = 0$ as follows:

$$x_n^o = \frac{n\log(\rho/\phi) + \log M}{\log(\delta/\phi) + \log(\rho/\tau)} = \frac{n\log(\rho/\phi) + \log M}{\log(\rho/\phi) + \log(\delta/\tau)} \tag{4.6}$$

where $M \equiv \dfrac{h_0}{\beta(1-\phi\alpha)v_0} \cdot \dfrac{\log(\rho/\tau)}{\log(\delta/\phi)}$. Hence, we can substitute

$$x_n^o \geqq n \quad \text{for} \quad \forall n=1,2,\ldots,T-1 \tag{4.7}$$

for the sufficient condition (4.3). Noticing that x_n^o is a linear function of n in (4.6), it can be seen that (4.7) holds in the case of

$$M \geqq 1 \rightarrow C(x,n) \downarrow x \quad \text{for all} \quad n\leqq n^* \tag{4.8}$$

where $n^* \equiv \dfrac{\log M}{\log(\delta/\tau)}$.

Thus, the sufficient condition of the corollary in the section 3.2 is rewritten as

(4.9)
1. β=0
2. $0<\beta<1$, $M\geqq 1$, $T-1\leqq n^*$.

Note that the relation $0\leqq\beta<1$ is necessitated by the assumption (3)(ii). With regard to the sufficient conditions of (4.9), the condition of $M\geqq 1$ and the size of n^* in the second condition are brought into question. As shown in the numerical examples stated later, the relation of (4.8) becomes realistic since M is greater than 1 and n^* becomes large for larger values of $h_0/\{\beta(1-\phi\alpha)v_0\}$. This fact implies that M and n^* can be considered as reflecting the essential characteristics of equipment replacement.

4.3 Comparison with the research by Kusaka

In order to relate this study with Kusaka's, we quote the criterion derived by him in the case of $\rho>\phi$ (see [4], pp.139-141). Notice that the planning periods $[0,T]$ and the operating cost $H(x,n)$ of this paper are respectively represented by the planning periods $[1,T+1]$ and $h(x,n)$ in [4]. By using our notations, the criterion of [4] is written as

$$\text{(4.10)} \qquad \eta_x(n) \begin{array}{l} > E_2^*(n) \rightarrow A_n=R \\ < E_1^*(n) \rightarrow A_n=K \end{array}$$

where

$$\eta_x(n) \equiv \frac{H(x,n) - H(n,n)}{I(n) - V(x,n)} = \frac{h_0}{v_0} \cdot \frac{\tau^x(\rho^{n-x} - \tau^{n-x})}{\delta^x(\delta^{n-x} - \beta\phi^{n-x})}$$

$$\text{(4.11)} \qquad E_1^*(n) \equiv \frac{1-(\phi\alpha)^{T-n}}{\sum_{r=n}^{T-1} (\rho\alpha)^{r-n}}$$

$$E_2^*(n) \equiv 1-\phi\alpha \ .$$

The criterion of (4.10) and (4.11) constitutes a practical method which enables us to quickly calculate $\eta_x(n)$, $E_1^*(n)$ and $E_2^*(n)$ and easily ascertain the state of obsolescence. However, a decision may be impossible in the case of $E_1^*(n)<\eta_x(n)<E_2^*(n)$ because of the relation $E_1^*(n)<E_2^*(n)$. That is to say, as shown in Fig. 5, we cannot determine whether a K or R action should be adopted at any time n such that $n_K<n<n_R$, where n_K and n_R are defined by the values of n such that $\eta_x(n)=E_1^*(n)$ and $\eta_x(n)=E_2^*(n)$, respectively.

Now in the case where there exists a control limit policy $\{x_n^*\}$, we will relate the replacement method using the control limit policy to that of (4.10) and (4.11).

We construct the monotonous non-decreasing progression $\{y_{n+1}^*, y_{n+2}^*, \ldots, y_T^*\}$ from the control limit policy $\{x_{n+1}^*, x_{n+2}^*, \ldots, x_T^*\}$ by making use of a rule such that $y_n^*=0$ and $y_i^*=\max\{x_i^*, y_{i-1}^*\}$ $(i=n+1,\ldots,T)$. Then the optimal replacement times on or after time $n+1$ can be completely determined by this monotonous non-decreasing progression. That is, the optimal first replacement time s_x^* for the present equipment purchased at time x is given by the smallest

i such that $y^*_{i-1}<x\leqq y^*_i$ ($i=n+1$, $n+2$, ..., T). From this, it can be seen that the relation $s^*_x \leqq s^*_n$ holds since $x\leqq n$, where s^*_x and s^*_n ($n+1 \leqq s^*_x, s^*_n \leqq T$), which are respectively the optimal first replacement times on or after time $n+1$ of $P(x,n+1)$ and $P(n,n+1)$ in (2.2). Noticing that there occurs no replacement until time s^*_x-1 for either case $P(x,n+1)$ or $P(n,n+1)$, and denoting s^*_x by j, then $\alpha P(x,n+1)$ and $\alpha P(n,n+1)$ in (2.2) can be expressed by

(4.12)

$$\alpha P(x,n+1) = \begin{cases} \alpha^{j-n}P(x,j) & \text{for } j=n+1 \\ H(x,n)\sum_{r=n+1}^{j-1}(\rho\alpha)^{r-n} + \alpha^{j-n}P(x,j) & \text{for } \forall\, j=n+2,\dots,T \end{cases}$$

$$\alpha P(n,n+1) = \begin{cases} \alpha^{j-n}P(n,j) & \text{for } j=n+1 \\ H(n,n)\sum_{r=n+1}^{j-1}(\rho\alpha)^{r-n} + \alpha^{j-n}P(n,j) & \text{for } \forall\, j=n+2,\dots,T\,. \end{cases}$$

Substituting (4.12) into (2.2) and using the relations of (3.2) and (3.4), then (2.2) is rewritten as

$$\text{(4.13)}\qquad \eta_x(n) \;\genfrac{}{}{0pt}{}{>}{<}\; \frac{1-(\phi\alpha)^{j-n}}{\sum_{r=n}^{j-1}(\rho\alpha)^{r-n}} + \Delta_x(n) \equiv E^*_x(n) \qquad \begin{matrix} \to A_n=R \\ \to A_n=K, \end{matrix}$$

where

$$\Delta_x(n) \equiv \frac{(1-\beta)(\phi\alpha)^{j-n}\cdot I(n)-\{R(x,j)-R(n,j)\}\alpha^{j-n}}{\{1-\beta(\phi/\delta)^{n-x}\}\sum_{r=n}^{j-1}(\rho\alpha)^{r-n}\cdot I(n)}.$$

Therefore, the first replacement time s^*_x on or after time $n+1$ for the present equipment purchased at time x can be considered as being the value of n at the intersection of $\eta_x(n)$ and $E^*_x(n)$ (minimum value of n in the case of multiple intersections) as shown in Fig. 5. This implies that the decision criterion of $x>x^*_n \to A_n=K$ and $x\leqq x^*_n \to A_n=R$ can be rewritten as $n<s^*_x \to A_n=K$ and $n\geqq s^*_x \to A_n=R$ when we vary n at a fixed x. Adding Kusaka's result to this fact, it is shown that

$$\text{(4.14)}\qquad n_K \leqq s^*_x \leqq n_R$$

holds as shown in Fig. 5. Relation (4.14) implies that the situation where the replacement decision is impossible (i.e., $E^*_1(n)<\eta_x(n)<E^*_2(n)$ or equivalently $n_K<n<n_R$) will cancel itself out.

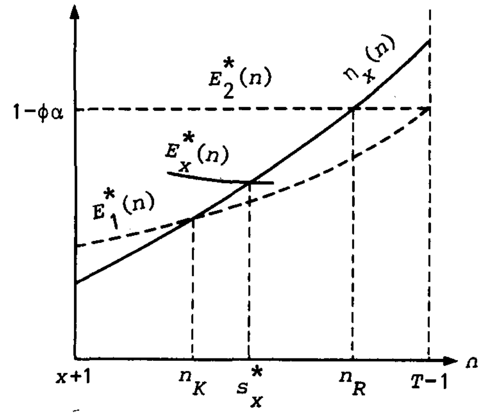


Fig. 5 Relation between $\eta_x(n)$ and X_n

5. Case Studies

For case studies, we shall consider investigations made for a certain bolt manufacturing firm. We discuss the problem of modernizing a cold forging machine and of replacing a continuous heat treatment furnace (hardening and annealing) as follows:

Case 1: With regard to cold forging machines, it is expected that production capacity and availability will increase due to advances in automation technology. Therefore, by introducing such technological advances, an improvement of the existing equipment would reduce labor costs and electricity costs as a result of shorter operating hours. Considering the sum of labor costs and electricity costs as those concerned with improvement decision, we should like to examine whether the improvement is economical or not and, if so, when it should be made. The existing equipment was purchased in 1981 and was planned to produce six hundred thousand bolts per month for 10 years. Expressed in terms of 3-month periods, the equipment was purchased in the 0-th period and expected to be salvaged in the 40-th period, the planning borizon totaling 40 periods. The man-hours required to achieve the planned production were calculated based on the production capacity and availability of equipment updated with the new technology. Further, the annual changes in initial operating costs were estimated with consideration for annual increase of labor costs. Annual changes in operating costs considering changes in labor costs were also estimated for the existing equipment. Parameters τ and ρ were estimated based on these data. The values of δ, ϕ and β were estimated based on the experience of the user in consideration of the characteristics of the equipment and market. For the value of α, the user's data were adopted. The set of estimated parameters is given in Table 1. Notice that in sensitivity analysis these values, with the exception of parameters to be varied, are assumed to be standard parameters.

Table 1 Parameter Values

h_0 = 960 thousand yen/period (3.84 million yen/year),	v_0 = 3.5 million yen
τ = 0.989846/period (0.96/year),	ρ = 1.00496/period (1.02/year)
δ = 1.00496/period (1.02/year),	ϕ = 0.927842/period (0.05/10 years)
α = 0.974004/period (0.90/year),	β = 0

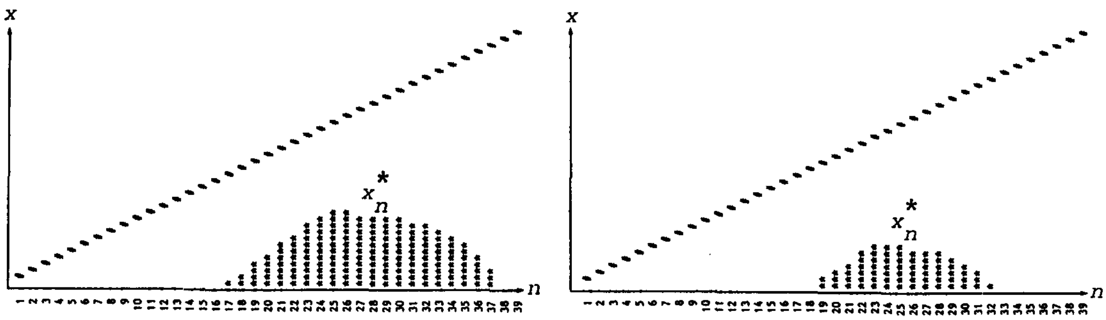
With respect to M in condition (4.9), the greater $h_0/\{\beta(1-\phi\alpha)v_0\}$ and $\log(\rho/\tau)/\log(\delta/\phi)$ are (i.e., operating costs are higher in relation to the decrease in equipment value) greater the possibility for existence of the control limit policy becomes. Table 2 shows the relation between the two quantities M and n^* related to the sufficient conditions for the existence of control limit policy, and the two rates h_0/v_0 and β(the decreasing rate of salvage value immediately after replacement). The greater h_0/v_0 and the smaller β, the greater the possibility of existence of the control limit policy. Even in the case of β=0.9, which has the least possibility of existence of a control limit policy, n^* is about 32 periods (8 years) or longer when $h_0/v_0 \geqq 0.75$. Thus, for the case of $h_0/v_0 \geqq 0.75$, it can be assumed that there exists a control limit policy for a planning horizon of 8 years. Note that the control limit policy always exists from condition 1 of (4.9) in the case of depreciating equipment completely (β=0).

Fig. 6 and Fig. 7 show two control limit policies at (i) β=0.9 and (ii) β=0.5 respectively, and with h_0=960 and h_0=1280 (monetary unit represents ten thousand yen). Here the cases of h_0=960 and h_0=1280 are equivalent to production volumes of 600 thousand and 800 thousand bolts per month, respectively. We denote the optimal time of the j-th replacement for the planning horizon as t_j. Then, it is optimal to replace once at t_1=17 and t_1=19 in the cases of β=0.9 and β=0.5 respectively, as in Fig. 6. On the other hand, replacement at t_1=11 and t_2=24 becomes optimal in the case of β=0.9, at t_1=18 in the case of β=0.5 as in Fig. 7.

Fig. 8 shows that the optimal replacement times in this study for β=0 are given by t_1=20 and t_1=19 in the cases of h_0=960 and h_0=1280, respectively.

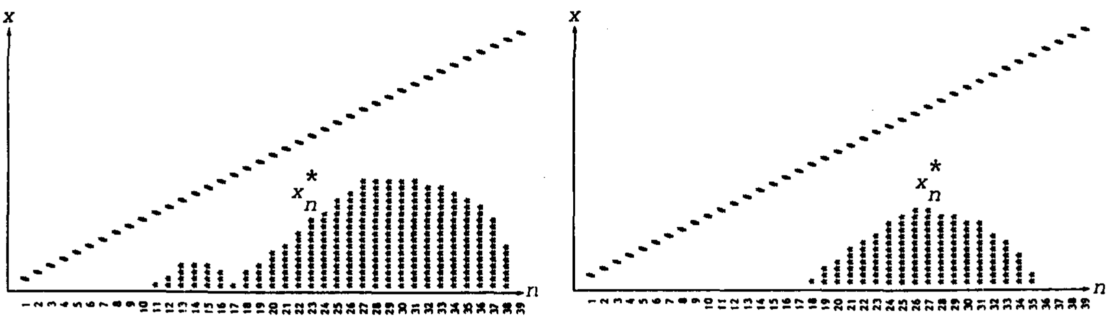
Table 2 Relations between β, h_0/v_0 and M, n^*

	h_0/v_0=0.25		h_0/v_0=0.50		h_0/v_0=0.75		h_0/v_0=1.00		h_0/v_0=1.25	
	M	n^*	M	n^*	M	n^*	M	n^*	M	n^*
β=0.9	0.55	-	1.10	5	1.64	32	2.19	51	2.74	66
β=0.5	0.99	-	1.97	44	2.96	71	3.94	90	4.93	105
β=0.1	4.93	105	9.86	150	14.78	177	19.71	196	24.94	211



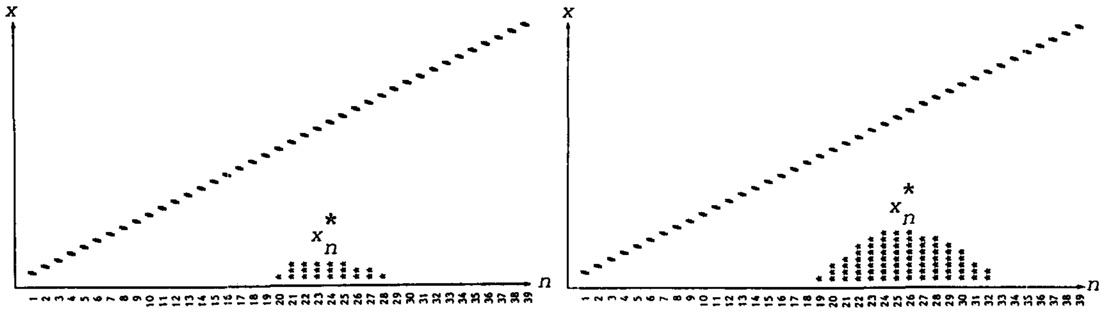
(i) β=0.9 (ii) β=0.5

Fig. 6 Control limit policy at h_0=960



(i) β=0.9 (ii) β=0.5

Fig. 7 Control limit policy at h_0=1280



(i) h_0=960 (ii) h_0=1280

Fig. 8 Control limit policy at β=0

For the case of h_0=960, the decision becomes $A_t=K$ at present time t=30 if replacement has not been made until t=29. On the other hand, for the case of h_0=1280, replacement at present time t=30 remains advantageous even when no replacement has been previously made.

As stated in the section 3.1, the fact that the control limit policy possesses a decreasing region with time has an important meaning in the replacement decision. It is by far more flexible and practical to examine replacement times with this figure as compared to the method of determining only the sequence of an optimal policy.

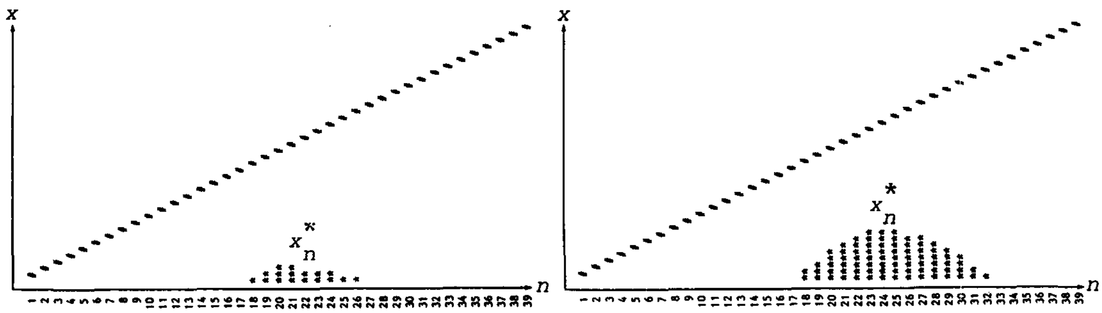
Case 2: For continuous heat treatment furnaces, energy saving has been a major motive for replacement in an era of low economic growth. In order to reduce heat losses in the furnace, such technologies as utilization of waste gas, change in heating method, computerized control of operating conditions, and so forth have been developed. The existing equipment was purchased in 1984 and was planned to operate for 10 years. Denoting 3 months as one period, the equipment was purchased in the 0-th period and is planned to be used up to the 39-th period, the planning horizon totaling 40 periods. We should like to examine whether the replacement is needed or not in this horizon with the above-mentioned technological advances taken into consideration.

A set of estimated parameters, based on the investigations of both the user and maker, is given in Table 3. Here, the decreasing rate τ of initial operating costs due to technological advances was estimated by the maker. Since the salvage value of equipment can be considerecd as 0 (β=0), the control limit policy exists from condition 1 of (4.9). Fig. 9 shows the control limit policies in the cases of h_0/v_0=0.2044 and 0.30, respectively. The case of v_0=58.95 in Table 3 corresponds to Fig. 9 (i). Each of the cases (i) and

(ii) of Fig. 9 shows that it is economical to replace the equipment once. Heat treatment furnaces are replaced after 6-7 years since purchase in most cases, and this is considered to be due to factors such as the shortened usage periods under technological advances, and a leasing term of 6 years.

Table 3 Parameter Values

h_0 = 12.05 million yen/period (48.2 million yen/year),	v_0 = 58.95 million yen
τ = 0.98465/period (0.94/year),	ρ = 1.00742/period (1.03/year)
δ = 1.01227/period (1.05/year),	ϕ = 0.927842/period (0.05/10 years)
α = 0.974004/period (0.90/year),	β = 0



(i) h_0/v_0=0.2044 (ii) h_0/v_0=0.30

Fig. 9 Control limit policy and h_0/v_0

6. Method for Revised Forecast

If the parameters forecasted in the past are applicable in the future, the replacement decision can be made based on a control limit policy obtained by extrapolating the parameters into the future. However, if the environment has changed, it may lead to incorrect decisions, and we need new forecasts. We shall briefly mention a fundamental method for dealing with such cases based on the results analyzed in this paper.

To simplify the description, let the present time be the 0-th period. We adopt the same notations for cost functions and parameters as in the case of new equipment starting at present time 0. However, for existing equipment that is to be kept, we express the newly re-forecasted cost functions and parameters using an apostrophe "'" as follows:

$H'(0,m)$: operating cost at time $m \geqq 0$ for the existing equipment re-estimated at present time 0.

$V'(0,m)$: salvage value at time $m \geqq 0$ for the existing equipment re-estimated at present time 0.

ρ', ϕ', β', h_0', v_0': newly re-estimated values for the existing equipment in the case of exponential funcitons.

Suppose that there exists an optimal control limit policy $\{x_n^*\}$ when starting with new equipment at time 0, and that it has been obtained. If we replace the existing equipment at time m ($m=0,1,\ldots,T$), we can make replacement decisions after m according to the $\{x_n^*\}$ obtained under a new environment. Thus, we can utilize all the method as proposed up to this point. When $m=0$, this implies that R action should be taken at present time, and $m=T$ means that K action should be taken for the duration of the planning horizon. Let $Q(m)$ be the present value of the total cost for $[0,T]$ when starting with the existing equipment at present time 0, replace it at the first replacement time m ($m=0,1,\ldots,T$) and follow the optimal policy. Then $Q(m)$ is given by

$$(6.1)\qquad Q(m) = \begin{cases} I(0) - V'(0,0) + H(0,0) + \alpha P(0,1) & \text{for } m=0 \\ \sum_{r=0}^{m-1} \alpha^r H'(0,r) + \alpha^m\{I(m) - V'(0,m) + H(m,m) + \alpha P(m,m+1)\} & \\ & \text{for } \forall\, m=1,2,\ldots,T-1 \\ \sum_{r=0}^{T-1} \alpha^r H'(0,r) - \alpha^T V'(0,T) & \text{for } m=T\ . \end{cases}$$

If the cost functions are given in the exponential forms of $H'(0,r)=h_0'\rho'^r$, $H(r,r)=h_0\tau^r$, $V'(0,r)=\beta' v_0'\phi'^r$ and $I(r)=v_0\delta^r$, then (6.1) is rewritten as

$$(6.2)\qquad Q(m) = \begin{cases} v_0 - \beta' v_0' + h_0 + \alpha P(0,1) & \text{for } m=0 \\ h_0' \sum_{r=0}^{m-1} (\rho'\alpha)^r + \beta' v_0'\{(1-\phi'\alpha) \sum_{r=0}^{m-1} (\phi'\alpha)^{r-1}\} & \\ \qquad + \alpha^m\{v_0\delta^m + h_0\tau^m + \alpha P(m,m+1)\} & \text{for } \forall\, m=1,2,\ldots,T-1 \\ h_0' \sum_{r=0}^{T-1} (\rho'\alpha)^r - \beta' v_0'(\phi'\alpha)^T & \text{for } m=T \end{cases}$$

Here, $P(x,n)$ ($n=1,2,\ldots,T-1$; $x=0,1,\ldots,n$) is known since the control limit policy has been already determined.

Hence the minimal total cost under the optimal policy is given by

$$\min_{m \in \{0,1,\ldots,T\}} Q(m) \qquad (6.3)$$

and let m satisfying (6.3) be m^*, which represents the next replacement time under the new environment. Assume that the control limit policy $\{x_n^*\}$ is determined under the new environment as shown in Fig. 10. Since the existing equipment is replaced at time m^*, we can determine the replacement times on or after m^* based on the new control limit policy $\{x_n^*\}$ as illustrated by the arrow line in the figure.

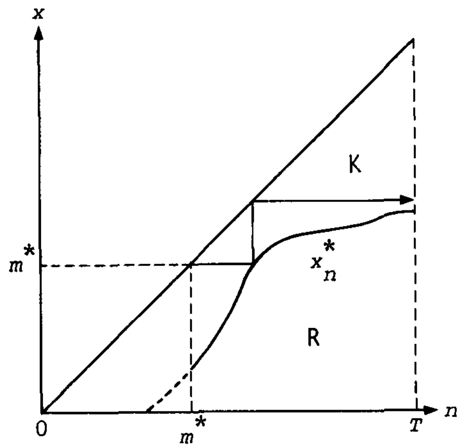


Fig. 10 Use of $\{x_n^*\}$ in revised forecast

7. Conclusion

In this paper, we have discussed the problem of determining economical replacement times of equipment using "control limit policy" and clarified the sufficient conditions that the optimal policy is given by the control limit policy. Next, we have shown the relation between the control limit policy and the conventional research [4], and discussed the role that the control limit policy plays in replacement decisions. Further, we have applied the method to some practical cases. The results show that the method is convenient and practical for evaluating replacement with consideration of technological advances.

In general, since technological advances have a tendency to shorten the economic life of equipment, we need to constantly forecast replacement times of equipment for the planning horizon. The proposed method is proven to be effective in meeting this need.

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