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THE NASH BARGAINING SOLUTION AS MUTUAL EXPECTED-UTILTY MAXIMIZATION

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Abstract The Nash bargaining problem is considered as a one-shot decision problem by two expected-utility maximizers with beliefs in the form of probability distributions over the opponent's strategy choices. Assuming three conditions as axioms on the formation of beliefs of both players, we derive the Nash solution as the optimal decisions of the mutual maximization problems.

1. Introduction

The Nash bargaining solution, characterized as the utility-product maximization, can be derived in several ways; the well-established axiomatic approach and the noncooperative-game approach due originally to Nash [1950, 1953], Bayesian negotiation models due to Harsanyi [1956, 1977], the probabilistic model of Anbar and Kalai [1978], and a more recent, new axiomatization by Binmore [1984] from a viewpoint of convention of the bargaining.

Among these models, the one-shot probabilistic model of Anbar and Kalai [1978] derives the Nash solution as an optimal outcome of mutual expected-utility maximization. In their model, the two players are assumed to have beliefs over the opponents' strategy sets in the form of cumulative probability distribution functions, and choose the strategies that maximize the expected utilities. Their main result is a complete characterization of the pairs of beliefs that generate Pareto optimal outcomes for *all* feasible sets of utilities. The Nash solution is obtained for each feasible set of utilities

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if the pair of beliefs is given by the uniform distribution.

The question to be raised here is that what is the rationale for the players to have particular beliefs in the one-shot bargaining situation, without which the justification of the Nash solution through such a probabilistic model would not be complete. There is a critical discussion that such a probabilistic model lacks a theory of how players come to draw particular beliefs (see, Roth [1979]).

In this paper, we shall try to justify a particular formation of beliefs in the one-shot bargaining situation, and derive the Nash solution thereby. In doing so, we do not assume the beliefs to be invariant for all feasible sets of utilities. This would be a desirable prerequisite, since the formation of beliefs will in general depend on the given feasible set of utilities. Instead, we assume that the beliefs are invariant only for all feasible sets which are *near-enough* to a given one. This assumption may be justified as a slight imperfection or insensitivity inherent in the mechanism of forming beliefs; namely, small-enough differences in the feasible sets cannot be reflected in the formation of beliefs, so that players may continue to hold the same beliefs if such a perturbation of the Pareto surface should occur. The perfect mechanism is the one that can be reached only through imperfect ones by reducing the degree of imperfectness (insensitivity).

A correspondence that associates to each bargaining game a nonempty set of pairs of beliefs will be called a *mechanism* of forming beliefs. Three requirements will be imposed, as axioms, on the mechanism. The first axiom states that the mechanism should generate those pairs of beliefs which are *optimality-consistent*; namely, each pair should generate a unique Pareto optimal payoff vector as a result of mutual expected-utility maximization. Secondly, we require that the resulting Pareto optimal payoff vector should be *belief-compatible* in the sense that it is most probable as an outcome of the game under the pair of beliefs. We call this property the *outcome belief-compatibility*. The third axiom is a formal statement of the above mentioned imperfectness property that the mechanism is insensitive to small-enough differences in the feasible sets. The treatment of beliefs due to Anbar and Kalai [1978] corresponds to the case where the insensitivity is global; that is, the mechanism is insensitive to any variation of the feasible set in the unit square.

With these conditions on the mechanism, we derive a necessary condition for a pair (F,G) of beliefs to satisfy. It will be shown that (F,G) must be linear in a neighborhood of the Nash solution, and the outcome of the mutual maximization coincides with the Nash solution.

The basic bargaining game and the maximization problem of each player are stated in the next section. In Section 3, we give the motivation and definitions of the requirements, and state the theorem. Finally, in Section 4, we give the proof of the theorem.

2. The Model

The bargaining game considered in this paper is a pair (S,c), where S is the set of feasible payoff vectors and c is the conflict payoff vector.
We assume:

(a) S is a compact, convex subset of the unit square,

(b) $c=(0,0)\in S$, $(0,1)\in S$ and $(1,0)\in S$.

The payoffs are given by the normalized von Neumann-Morgenstern utilities. The rule of the game is as follows: players 1 and 2 independently choose x in A and y in B, respectively, where A and B are the same closed unit interval [0,1]. If the chosen strategy pair (x,y) is in S, then they each receive x and y. Otherwise, the conflict payoffs $c=(0,0)$ result. The Nash solution (x^0,y^0) is a point in S with the property:

$$x^0y^0 = \max \{xy \mid (x,y)\in S\}.$$

Consider, hereafter, the bargaining game (S,c) as one played by expected-utility maximizers. That is, each player is assumed to have a belief about the opponent's choices in the form of a probability distribution over the strategy set of the opponent. So, let F and G be cumulative probability distribution functions over B and A, respectively. By $F(y)$, we express the belief of player 1 that player 2's strategy choice is less than or equal to y, and $G(x)$, the belief of player 2 that player 1's strategy choice is less than or equal to x.

By choosing suitably two continuous, concave non-increasing functions $\psi(x)$ and $\phi(y)$ on A and B, respectively, S can be represented by

$$\begin{aligned} S &= \{(x,y)\in E_+^2 \mid y\leq\psi(x),\ x\in \mathrm{A}\} \\ &= \{(x,y)\in E_+^2 \mid x\leq\phi(y),\ y\in \mathrm{B}\}. \end{aligned}$$

Let (x,y) be a pair of strategy choices of player 1 and 2. Then, under the rule of the game, player 1 obtains the payoff x if $y\leq\psi(x)$ and obtains 0 otherwise. Similarly, player 2 obtains y if $x\leq\phi(y)$ and 0 otherwise. Thus, the expected payoff to each player when they each choose x and y are given respectively by

$$xF(\psi(x)) + 0(1-F(\psi(x))) = xF(\psi(x)),$$

$$yG(\phi(y)) + 0(1-G(\phi(y))) = yG(\phi(y)).$$

Each player then tries to maximize his own expected payoff. Thus, following Anbar and Kalai [1978], we define as follows:

Definition 1. A point $(x,y) \in A\times B$ is called $(F,G;S)$-optimal, or simply (F,G)-optimal if

$$xF(\psi(x)) = \max \{x'F(\psi(x')) \mid x' \in A\},$$

$$yG(\phi(y)) = \max \{y'G(\phi(y')) \mid y' \in B\}.$$

Note that there always exists an (F,G)-optimal point. We say a point $(x,y) \in S$ is *Pareto optimal* if there is no other point $(x',y') \in S$ such that $x' \geq x$ and $y' \geq y$. An (F,G)-optimal point (x,y) may not be Pareto optimal, nor even feasible. This depends on the pair of beliefs (F,G).

Definition 2. A pair of beliefs (F,G) is said to be *optimality-consistent* if *every* (F,G)-optimal point (x,y) is Pareto optimal.

An immediate example of an optimality-consistent pair is one such that F and G are uniform. Another example would be, say, the pair (F,G) such that $F(y)=y^2$ and $G(x)=x^{1/2}$.

Notice that the optimality-consistency is defined relative to the given bargaining game (S,c). The globally optimal pair (F,G) defined by Anbar and Kalai [1978] corersponds to the case where (F,G) is optimality-consistent for *every* (S,c) satisfying (a) and (b). We do not need such a strong requirement.

Finally, we state a preliminary result which follows immediately from Lemma 1 in Anbar and Kalai [1978].

Lemma 1. Let (S,c) be given, and let (F,G) be an optimality-consistent pair of beliefs. Then, there is a unique (F,G)-optimal point (x,y). Moreover, $xF(\psi(x))>0$ and $yG(\phi(y))>0$.

3. Plausible Beliefs

As assumed in the previous section, each player is an expected-utility maximizer. They know each other that the opponent is also an expected-utility maximizer. To be consistent with this knowledge, therefore, each player must take into account the belief which the opponent might have over his choice of strategies. Thus, each player must assume a pair of beliefs; one for his opponent's choices, and the other for his own choices which he thinks the opponent might have over his own choices. With this pair of beliefs in mind, each player then tries to infer what is a probable outcome of the game.

To consider the game in this scenario, we need criteria on what are plausible pairs of beliefs to each player. We shall formulate three conditions on the formation of such pairs of beliefs.

Let Σ be the class of all feasible sets satisfying condition (a) and (b) in Section 2. Given $S\in\Sigma$, let $f(S)$ be a nonempty set of pairs of beliefs (F,G). We call f the *mechanism* of forming beliefs and interpret (F,G) to be plausible if $(F,G)\in f(S)$. The first requirement of f is the following.

Axiom 1. (Optimality-Consistency). Let $S\in\Sigma$ and let $(F,G)\in f(S)$. Then (F,G) is optimality-consistent.

This axiom simply requires that a plausible pair of beliefs should be one that generates a Pareto optimal outcome. Each player must consider which pair of beliefs is plausible. But, whatever it may be, it would be irrational for each player to assume a pair that does not generate a Pareto optimal outcome.

For each $S\in\Sigma$, denote by $P(S)$ the Pareto optimal surface of S. Also, denote by (x^S,y^S) any $(F,G;S)$-optimal point which may be Pareto optimal or not (see the remark of Definition 1). Then, the second condition on f can be stated as follows:

Axiom 2. (Outcome Belief-Compatibility). Let $S\in\Sigma$ and let $(F,G)\in f(S)$. If $(x^S,y^S)\in P(S)$, then there exists no point $(x,y)\in P(S)$ such that

$$F(y)G(x) > F(y^S)G(x^S).$$

To motivate this axiom, consider any point $(x,y)\in P(S)$. Then, under the rule of the game, we have

Prob $[(x,y)\in P(S)$ is the outcome of the game$]$
= Prob [player 1 obtains x *and* player 2 obtains y]
= Prob [player 1 thinks player 2's choice y' to be $y'\le y$
and player 2 thinks player 1's choice x' to be $x'\le x$]
= $F(y)G(x)$.

Thus, Axiom 2 requires that a plausible pair (F,G) should be one with the property that the (F,G)-optimal point (x^S,y^S) is compatible with the maximal probability of obtaining the point (x^S,y^S) as the outcome of the game. In this case, we say the $(F,G;S)$-optimal point is *belief-compatible*.

The outcome belief-compatibility may be justified in our scenario, because in the one-shot situation any player cannot revise the belief at all so that he must rely on the belief the outcome by which he thinks is most probable.

To state the last condition, we need some notations. Let $S \in \Sigma$ be given fixed, and let $\varepsilon>0$ be a sufficiently small number. Consider, then, the subclass $\Sigma(\varepsilon)$ of Σ with the Hausdorff distance δ from S less than ε, i.e.,

$$\Sigma(\varepsilon) = \{ T \in \Sigma \mid \delta(S,T) < \varepsilon \} .$$

$\Sigma(\varepsilon)$ is an expression of possible small variations of the given feasible set S, Then:

Axiom 3. (ε-insensitivity). Let $S \in \Sigma$ be fixed, and let $(F,G) \in f(S)$. Then, there exists $\varepsilon>0$ such that for any $T \in \Sigma(\varepsilon)$, $(F,G) \in f(T)$.

Axiom 3 states that the mechanism of forming beliefs may have a slight *imperfection* or *inertia* such that once a particular belief is formed, it cannot be sensitive to *small-enough* differences from the given feasible set S. As we mentioned in the Introduction, it would be a realistic characterization to assume the imperfectness as an inherent nature of the mechanism. The perfect mechanism is the one corresponding to the limit when the degree of the imperfectness, i.e., $\varepsilon>0$, tends to zero.

We can now state our theorem.

Theorem 1. Given $S \in \Sigma$, let $(F,G) \in f(S)$. Then, under axioms 1,2 and 3, we have:

(i) The $(F,G;S)$-optimal point coincides with the Nash solution for (S,c).

(ii) Let (x^0,y^0) be the Nash solution for (S,c). Then there are intervals $d(y^0) \subset B$ containing y^0, and $d(x^0) \subset A$ containing x^0 such that

$$F(y) = [F(y^0)/y^0]y \quad \text{for all } y \in d(y^0), \text{ and}$$
$$G(x) = [G(x^0)/x^0]x \quad \text{for all } x \in d(x^0).$$

The proof will be given in the next section. In Figure 1, we illustrate Theorem 1. This theorem implies in particular that any pair (F,G) such that $F(y)=y^h$, $G(x)=x^{1/h}$ with $h>1$ is *not* in $f(S)$, though it satisfies Axiom 1.

Theorem 1 gives a necessary condition for a pair (F,G) to be in $f(S)$. It imposes no restriction on the beliefs away from the neighborhood of the Nash solution. The pair of uniform distributions is a typical example that satisfies the condition, but, of course, the theorem does not say that it necessarily belongs to $f(S)$. In fact, there are cases in which the pair of uniform distributions does not a priori appear plausible. This occurs when there exists $y'>0$ with $(1,y') \in S$, in which case player 1 may expect $F(y)=0$ for all y with $0 \leq y < y'$.

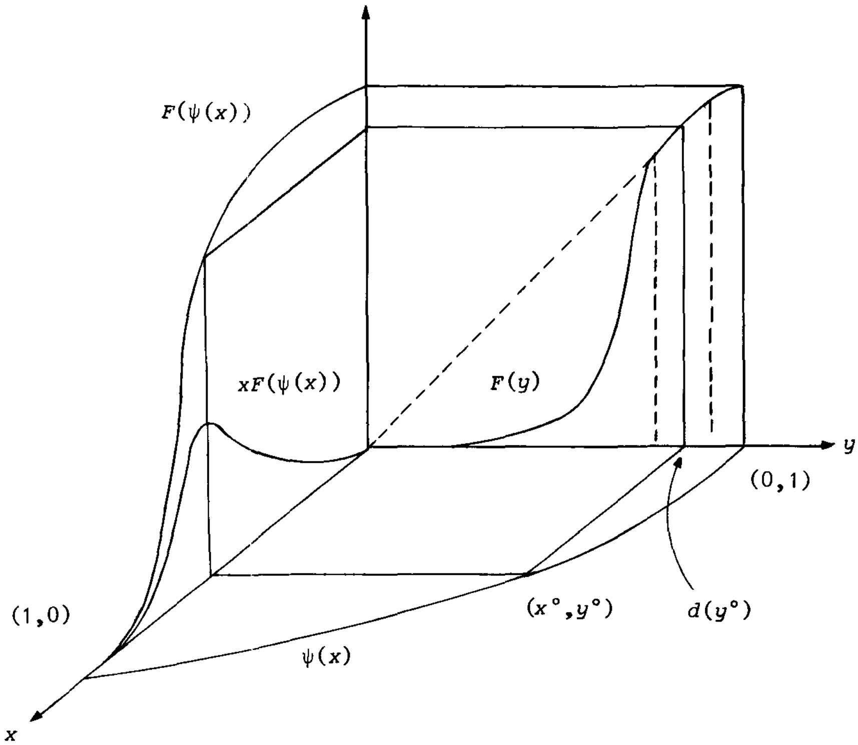

Figure 1

An example of the class of pairs of beliefs satisfying axioms 1 and 2 would be one consisting of all (F,G) such that for all $T \in \Sigma(\varepsilon)$, the Nash solution (x^0,y^0) for T satisfies,

$$F(y^0)/y^0 \geq F(y)/y \quad \text{for all } y \in \mathrm{B}, \quad \text{and}$$

$$G(x^0)/x^0 \geq G(x)/x \quad \text{for all } x \in \mathrm{A}.$$

The proof is straightforward, and it can be seen that any pair in this class has the necessary linear portion around the Nash solution. The pair of uniform distributions is in this class, but, in a sense, it is the *least* plausible one in this class. A more plausible pair in this class would be one that approximates the degenerate pair of beliefs with all the mass at the Nash solution. Such an approximation can be considered consistent with the whole of the optimal pairs of choices (x^0,y^0) for *all* $T \in \Sigma(\varepsilon)$.

4. Proofs

Proposition 1. Given $S \in \Sigma$, let $(F,G) \in f(S)$ with axioms 1 and 2. Assume that the $(F,G;S)$-optimal point $(x^*,\psi(x^*))$ satisfies $x^*<1$ and $y^*<1$, and that G and ψ are differentiable at $x=x^*$, and F is differentiable at $y=\psi(x^*)$. Then, the point $(x^*,\psi(x^*))$ is the Nash solution for (S,c).

Proof: Let us denote the differentiation by the prime. Then, by differentiating at $x=x^*$, it follows from Axiom 1 that

(1) $$F(\psi(x)) + xF'(\psi(x))\psi'(x) = 0$$

(2) $$\psi'(x)G(x) + \psi(x)G'(x) = 0$$

Also, by differentiating at $x=x^*$, it follows from Axiom 2 that

(3) $$F'(\psi(x))\psi'(x)G(x) + F(\psi(x))G'(x) = 0$$

Note that, by Lemma 1, we have

$$F(\psi(x)) = -xF'(\psi(x))\psi'(x) \neq 0 \quad \text{at } x=x^*.$$

Then (1) and (3) imply

(4) $$G(x) - xG'(x) = 0$$

By Lemma 1 we have $G(x^*)>0$, so that (4) and (2) imply

$$x\psi'(x) + \psi(x) = 0 \quad \text{at } x=x^*.$$

But, this implies that the point $(x^*,\psi(x^*))$ is the Nash solution.

Proposition 1 shows that with the differentiability, axioms 1 and 2 are enough to obtain the result.

The next lemma is a variant of Lemma 2 in Anbar and Kalai [1978], but we give the proof for completeness.

Lemma 2. Given $S\in\Sigma$, let $(F,G)\in f(S)$ with axioms 1 and 3. Assume that the $(F,G;S)$-optimal point (p,q) satisfies $p<1$ and $q<1$. Then, there is a number $t_q>1$ such that $t_q q\leq 1$ and for any t with $1\leq t<t_q$,

(i) $S_t\in\Sigma(\varepsilon)$, where $S_t = \{(x,ty)\mid(x,y)\in S\}\cap(A\times B)$, and

(ii) (p,t_q) is $(F,G;S_t)$-optimal.

Similarly, there is a number $t_p>1$ such that $t_p p\leq 1$ and for any t with $1\leq t<t_p$,

(iii) $R_t\in\Sigma(\varepsilon)$, where $R_t = \{(tx,y)\mid(x,y)\in S\}\cap(A\times B)$, and

(iv) (tp,q) is $(F,G;R_t)$-optimal.

Proof: Let $\varepsilon>0$ be chosen by Axiom 3. Define t_q by

$$t_q = \sup\{t\mid(p,tq)\in P(T),\ T\in\Sigma(\varepsilon)\}.$$

Then (i) follows. To show (ii), let S_t be given by

$$S_t = \{(x,z)\,|\,x\le\phi_t(z),\ \ z\in B\}.$$

Notice that $\phi_t(z) = \phi(z/t)$ if $z<1$, and that $q<1/t$ because $1\le t<t_q$. Then,

$$\max\ \{zG(\phi_t(z))\,|\,z\in B\} = t\cdot\max\ \{yG(\phi(y))\,|\,0\le y\le 1/t\}$$
$$= tqG(\phi(q)) = (tq)G(\phi_t(tq)).$$

By axioms 3 and 1, it follows that $(F,G)\in f(S_t)$ and (F,G) is optimality consistent for S_t. Then, by the uniqueness of the $(F,G;S_t)$-optimal point (Lemma 1), the above equality implies that (p,tq) is the $(F,G;S_t)$-optimal point. The proofs for (iii) and (iv) are analogous.

Lemma 3. Given $S\in\Sigma$, let $(F,G)\in f(S)$ with axioms 1 and 3. Assume that the $(F,G;S)$-optimal point (p,q) satisfies $p<1$ and $q<1$. Then, there are intervals $[q,y^+)$ and $[p,x^+)$ such that F and G are respectively continuous. Moreover, if the function ψ(and ϕ) is differentiable at $x=p$ (at $y=q$), then F and G are differentiable in (q,y^+) and (p,x^+), respectively, and

(i) $\quad F'(y) = [F(y)/y][q/p](-\phi'(y)|_{y=q})$, for all $y\in(q,y^+)$,

(ii) $\quad G'(x) = [G(x)/x][p/q](-\psi'(x)|_{x=p})$, for all $x\in(p,x^+)$.

Proof: Let $\varepsilon>0$ be chosen by Axiom 3. For any t, $(1\le t<t_q)$, consider the feasible set $S_t\in\Sigma(\varepsilon)$ defined in Lemma 2. By Lemma 2, the point (p,tq) is $(F,G;S_t)$-optimal. Hence,

$$pF(tq) \ge xF(t\psi(x)) = \phi(y/t)F(y) \quad \text{for } 0\le y\le 1.$$

Also, for any s $(1\le s<t_q,\ s\ne t)$,

$$pF(sq) \ge xF(s\psi(x)) = \phi(y/s)F(y) \quad \text{for } 0\le y\le 1.$$

Since t and s are arbitrary with $1\le t,\ s<t_q$, we have

(5) $\quad pF(tq) \ge \phi(sq/t)F(sq) \quad$ for $1\le s<t_q$

and

(6) $\quad pF(sq) \ge \phi(tq/s)F(tq) \quad$ for $1\le t<t_q$

It then follows from (5) that

(7)
$$F(sq) - F(tq) \le F(sq)(-1/p)[\phi(sq/t)-\phi(q)]$$
$$= F(sq)(-1/tp)(sq-tq)[\phi(sq/t)-\phi(q)]/[sq/t - q]$$

Also, from (6),

(8)
$$F(sq) - F(tq) \ge F(tq)(-1/p)[\phi(q)-\phi(tq/s)]$$
$$= F(tq)(-1/sp)(sq-tq)[\phi(tq/s)-\phi(q)]/[tq/s - q]$$

Combining (7) and (8), and letting $s \to t$, we see that F is continuous at $y=tq$ $(1 \leq t < t_q)$ because ϕ is continuous by assumption. If ϕ is differentiable at $y=q$, then F is differentiable at $y=tq$, $(1<t<t_q)$. Hence, (i) follows with $y^+=t_q q$. For (ii), the proof is analogous.

Proof of Theorem 1: Let $(F,G) \in f(S)$. By Axiom 1, (F,G) is optimality consistent, so that there is a unique $(F,G;S)$-optimal point (p,q).

Case 1: $p<1$ and $q<1$. Assume first that ψ (and ϕ) is differentiable at $x=p$ (at $y=q$). Choose $t>1$ such that $tp<1$, $tq<1$ and $S_t, S_{tt} \in \Sigma(\varepsilon)$ where

$$S_t = \{(x,ty) \mid (x,y) \in S\} \cap (A \times B),$$

$$S_{tt} = \{(tx,ty) \mid (x,y) \in S\} \cap (A \times B).$$

By axioms 1, 3 and Lemma 2, (p,tq) is $(F,G;S_t)$-optimal, and (tp,tq) is $(F,G;S_{tt})$-optimal. Then, by Lemma 3, F and G are differentiable, respectively, at $y=tq$ and $x=tp$. Hence, by Axiom 2 and Proposition 1, (tp,tq) is the Nash solution for S_{tt}, that is,

$$(tp)(tq) = \max \{(tx)(ty) \mid (x,y) \in S\}.$$

This implies that (p,q) is the Nash solution for S.

Next, by differentiating $x\psi(x)$ and $y\phi(y)$ at $x=p$ and $y=q$, respectively, we obtain

(9) $\quad x\psi'(x) + \psi(x) = 0 \quad$ at $x=p$

(10) $\quad y\phi'(y) + \phi(y) = 0 \quad$ at $y=q$

Then, it follows from (9), (10) and (i), (ii) of Lemma 3 that

$$F'(y) = F(y)/y \quad \text{for all } y \in (q,y^+),$$

$$G'(x) = G(x)/x \quad \text{for all } x \in (p,x^+).$$

Hence, we obtain

(11) $\quad F(y) = [F(q)/q]y \quad$ for all $y \in [q,y^+]$

(12) $\quad G(x) = [G(p)/p]x \quad$ for all $x \in [p,x^+]$

When ψ is not differentiable at $x=p$, we approximate S by a smooth $T \in \Sigma(\varepsilon)$ such that

$$T \subset S,\ \psi_t(1)=\psi(1) \text{ and } \phi_t(1)=\phi(1).$$

Then, by Axiom 3, $(F,G) \in f(T)$. By Axiom 1 and Lemma 1, there exists a unique $(F,G;T)$-optimal point, which we denote by (p_t,q_t). Then, we may assume that (p_t,q_t) is in the interior of A×B, which can be seen as follows: Since $q \neq 1$, we have

$$\phi(q)F(q) > \phi(1)F(1) = \phi_t(1)F(1).$$

But, we may choose ϕ_t which is near-enough to ϕ so that

$$\phi(q)F(q) > \phi_t(q)F(q) > \phi_t(1)F(1).$$

Hence $q_t \neq 1$, that is $q_t<1$. That $p_t<1$ is similarly verified.

Thus, we have a sequence of smooth approximations T with each $(F,G;T)$-optimal point being the interior Nash solution for T. This proves that (p,q) is the limit of the sequence of the Nash solutions, so that (p,q) is also the Nash solution for S.

The linearity of F and G can be proved by choosing a smooth $T \in \Sigma(\varepsilon)$ with the $(F,G;T)$-optimal point being equal to (p,q).

Case 2: $p=1$ or $q=1$. Let $q<1$ and assume that the curve $x=\phi(y)$ is kinked at the $(F,G;S)$-optimal point $(1,q)$. But, this case is essentially the same as the above non-differentiable case by considering the approximations such that $\psi_t(1)=0$.

Assume, next, that $x=\phi(y)$ is differentiable at $y=q$, so that $\phi'(y)=0$ at $y=q$. We can show, however, that this case cannot occur. By (i) of Lemma 3, it must be true that

$$F'(y) = 0 \quad \text{for all } y \text{ with } q<y<y^+ \tag{13}$$

Consider the feasible set $T = \{(x,y) \mid x \le \phi_t(y)\} \in \Sigma(\varepsilon)$ such that

$$\phi_t(y) < \phi(y) \quad \text{for all } y \text{ with } q<y\le 1,$$

$$\phi_t(y) = \phi(y) \quad \text{for all } y \text{ with } 0\le y\le q, \quad \text{and}$$

$$\lim_{y\to q} \phi_t'(y) \neq 0.$$

Then, clearly, $(1,q) \in T \subset S$ and the curve $x=\phi(y)$ is kinked at $(1,q)$. Moreover, the point $(1,q)$ is $(F,G;T)$-optimal as shown below:

$$1F(\psi_t(1)) = \phi(q)F(q) \ge \phi(y)F(y)$$

$$\ge \phi_t(y)F(y) = xF(\psi_t(x)) \quad \text{for all } x \in A, \quad \text{and}$$

$$qG(\phi_t(q)) = qG(\phi(q))$$

$$\ge yG(\phi(y)) \ge yG(\phi_t(y)) \quad \text{for all } y \in B.$$

Hence, approximating T by a smooth $R \in \Sigma(\varepsilon)$ such that $R \subset T$ and

$$(1,y) \notin R \quad \text{for all } y>0,$$

we obtain, by axioms 1 and 3, the $(F,G;R)$-optimal point (p_r,q_r) satisfying $p_r<1$ and $q_r<1$. Then, since $\phi_r'(y)\neq 0$ for all $y \in (0,1)$, it follows from (i) of Lemma 3 that for some $y_r^+>q_r$,

$F'(y) \neq 0 \quad$ for all $y \in (q_r, y_r^+)$.

Hence, letting $\delta(T,R) \to 0$, we can find $y \in (q, y^+)$ that satisfies $F'(y) \neq 0$. This contradicts (13).

Thus we have proved that $(F,G;S)$-optimal point coincides with the Nash solution for (S,c), and that F and G satisfy (11) and (12).

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