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SEARCH MODELS WITH CONTINUOUS EFFORT UNDER VARIOUS CRITERIA

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Abstract The purpose of this paper is to make up for lacks of the past researches with respect to basic search models with continuous effort. We suppose a discrete search space composed of n boxes and an exponential-type detection function in each box. First an explicit solution is derived for a detection search game. Secondly we consider an information search problem and propose a sequential method of constructing the optimal policy. Thirdly we consider a certainty search game in which the payoff function is given by the posterior uncertainty with respect to the position of player I (hider) and obtain the solution in the form of a solution of a certain simultaneous equations. Finally we consider a whereabouts search model and derive the same result as the discrete effort case.

1. Introduction

The research area of the search theory is divided broadly into two : One is the one-sided search model in which decision is made only by the searcher, and another is the two-sided model in which decisions are made by both the searcher and the object (or hider). For example, a search model for a stationary object or a target moving according to a given rule is the former case and the case that the object can evade (or hide) of his own free will is the latter. Mathematical treatments of models are different as the search effort is discrete or continuous. If the search effort is discrete, most of one-sided search models can be formulated by non-linear programming or dynamic programming, on the other hand, if the search effort is continuous, they become variational problems. For the two-sided search model, the game theory is generally used. Moreover various criteria for decision making have been considered in each case, that is,

(i) Maximizing the detection probability under a given search effort (the detection search problem).

(ii) Minimizing the expected search effort until detection of the object.

(iii) Maximizing the expected amount of information about the location of the object obtained by allocating a given search effort (the information search problem).

(iv) Maximizing the probability of collectly guessing the whereabouts of the object after allocating a given search effort (the whereabouts search problem).

Usually it is assumed that the objective of the search is detection of object and therefore criteria (i) and (ii) are used broadly. But other possible objectives can be obtianed by considering the contemplated post-search action. For example, in a reconnaissance problem the main duty of a scout is not to discover but to locate the enemy correctly. Therefore his true objective is to maximize the expected information gain about the position of the enemy or the probability of correct guess for its position. In such a case the criterion (iii) or (iv) is used. For example, Danskin [4] used the expected information gain as a measure of the effectiveness of a reconnaissance scheme. By simple examples, Mela [10] and Pollock [11] show that the optimal policies for detection search, information search and whereabouts search need not coincide with each other. Since each optimal policy is based on a different criterion, it cannot be considered that one is an improvement for another. The above examples seem to show that the connection between information theory and search theory is at best tenuous. Nevertheless in the case of an exponential detection function Barker [1] shows that the search that maximizes the probability of detection also maximizes the entropy of the posterior distribution of the object. The first paper concerning whereabouts search problem is Tognetti [17]. In the case of discrete search effort Kadane [9] shows that the optimal whereabouts search policy allocates the given search effort according to the optimal detection search policy among all boxes except the box to be guessed at the end of an unsuccessful search. Stone and Kadane [16] treats a whereabouts search problem for a moving target.

Another well-known criterion is to minimize the expected risk, that is, the expected search cost minus the expected reward for detecting the object. In this case we need to consider the stop of search when the expected future reward is smaller than the expected search cost. The first paper which introduces this criterion and considers the problem of search and stop is Ross [13]. He derives many important results for the discrete search model. This criterion containes the above-mentioned criteria (i) and (ii) as special cases. That is to say, letting c_i be a search cost in box i and R_i be a reward for the detection in box i, when $c_i = 0$ and $R_i = 1$ ($i=1,\ldots,n$), the risk criterion coincides the detection search criterion and when $R_i = 0$ ($i=1,\ldots,n$), it coincides the minimum cost (effort) criterion. In this sense, the risk criterion seems to be more complicated than the above-mentioned criteria. Therefore we exclude the risk criterion from basic (simple) criteria. Considering the combination of decision, effort and criterion, various cases are possible which are classified in Table 1. In each case of Table 1 the following basic search problem is considered: In the one-sided search model, we consider that the object is in one of the n boxes with the prior distribution $p=\langle p_1,\ldots, p_n\rangle$ and therefore the problem is formulated as a maximizing (or minimizing) problem with constraints. In the two-sided search model, we suppose that the object (or the hider) hides in one of the n boxes of his own free will and does not move during search periods by the searcher. The problem is formulated as a zero-sum two-person game. We assume that the conditional probability of detection is given, that is, in a discrete search effort case, the conditional detection probability β_i (the probability that the object is detected by one look in box i, given that it is in box i) is given, and in a continuous search effort case, the exponential detection function $1-\exp(-\lambda_i z)$ (the probability that the searcher detects the object by allocating search effort z into box i) is given where λ_i is a known positive constant. Furthermore in the cases of criteria (i), (iii) and (iv), the total search effort E is supposed to be given.

Table 1 summarizes the present status of research of each basic model, that is, the symbol "o" in Table 1 indicates that the model has been already solved, the symbol "Δ" indicates that the model is unsolved but has relevant papers and the symbol "x" indicates that no relevant paper exists. Papers related to each basic model are as follows:

model [1] ... Kadane [8]
model [2] ... Charnes and Cooper [3]
model [3] ... Suberman [17]
model [5] ... Blackwell [2]
model [6] ... Dobbie [5]
model [7] ... Sakaguchi [14], Roberts and Gittins [12], Gittins and Roberts [7]
model [8] ... Gittins [6]
model [9] ... Sakaguchi [15]
model [13] ... Kadane [9].

Table 1. Classification of the basic search models

criterion \ effort \ decision	one-sided		two-sided	
	discrete	continuous	discrete	continuous
detection search	[1] ○	[2] ○	[3] ○	[4] ×
exp. effort→min	[5] ○	[6] ○	[7] △	[8] ○
information search	[9] △	[10] ×	[11] ×	[12] ×
whereabouts search	[13] △	[14] ×	[15] ×	[16] ×

In this paper we intend to supplement lacks of the past researches. In Section 2 a solution of the detection search game (model [4]) is given explicitly. In Section 3 we propose a sequential method of constructing an optimal policy for the information search problem (model [10]). In Section 4 we consider a certainty search game in connection with the information search game (model [12]) and obtain its solution in the form of a solution of certain simultaneous equations. Finally in Section 5 we consider a whereabouts search problem (model [14]) and derive a similar result to the discrete search effort case.

2. Detection Search Game

Player I (hider) selects one of n boxes of his own free will, hides in it and never moves during the search by player II. Player II (searcher) searches player I by dividing the given total continuous search effort E and allocating it in each box. For each box i the exponential detection function $1-\exp(-\lambda_i z)$ is given and known to both players (which is defined in the previous section). The strategies for player I and II can be represented by $p=\langle p_1,\ldots, p_n\rangle$ and $x=(x_1,\ldots, x_n)$ respectively where p_i is the probability that player I hides in box i and x_i is the amount of effort allocated in box i $(\sum_{i=1}^{n} x_i = E\ ;\ x_i \geq 0,\ i=1,\ldots, n)$ by player II. The payoff function for player I is given by

$$(2.1)\quad M(p,x) = \sum_{i=1}^{n} p_i \exp(-\lambda_i x_i),$$

that is, the probability that player I is not detected given that strategies

p and x are used. Solve this zero-sum two-person search game.

Theorem 1. The optimal strategies for player I and II are given by $p^*=\langle p_1^*,\cdots, p_n^*\rangle$ and $x^*=(x_1^*, \cdots, x_n^*)$ respectively where

$$(2.2)\quad p_i^* = \lambda_i^{-1}/\sum_{j=1}^{n}\lambda_j^{-1},\qquad x_i^* = \lambda_i^{-1}\,E/\sum_{j=1}^{n}\lambda_j^{-1}\qquad (i=1,\cdots, n).$$

The game value is $v=\exp(-E/\sum_{j=1}^{n}\lambda_j^{-1})$.

Proof: It is sufficient to show

$$(2.3)\quad M(p,x^*) \leqq v \leqq M(p^*, x)\qquad \text{for all } x, \text{ all } p.$$

The left-side of the relation (2.3) is obvious because

$$(2.4)\quad M(p,x^*) = \sum_{i=1}^{n} p_i \exp(-E/\sum_{j=1}^{n}\lambda_j^{-1}) = v \text{ for all } p.$$

To show the right-side of the relation (2.3), we consider the following concave programming:

$$(2.5)\quad M(p^*,x) = (\sum_{j=1}^{n}\lambda_i^{-1})^{-1}\sum_{i=1}^{n}\lambda^{-1}\exp(-\lambda_i x_i)\qquad \to \min_x$$

subject to

$$\sum_{i=1}^{n} x_i = E$$

$$x_i \geqq 0\ (i=1, \cdots,n).$$

The Kuhn-Tucker theorem gives the necessary and sufficient condition for $x^0=(x_1^0, \cdots, x_n^0)$ to be optimal as follows: For a Lagrangian multiplier u,

$$(2.6)\quad (\sum_{j=1}^{n}\lambda_j^{-1})^{-1}\exp(-\lambda_i x_i^0) \leqq u\qquad (i=1, \cdots, n)$$

$$(2.7)\quad \sum_{i=1}^{n} x_i^0[(\sum_{j=1}^{n}\lambda_j^{-1})^{-1}\exp(-\lambda_i x_i^0)-u] = 0$$

$$(2.8)\quad E - \sum_{i=1}^{n} x_i^0 = 0.$$

Because $E>0$, there is such a box k that $x_k^0>0$ and hence by the relations (2.6) and (2.7), we obtain

$$(2.9)\quad u = (\sum_{j=1}^{n}\lambda_j^{-1})^{-1}\exp(-\lambda_k x_k^0) < (\sum_{j=1}^{n}\lambda_j^{-1})^{-1}.$$

Therefore by the relation (2.6), $x_i^0 > 0$ for all i. Then by the relations (2.6) and (2.7),

$$(\sum_{j=1}^{n} \lambda_j^{-1})^{-1} \exp(-\lambda_i x_i^0) = u \qquad (i=1,\cdots, n)$$

which denotes that $\lambda_i x_i^0$ is constant in i. Therefore x_i^0 is proportional to λ_i^{-1}. From the relation (2.8), we obtain $x^0 = x^*$ and hence the relation (2.3) is proved. (q.e.d.)

Theorem 1 states that player I (II) should hide (allocate the total effort) in each box with ratio in proportion to the inverse of the conditional detection rate. In other words, player I should hide in such a manner that the detection rate $p_i \lambda_i$ becomes constant for all boxes to confuse the searcher and player II should search in such a manner that the conditional probability of no detection $\exp(-\lambda_i x_i)$ becomes constant for all boxes.

3. Information Search Problem

A stationary object exists in one of n boxes with the prior distribution $p = \langle p_1, \cdots, p_n \rangle$ and the searcher wants to obtain as much information gain as possible concerning the location of the object by allocating the given total search effort E. We assume the exponential detection function $1-\exp(-\lambda_i z)$. The expected amount of information concerning the location of the object under the search policy $x=(x_1, \cdots, x_n)$ is given by

(3.1) $$I_x(p) \equiv H(p) - [D_x(p)\cdot 0 + \{1-D_x(p)\}H(T_x p)]$$

where

(3.2) $$H(p) \equiv -\sum_{i=1}^{n} p_i \log p_i$$

(3.3) $$D_x(p) \equiv 1 - \sum_{i=1}^{n} p_i \exp(-\lambda_i x_i)$$

(3.4) $$T_x p = \langle (T_x p)_1, \cdots, (T_x p)_n \rangle$$

$$(T_x p)_i = p_i \exp(-\lambda_i x_i) / \sum_{j=1}^{n} \{p_j \exp(-\lambda_j x_j)\} \qquad (i=1, \cdots, n)$$

$H(p)$ is a well-known entropy function used by Shannon in communications work. $D_x(p)$ is the probability of detecting the object by the policy x given that the prior distribution of the object is p. $T_x p$ is the posterior distribution of the position of the object given that the prior distribution is p and that it is not detected by the policy x. If the object is (is not) detected, then the entropy of its positin is zero ($H(T_x p)$) and hence the right-hand side of the relation (3.1) represents the difference between the prior entropy and the posterior entropy, that is, the expected amount of information obtained by the policy x. The objective is to obtain the optimal policy x^* maximizing $I_x(p)$. The problem becomes a nonlinear programming :

$$(3.5)\quad f(x) \equiv \sum_{i=1}^{n} p_i \exp(-\lambda_i x_i) \times \log \left[\sum_{k=1}^{n} p_k \exp(-\lambda_k x_k)/\{p_i \exp(-\lambda_i x_i)\} \right] \to \min_x$$

subject to

$$(3.6)\quad \sum_{i=1}^{n} x_i = E \quad ; \quad x_i \geqq 0 \quad (i=1, \cdots, n).$$

Lemma 1. The function $f(x)$ is convex in $x_i(\geqq 0)$ for all $i(=1,\ldots,n)$.

Proof: By a simple calculation, we can obtain

$$(3.7)\quad \frac{\partial f}{\partial x_i} = -p_i\lambda_i \exp(-\lambda_i x_i) \log t$$

and

$$(3.8)\quad \frac{\partial^2 f}{\partial x_i^2} = p_i\lambda_i^2 \exp(-\lambda_i x_i)\left(\log t + t^{-1} - 1\right)$$

where

$$(3.9)\quad t = t(x_i) = \sum_{k=1}^{n} p_k \exp(-\lambda_k x_k)/\{p_i \exp(-\lambda_i x_i)\}.$$

Since

$$t = 1 + \sum_{k \neq i} p_k \exp(-\lambda_k x_k)/\{p_i \exp(-\lambda_i x_i)\},$$

$t \geqq 1$ and $t(x_i)$ is nondecreasing in $x_i(\geqq 0)$.
Put $g(t) = \log t + t^{-1} - 1$. Since $g(1)=0$ and $g'(t)=(t-1)/t^2 \geqq 0$ for any $t(\geqq 1)$, we know that $g(t) \geqq 0$ for any $t(\geqq 1)$. Therefore $\partial^2 f/\partial x_i^2$ is nondecreasing in $x_i(\geqq 0)$. (q.e.d.)

Theorem 2. A necessary and sufficient condition for a policy $x^*=(x_1^*,\cdots, x_n^*)$ to be optimal is given as follows : There exists a positive constant such that

$$(3.10)\quad L_i[x^*] \begin{Bmatrix} = \\ \leqq \end{Bmatrix} \mu \qquad \text{if } x_i^* \begin{Bmatrix} > \\ = \end{Bmatrix} 0$$

where

$$(3.11)\quad L_i[x] \equiv p_i\lambda_i \exp(-\lambda_i x_i) \log \left[\sum_{k=1}^{n} p_k \exp(-\lambda_k x_k)/\{p_i \exp(-\lambda_i x_i)\} \right].$$

Proof: Because $f(x)$ is convex by Lemma 1, the Kuhn-Tucker theorem gives a necessary and sufficient condition for a policy x^* to be optimal, that is,

$$(3.12)\quad L_i[x^*] \leqq \mu \quad (i=1,\dots, n)$$

$$(3.13)\quad \sum_{i=1}^{n} x_i^*\{L_i[x^*]-\mu\} = 0$$

where μ is a Lagrangian multiplier. The result (3.10) cna be derived directly from the relations (3.12), (3.13) and $x_i^* \geqq 0 (i=1,\dots, n)$. Moreover it is clear that $\mu \geqq L_i[x^*] \geqq 0$. If $\mu=0$, then by the relation (3.10) and (3.11) we obtain that $p_i=0$ for all i, which is contradictory to $\sum_{i=1}^{n} p_i=1$. Hence $\mu>0$. (q.e.d.)

Lemma 2. the function $L_i[x]$ is nonincreasing in x_k ($k=1, \cdots, n$).

Proof: Put

$$(3.14)\quad y_i \equiv p_i \exp(-\lambda_i x_i), \quad z_i \equiv \sum_{k\neq i} p_k \exp(-\lambda_k x_k).$$

Then

$$(3.15)\quad L_i[x] = \lambda_i y_i \log(1+z_i/y_i).$$

Note that y_i is independent of x_k ($k\neq i$) and nonincreasing in x_i and that z_i is independent of x_i and nonincreasing in x_k ($k\neq i$). Therefore it is clear that $L_i[x]$ is nonincreasing in $x_k (k\neq i)$. In order to prove that $L_i[x]$ is nonincreasing in x_i, it is sufficient to show that $L_i[x]$ is nondecreasing in y_i. By simple calculations, we obtain

$$(3.16)\quad \frac{\partial L_i}{\partial y_i} = \lambda_i \log\left(1+\frac{z_i}{y_i}\right) - \frac{\lambda_i z_i}{y_i+z_i}$$

$$(3.17)\quad \frac{\partial^2 L_i}{\partial y_i^2} = -\frac{\lambda_i z_i^2}{y_i(y_i+z_i)^2}$$

Since $\partial^2 L_i/\partial y_i^2$ is nonpositive, $\partial L_i/\partial y_i$ is nonincreasing in y_i. Futhermore $\partial L_i/\partial y_i$ converges to zero as y_i tends to infinity. Therfore $\partial L_i/\partial y_i$ is non-negative. Hence $L_i[x]$ is nondecreasing in y_i. (q.e.d.)

Theorem 2 states that if the optimal amount of search effort allocated in box i is positive (zero), then at the end of the optimal search x^* the value of the function $L_i[x^*]$ is equal to (is not larger than)a constant level μ which depends on the total search effort. From this fact we think of such a sequential method for constructing an optimal policy that at each time next minute effor is allocated into boxes having the maximum of $L_i[\tilde{x}]$ where $\tilde{x}$ is the accumulated allocation until now. The realizability of this sequential method is guaranteed by Lemma 2. Thus we can obtain the next theorem.

Theorem 3. An optimal policy can be obtained by the following sequential method : Suppose that the effort $E_1(<E)$ has been already allocated optimally by now and that as a result the allocation

$$x^*(E_1)=(x_1^*(E_1),\ldots, x_n^*(E_1)) \ (\sum_{i=1}^{n} x_i^*(E_1)=E_1 \ ; \ x_i^*(E_1)\geqq 0 \quad i=1,\ldots,n)$$

has been obtained. Then the additional minute effort must be allocated in boxes of a set

$$(3.18) \quad I(E_1) \equiv \{i \mid L_i[x^*(E_1)] = \max_{1\leqq j\leqq n} L_j[x^*(E_1)]\}$$

at the rates satisfying the relation

$$(3.19) \quad L_i[x^*(E_1)+y] = L_j[x^*(E_1)+y] \text{ for all } i,\ j\in I(E_1)$$

where $y=(y_1,\ldots,y_n)$ is the aditional effort. The process of this additional allocation should be continued until the relation

$$(3.20) \quad L_i[x^*(E_1)+y^*] = \max_{k(\notin I(E_1))} L_k[x^*(E_1)+y^*] \text{ for all } i\in I(E_1)$$

is satisfied. At this time, regard $E_1+\sum_{i=1}^{n} y_i^*$ as new E_1 and repeat the above procedure until the total search effort E is completely allocated.

Proof: If $x_i^*(E)>0$, then box i should be searched at a certain time and hence by (3.18) there exists a certain accumulated effort $E_1(\leqq E)$ such that $i\in I(E_1)$. From the sequential construction, if once $i\in I(E_1)$ occurs, this property is kept until last time, that is $i\in I(\tilde{E})$ for all $\tilde{E}(E_1\leqq\tilde{E}\leqq E)$. Therefore

$$(3.20) \quad L_i[x^*(E)] = \max_{1\leqq j\leqq n} L_j[x^*(E)].$$

If $x_i^*(E)=0$, then box i is not searched at any time and hence $i\notin I(\tilde{E})$ for all $\tilde{E}(0\leqq\tilde{E}\leqq E)$. Therefore

$$(3.21) \quad L_i[x^*(E)] < \max_{1\leqq j\leqq n} L_j[x^*(E)].$$

Then by (3.20) and (3.21) the sufficient condition (3.10) for an optimal policy in Theorem 2 is satisfied for $\mu=\max_{1\leqq j\leqq n} L_i[x^*(E)]$. Hence the policy obtained by the sequential method in Theorem 3 is optimal. (q.e.d.)

The sequential method given in Theorem 3 is characterized by following properties :

(i) At each time the optimal policy searches only in boxes having the maximum value of the function $L_i[\tilde{x}]$ where $\tilde{x}$ is the accumulated allocation until now.

(ii) After an allocation $\tilde{x}$ has been carried out, the additional minute effort should be allocated with such rates as the values of $L_i[\tilde{x}+y]$ equal each other for all searched boxes where y is the new additional allocation.

(iii) By Lemma 2, $L_i[x]$ is nonincreasing in x_k and therefore as the allocation of search effort progresses, new boxes enter into the set of searched boxes one after another.

Let us consider the meaning of the function $L_i[x]$. Let $p(x)=\langle p_1(x),\ldots, p_n(x)\rangle$ be the posterior distribution of the object's location given that the allocation x fails to detect it, that is,

$$(3.22)\quad p_i(x) = p_i\exp(-\lambda_i x_i)/\sum_{k=1}^{n} p_k\exp(-\lambda_k x_k).$$

Then $L_i(x)$ is represented as follows:

$$(3.23)\quad L_i(x) = [\sum_{k=1}^{n} p_k\exp(-\lambda_k x_k)][-p_i(x)\log p_i(x)]\lambda_i$$

$$\propto [-p_i(x)\log p_i(x)]\lambda_i$$

Taking the information criterion into account, define the value of the fact that the object is in box i with probability p_i by $U(p_i)= -p_i\log p_i$ and hence $U[p_i(x)]= -p_i(x)\log p_i(x)$ denotes the value of the posterior existence probability of the object in box i. Therefore by means of the value of the existence probability, it is permitted to say roughly that $L_i[x]$ is proportional to the posterior detection rate, that is,

$$(3.24)\quad L_i[x]\propto U[p_i(x)]\lambda_i.$$

4. Certainty Search Game

Let us consider an information search game in which the payoff for the maximizing player I (hider) is given by the negative sign of the expected information gain $I_x(p)$ in (3.1). In spite of the searcher's policy it is obviously optimal for the hider to hide in any box with probability one since $H(p)=H(T_x p)=0$ in (3.1). Of course the game value is zero. Then the information search game is trivial. In connection with the information search game, we consider a certainty search game which is the same as the detection search game in Section 2 except that the payoff for player I (hider) is the uncertainty (entropy) of the posterior distribution of the hider's location. In other words, player II (searcher) intends to know the hider's location as certainly as possible. The payoff function is given by

(4.1) $M(p,x) \equiv D_x(p)\cdot 0 + \{1-D_x(p)\}]H(T_x p)$

which is equal to $f(x)$ given by (3.5).

Theorem 4. A necessary and sufficient condition for a pair of strategies $p^* = \langle p_1^*, \cdots, p_n^* \rangle$ and $x^* = (x_1^*, \cdots, x_n^*)$ to be optimal is given as follows: There exist two positive constants ξ and μ such that

(4.2) $K_i[p^*, x^*]\begin{Bmatrix} = \\ \leqq \end{Bmatrix}\xi \qquad \text{if } p_i^*\begin{Bmatrix} > \\ = \end{Bmatrix}0$

(4.3) $p_i^* \lambda_i K_i[p^*, x^*]\begin{Bmatrix} = \\ \leqq \end{Bmatrix}\mu \qquad \text{if } x_i^*\begin{Bmatrix} > \\ = \end{Bmatrix}0$

where

(4.4) $K_i[p, x] \equiv \exp(-\lambda_i x_i) \log[\sum_{k=1}^{n} p_k \exp(-\lambda_k x_k)/\{p_i \exp(-\lambda_i x_i)\}]$.

Proof: It is sufficient to show that

(4.5) $M(p,x^*) \leqq M(p^*, x^*) \leqq M(p^*, x)$ for any p,x.

first we consider a maximizing problem

(4.6) $M(p,x^*) \to \max_p$.

Since the function $M(p,x^*)$ is concave in $p_i (i=1,\cdots, n)$, the Kuhn-Tucker theorem gives the necessary and sufficient condition for p^* to be optimal, that is,

(4.7) $K_i[p^*, x^*] - \xi \leqq 0 \qquad (i=1, \cdots, n)$

(4.8) $\sum_{i=1}^{n} p_i^*\{K_i[p^*, x^*] - \xi\} = 0$

from which the relation (4.2) can be derived. Next we consider a minimizing problem

(4.9) $M(p^*, x) \to \min_x$.

By Lemma 1, the function $M(p^*, x)$ is convex in $x_i (i=1, \cdots, n)$.
Therefore the necessary and sufficient condtion for an optimal solution is given by Kuhn-Tucker theorem as follows:

(4.10) $p_i^* \lambda_i K_i[p^*, x^*] - \mu \leqq 0 \qquad (i=1, \cdots, n)$

$$(4.11) \quad \sum_{i=1}^{n} p_i^*\{p_i^*\lambda_i K_i[p^*, x^*] - \mu\} = 0$$

from which the relation (4.3) can be derived. There exists at least one box for which $p_i^* > 0$ and therefore $K_i[p^*, x^*]$ is positive for such a box i. Hence ξ and μ must be positive. (q.e.d.)

Both the relations (4.2) and (4.3) have similar interpretations to the relation (3.10).

Lemma 3. (i) If $p_i^* = 0$, then $x_i^* = 0$. (ii) If $p_i^* > 0$, then p_i^* has the following form:

$$(4.12) \quad p_i^* = \begin{cases} \alpha\lambda_i^{-1} \\ \beta \end{cases} \quad \text{if } x_i^* \begin{Bmatrix} > \\ = \end{Bmatrix} 0$$

where α and β are constant in i.

Proof: (i) The assertion is obvious from the relation (4.3).

(ii) If $p_i^* > 0$ and $x_i^* > 0$, the relations (4.2) and (4.3) hold in equality from which we can derive $p_i^*\lambda_i\xi = \mu$. Hence p_i^* is proportional to λ_i^{-1}. If $p_i^* > 0$ and $x_i^* = 0$, then the relation (4.2) becomes

$$\log\,[\sum_{k=1}^{n} p_k^* \exp(-\lambda_k x_k^*)/p_i^*] = \xi.$$

Hence p_i^* is independent of i. (q.e.d.)

Theorem 5. Without loss of generality, we can suppose that

$$(4.13) \quad \lambda_1 \geqq \lambda_2 \geqq \cdots \geqq \lambda_n.$$

Then there is a sequence $\{E_i\}_{i=0}^{n}$ such that

$$(4.14) \quad (0\equiv)\ E_0 \leqq E_1 \leqq E_2 \leqq \cdots \leqq E_{n-1} \leqq E_n\ (\equiv\infty)$$

and the solution of the certainty search game is given as follows :

If $E_{\ell-1} < E \leqq E_\ell$, then

$$(4.15) \quad p^* = \langle \alpha\lambda_1^{-1}, \cdots, \alpha\lambda_\ell^{-1}, \beta, \cdots . \beta \rangle$$

$$(4.16) \quad x^* = (x_1^*, \cdots, x_\ell^*, 0, \cdots, 0)$$

are optimal strategies for players I and II respectively and the game value is given by ξ, where α, β, $x_1^*, \cdots, x_\ell^*$ and ξ are roots of the following simultaneous equations ;

$$(4.17) \quad \exp(-\alpha_i x_i^*)\log[A(\alpha, x^*)/\{\alpha\lambda_i^{-1}\exp(-\lambda_i x_i^*)\}] = \xi \qquad (i=1, \cdots, \ell)$$

(4.18) $\log[A(\alpha,x^*)/\beta] = \xi$

(4.19) $\alpha \sum_{k=1}^{\ell} \lambda_k^{-1} + \beta(n-\ell) = 1$

(4.20) $\sum_{i=1}^{\ell} x_i^* = E$

where

(4.21) $A(\alpha,x^*) = \alpha \sum_{k=1}^{\ell} \lambda_k^{-1}\exp(-\lambda_k x_k^*)+1-\alpha \sum_{k=1}^{\ell} \lambda_k^{-1}.$

The value of E_ℓ is determined by $E_\ell = \sum_{k=1}^{\ell} x^0_k$ where $x^0_k (k=1,\cdots,\ell)$ are the solution of the simultaneous equations

(4.22) $\alpha \exp(-\lambda_i x^0_i)\log[A(\alpha,x^0)/\{\alpha\lambda_i^{-1}\exp(-\lambda_i x_i^0)\}]$

$= \beta\lambda_{\ell+1}\log[A(\alpha,x^0)/\beta] \qquad (i=1,\cdots,\ell).$

Hence if we increase the amount of the allocated effort from zero to infinity, then the values of E_i $(i=1,2,\cdots,n-1)$ can be determined successively and at last we can obtain the solution for a given total search effort E.

Proof: We sahll show that p^* and x^* given by (4.15)~(4.20) satisfy the relations (4.2) and (4.3). Substituting (4.15) and (4.16) into $K_i[p^*,x^*]$ and paying attention to (4.17) and (4.18), we can obtain that

(4.23) $K_i[p^*,x^*] = \xi \qquad (i=1,\cdots,n)$

and therefore the relation (4.2) holds in equality for any i. Next, substituting (4.15) and (4.16) into the left-hand side of (4.3) and using (4.17), we can obtain that

(4.24) $p_i^*\lambda_i K_i[p^*,x^*] = \alpha\xi \qquad (i=1,\cdots,\ell)$

and therefore if we put $\mu = \alpha\xi$, then the relation (4.3) holds in equality for $i=1,\cdots,\ell$. For $i=\ell+1,\cdots,n$,

(4.25) $p_i^*\lambda_i K_i[p^*,x^*] = \beta\lambda_i\xi$ by (4.18)

$\leqq \beta\lambda_{\ell+1}\xi$ by (4.13)

$= p_{\ell+1}^*\lambda_{\ell+1}K_{\ell+1}[p^*,x^*]$ by (4.18)

$\leqq p_\ell^*\lambda_\ell K_\ell[p^*,x^*]$ by $E_{\ell-1}<E\leqq E_\ell$ (See(4.22))

$= \alpha\xi$ by (4.17)

$= \mu$

and therfore the relation (4.3) holds for $i=\ell+1,\cdots,n$. Hence by Theorem 4,

p^* and x^* in (4.15) and (4.16) are optimal. Finally,

(4.26) $$M(p^*,x^*)$$

$$= \sum_{i=1}^{\ell} \alpha\lambda_i^{-1}\exp(-\lambda_i x_i^*)\log[A(\alpha,x^*)/\{\alpha\lambda_i^{-1}\exp(-\lambda_i x_i^*)\}] + \sum_{i=\ell+1}^{n} \beta \log[A(\alpha,x^*)/\beta]$$

$$= \xi[\alpha \sum_{i=1}^{\ell} \lambda_i^{-1} + \beta(n-\ell)] \qquad \text{by (4.17) and (4.18)}$$

$$= \xi \qquad \text{by (4.19).}$$

Therefore the game value is given by ξ. (q.e.d.)

Theorem 5 gives the solution of the certainty search game in the form of the solution of a simultaneous equations. Moreover it shows some properties of the solution:

(i) A box, which has higher conditional detection rate, has a larger possibility that player I hides and II searches in it.

(ii) The larger the total search effort is, the wider the range in which both players take actions becomes.

(iii) The optimal hiding-rate for player I is proportional to the inverse of the conditional detection rate in searched box and in constant in unsearched box.

5. Whereabouts Search Problem

A stationary object is in one of n boxes according to the prior distribution $p = \langle p_1,\cdots, p_n\rangle$. We assume an exponential detection function $1-\exp(-\lambda_i z)$ for each box. The searcher allocates the total effort E and if he detects the object, then he obtains payoff unit, on the other hand, if he fails to detect it, then he guesses the whereabouts of the object and if his guess is correct (incorrect), his payoff is unit (zero). The objective of the searcher is to determine optimally the allocation of the search effort and the guessed box in order to maximize his expected payoff. The searcher's policy is given by (x,i) where $x=(x_1, \cdots, x_n)$ is an allocation of total search effort E and i denotes a box guessed after the failure of detection. The maximum expected payoff $W(p)$ is given by

(5.1) $$W(p) = \max_{(x,\ i)} [D_x(p) + \{1-D_x(p)\}(T_x p)_i]$$

where $D_x(p)$ and $T_x p$ are given by (3.3) and (3.4) respectively.

We consider a relation between a detection search policy and a whereabouts search policy. The expected payoff by a whereabouts search policy (x, i) is $D_x(p) + \{1-D_x(p)\}(T_x p)_i$. On the other hand, the expected payoff by a detection search policy x is $D_x(p)$ since the payoff of detection is unit. Since $\{1-D_x(p)\}(T_x p)_i \geq 0$, it seems that a whereabouts search policy (x, i) is an improvement for a detection search policy x. But this is based on the fact that the payoff of correct guess is positive (in this case, it is assumed to be unit). If we let α be the payoff of correct guess, then the expected payoff by a whereabouts search policy (x, i) is given by $D_x(p) + \alpha\{1-D_x(p)\}(T_x p)_i$. Under a detection search criterion, since a correct guess with no detection is entirely worthless ($\alpha=0$), a whereabouts search policy is not an improvement for a detection search policy. Under a whereabouts search criterion in a wide sense ($\alpha>0$), a detection search policy is not worthy of note since no guess is obviously nonsense under this criterion. Substituting (3.3) and (3.4) into (5.1), we can obtain

$$(5.2)\quad W(p) = \max_{(x,\, i)}[1-\sum_{k(\neq i)} p_k \exp(-\lambda_k x_k)]$$

$$= 1 - \min_{1\leq i\leq n} (1-p_i) \min_{x} \sum_{k(\neq i)} \frac{p_k}{1-p_i} \exp(-\lambda_k x_k).$$

If box i should be guessed, then by the right-hand side of (5.2), it is not optimal to allocate the effort into box i, that is, $x_i^*=0$. Moreover if we put $p_k'=p_k/(1-p_i)$, then $\sum_{k(\neq i)} p_k'=1$. Therefore if we fix a guessed box i, the problem

$$(5.3)\quad \sum_{k(\neq i)} p_k' \exp(-\lambda_k x_k) \longrightarrow \min_{x}$$

becomes a detection search problem. Hence the optimal allocation policy for the whereabouts search problem is the same as the optimal detection search policy (model [2] in Table 1) for n-1 boxes except the guessed box. Thus we can obtain the following theorem.

Theorem 6. The optimal allocation of search effort is the same as the optimal detection search policy for all other boxes except the guessed box. The guessed box can be obtained as follows:
Solve a detection search problems (5.3) for each box i $(=1,\ldots, n)$, substitute the results into (5.2) and find a box attaining $\min_{1\leq i\leq n}$ in (5.2).

Theorem 6 is similar to the well-known result (Kadane[9]) in the case of discrete search effort.

6. Conclusion

Under the detection search criterion, this paper has resolved a unique remained problem, that is, a detection search game with continuous search effort. In Sections 3 and 4, the information search problem has been discussed in both a one-sided case and a two-sided case. Their solution have been obtained in such a form that numerical problems can be solved with the help of a computer. Finally in the case of continuous search effort the fundamental theorem for the whereabouts search problem has been obtained. In Table 1, some basic search models are still unsolved. As a result of attacking the models, it seems that if an analytic and explicit solution for a one-sided model is not obtained, then it is difficult to solve the corresponding two-sided model (game). For a moving target, we can consider the same basic search models as in Table 1. In fact, in recent fifteen years there has been a remarkable progress in the area of one-sided search models for a moving target. On the other hand, the two-sided search model becomes the evasion-search game in which player I (evader) can move of his own free will among search, but there are few papers in this area.

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