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THE OPTIMAL SEARCH PLAN FOR A MOVING TARGET MINIMIZING THE EXPECTED RISK

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Abstract This paper deals with an optimal search problem in which a target moves in a target space stochastically and the amount of search cost being continuously divisible is restricted in certain rate at each time. The optimal allocation of search effort and the stopping time of the search, which minimize the expected risk, are sought. Necessary and sufficient conditions for the optimal search plan are derived and physical meanings of the conditions are elucidated. An algorithm for numerical calculation of the optimal search plan and examples are also discussed.

1. Introduction

A problem of optimal search and stop for a moving target is dealt with in this paper. Suppose a target moves in a target space and a searcher wishes to find it efficiently. The target is assumed to move as a stochastic process with parameters known to the searcher. It is assumed that the search is started at t_0, and to be ended by T at the latest. His available search cost is limited to $m(t)$ per unit time at each time $t \in [t_0,T]$. The total search cost rate $m(t)$ is assumed to be divisible in any way and a regular detection function, the definition of which will be given later in the next section, is also assumed. The search cost is proportional to the search effort applied to the target space and the searcher earns a reward $R(t)$ when he successfully detects the target at t. It is assumed that the searcher wishes to minimize the expected risk of the search (the expected search cost minus the expected reward). The search plan which minimizes the expected risk of the search is called optimal.

The optimal search problems for a moving target have been studied by many authors. In Dobbie's paper [3] published in 1963, he discussed some unsolved problems in search theory and pointed out the necessity of studies of the moving target problems. There came to existence a number of papers on the optimal search problems for a moving target in the 1970's. In 1970, Pollock [10] formulated a search model for a target which moved between two regions in a Markovian fashion. He gave an optimal search plan that maximized the detection probability with a given number of looks and an optimal plan that minimized the expected number of looks to find the target. Later Dobbie [4] dealt with a time continuous version of Pollock's model, but the study was restricted to the two-box model as yet. In 1972, Hellman [6] studied a maximization problem of the detection probability for a target moving according to a diffusion process and found necessary conditions for the optimal search. Iida [7] derived necessary and sufficient conditions for the optimal search plan for the target moving along a path selected among a given set of paths with a known probability. Saretsalo [12] dealt with a search model in which the target motion belonged to a large class of Markov process. Stone and Richardson [13] investigated a search problem for a moving target of a special class called conditionally deterministic motion. Later Stone [14] generalized their model to include the above-mentioned Iida model. Kan [9] generalized the Pollock model to n-box problem and he also considered a problem with a stop option under the criterion of the expected net return (i.e. expected reward minus expected searching cost). Brown [1] considered an optimal search maximizing the detection probability with continuously divisible search effort and proposed an efficient algorithm for calculating the optimal search plan. The algorithm was essentially the same as that employed by Iida in computing his numerical example, and was later named FAB algorithm by Washburn. Washburn [19] dealt with a discrete search problem in which all search effort had to be placed in a single cell at each t and gave an upper bound on the detection probability [20]. He also investigated a search model with a penalty when the target was not detected and proposed a generalized FAB algorithm [21]. A whereabouts search model for a moving target was investigated by Stone and Kadane [16] and a surveillance search model for a moving target by Tierney and Kadane [18]. Eagle [5] investigated the optimal search that maximized the probability of detecting a moving target, in which the search path was constrained.

The theorems of the maximization problem of detection probability for moving targets were generalized by Stone [15] who derived necessary and sufficient conditions for the optimal search plan assuming a concave detection function and practically no restrictions on the stochastic process used to model the target motion. In 1981, Stromquist and Stone [17] generalized the theory to include a wide class of non-linear, non-separable functional and separable constraints. Their theorem can be applied not only to the maximization problem of the detection probability for a moving target, but also to a large variety of optimal search problems such as minimizing the expected search effort or the like.

As for the optimal search problem for a stationary target, many authors have dealt with the problem under various measures of effectiveness of the search operation such as the detection probability, the whereabouts probability, the expected time or cost until detection, and the expected risk or reward of the search. One of the authors, Iida [8], investigated the optimal search plan minimizing the expected risk of the search for a stationary target. We think similar models are worth investigating for a moving target. In this paper, applying the theorems given by Stromquist and Stone [17], we derive necessary and sufficient conditions for the optimal search plan which minimizes the expected risk of the search for a moving target.

In the next section, describing the assumptions of the model precisely, we formulate it as a minimizing problem of a functional. In Section 3, we derive necessary and sufficient conditions for the optimal search plan when the search time is limited to $[t_0,T]$ and then we give the conditions for the optimal stopping time when the search time is not restricted, namely $T \to \infty$. Section 4 is devoted to discuss FAB algorithm to calculate the optimal search plan and two examples are presented. Finally, various discussions are given in Section 5.

2. Formulation of the Problem

In this section, defining system parameters of the search precisely, we formulate the problem mentioned above as a variational problem. The detailed descriptions of the search problem are as follows.

1. We consider a target space $\boldsymbol{Y}$ with σ-finite measurer η and a continuous time interval $\boldsymbol{T} = [0,T]$ of the search time. Let μ be the product measure on $\boldsymbol{Y} \times \boldsymbol{T}$ and $\boldsymbol{Z}$ be a μ- measurable subset of $\boldsymbol{Y} \times \boldsymbol{T}$. Denote its t-section by $\boldsymbol{Z}_t = \{y \in \boldsymbol{Y} \mid (y,t) \in \boldsymbol{Z}\}$.
2. Suppose the target's motion is given by a stochastic process $X = \{X_t \in \boldsymbol{Z}_t\}$, where X_t is the position of the target at t. We assume that X_t has a probability density function $q_t(X_t = y)$ defined on $\boldsymbol{Z}_t$.
3. We assume that the search is started at t_0 and the searcher may stop the search at any time whenever the search does not pay. m: $\boldsymbol{T} \to (0,\infty)$ is assumed to be given, where $m(t)$ is a total search cost rate being available to the searcher at t and is assumed to be continuously divisible in the target space.

4. A search plan is a Borel measurable function ϕ_T: $Z \to [0,\infty)$ which specifies a density of search effort in both time and space. The set of $\phi_T(y,t)$ is denoted by $\phi_T(= \{\phi_T(y,t),\ (y,t) \in Z\})$. (Sometimes, the suffix T of ϕ_T or $\phi_T(y,t)$ is omitted if no confusion is expected.)

5. Let α: $Z \to (0,\infty)$ be μ-measurable. For each sample path of the process X, the probability of detecting the target given it follows that path is a function of the weighted total search effort density ψ $(= \int_{t_0}^{t} \alpha(X_\tau,\tau)\phi_T(X_\tau,\tau)d\tau)$ which is the cumulative search effort density on the target over the path until t. The weight $\alpha(y,t)$ represents the relative detectability to the target if it stays at point y at time t. There is a function b: $[0,\infty) \to [0,1]$, such that $b(\psi)$ is the probability of detecting the target by t given it follows the given path and effort density ψ is applied. We assume that $b(\psi)$ is a continuous, differentiable and strictly concave function of $\psi > 0$ with $b(0) = 0$ and $b(\infty) = 1$, and the derivative is positive, continuous and bounded, $0 < b'(\psi) < \infty$. (The function satisfying these properties is called a regular detection function.)

6. R: $T \to [0,\infty)$ is assumed to be given. $R(t)$ is a reward earned by the searcher if the target is detected at t. $R(t)$ is assumed to be a non-negative, non-increasing and differentiable function of t.

7. Let c_0: $Z \to (0,\infty)$ be μ-measurable. $c_0(y,t)$ is the cost of unit search effort density allocated to y at t.

8. The measure of effectiveness for a search plan is assumed to be the expected risk of the search, i.e., the expected search cost until detection or stopping, whichever comes first, minus the expected reward. (This measure of effectiveness was first employed by Ross [11] to investigate the optimal search and stop and later by many authors.)

Under the assumptions described above, conditions for the optimal search plan are derived in the following two steps. As the first step, we deal with a problem with a finite T (T is called the limit time of the search) and the optimal distribution ϕ_T^* of search effort is sought. Then, as the second step, we consider a problem with no restriction on the limit of the search time and the condition for the optimal stopping time T^* is obtained. It is obvious that the search plan $\phi_{T^*}^*$ is optimal when the search time is not limited.

Suppose a limit time T of the search is given. The expected risk of the search in which a search plan ϕ is employed is obtained as follows. The search plan $\phi = \{\phi(y,t),\ (y,t) \in Z\}$ must satisfy the next conditions from Assumption 3.

(1) $$\phi(y,t) \geq 0, \qquad \int_{Z_t} c_0(y,t)\phi(y,t)d\eta(y) \leq m(t), \qquad (y,t) \in \mathbf{Z},$$

for a.e. $t \in \mathbf{T}$. The search plan ϕ which satisfies the above conditions for all t is called a feasible plan and the set of the feasible plan is denoted by Ψ. Ψ_0 is defined as the set of $\phi \in \psi$ such that

(2) $$\int_{Z_t} c_0(y,t)\phi(y,t)d\eta(y) = m(t).$$

Our attention is concentrated on the plan $\phi \in \Psi$ or Ψ_0 hereafter. The detection probability $P_t(\phi)$ with plan $\phi \in \Psi$ by time t is given by

(3) $$P_t(\phi) = E[b(\int_{t_0}^{t} \alpha(X_\tau,\tau)\phi(X_\tau,\tau)d\tau)],$$

where $E[\cdot]$ is the expectation over the sample paths of X.

The cumulative search cost $C(t,\phi)$ by t is given by

(4) $$C(t,\phi) = \int_{t_0}^{t}\int_{Z_\tau} c_0(y,\tau)\phi(y,\tau)d\eta(y)d\tau,$$

and

(5) $$\frac{\partial C(t,\phi)}{\partial t} = \int_{Z_t} c_0(y,t)\phi(y,t)d\eta(y) = C'(t,\phi).$$

Using $P_t(\phi)$ and $C(t,\phi)$ presented above, we obtain the expected risk $f(\phi)$ as

(6) $$f(\phi) = \int_{t_0}^{T} (C(\tau,\phi)-R(\tau))dP_\tau(\phi) + C(T,\phi)(1-P_T(\phi)).$$

Since $b(\cdot)$ is a regular detection function, $P_t(\phi)$ is continuous and is a function of bounded variation. Integration by parts yields

(7) $$f(\phi) = \int_{t_0}^{T} (R'(\tau)-C'(\tau,\phi))P_\tau(\phi)d\tau + C(T,\phi) - R(T)P_T(\phi).$$

Therefore, the problem with a finite limit time T is formulated as a variational problem to find a function $\phi_T^* = \{\phi_T^*(y,t),\ (y,t) \in \mathbf{Z}\}$ which minimizes the functional $f(\phi_T)$ subject to the restriction (1).

(8) $$f(\phi_T^*) = \inf_{\phi_T} f(\phi_T), \qquad \phi_T \in \Psi.$$

ϕ_T^* is called T-optimal allocation of search effort.

In the second stage of our investigation, the limit time T is considered as a variable in $[t_0,\infty)$. The earliest T which minimizes $f(\phi_T^*)$, $f(\phi_{T^*}^*) = \inf_T \inf_{\phi_T} f(\phi_T)$, is defined as the optimal stopping time T^* of the search.

(9) $$T^* = \min\{T \mid \inf_T f(\phi_T^*)\}.$$

The optimal search plan is obviously given by $\phi^*_{T^*} = \{\phi^*_{T^*}(y,t),\ (y,t) \in Z\}$, if the search time is not limited.

3. Optimal Search Plan

In this section, we derive necessary and sufficient conditions for the T-optimal allocation of search effort and for the optimal stopping time of the search in which search time is not limited.

3.1 The T-optimal allocation of search effort

The derivation of necessary and sufficient conditions for the optimal search plan given a limit time T is similar to the pattern of reasoning by Stromquist and Stone [17]. They consider a maximization problem for a real-valued functional $P(\phi)$ under the constraint (1) and derive necessary and sufficient conditoins for ϕ to be optimal. Before presenting their theorem, we define the Gateaux differential of a real-valued functional $g(\psi)$ at ψ in a direction h by

(10) $$dg(\psi,h) = \lim_{\delta\to 0^+} \frac{1}{\delta}(g(\psi+\delta h) - g(\psi))$$

for $\psi \in \Psi$ and $\psi+\delta h \in \Psi$ for all sufficiently small positive δ. Suppose that there exists a function $d(\psi,y,t)$ defined on Z such that for every h, $\psi+\delta h \in \Psi$, the Gateaux differential is given by

(11) $$dg(\psi,h) = \int_Z d(\psi,y,t)h(y,t)d\mu(y,t).$$

Then $d(\psi,\cdot,\cdot)$ is called a kernel of the Gateaux differential at ψ.

The necessary and sufficient conditions for ϕ^* to be optimal given by Theorems 1 and 2 in Strömquist and Stone [17] are quoted as the following lemmas.

Lemma 1. Let Ψ be the set of measurable function ϕ: $Z \to [0,\infty)$ satisfying (1), let $\phi^* \in \Psi$, and let P be a real-valued functional on Ψ. Assume that P has a Gateaux differential at ϕ^* with kernel $d(\phi^*,\cdot,\cdot)$. Then a necessary condition for ϕ^* to be optimal is that there exists a measurable function λ: $T \to (-\infty,\infty)$ such that for a.e. (y,t)

(A) $$d(\phi^*,y,t) \le \lambda(t)c_0(y,t), \text{ and}$$
$$d(\phi^*,y,t) = \lambda(t)c_0(y,t), \text{ if } \phi^*(y,t) > 0.$$

Lemma 2. In addition to the hypotheses of Lemma 1, assume that $P(\phi)$ is concave and that $\phi^* \in \Psi_0$. Then the necessary conditions of Lemma 1 are also sufficient for ϕ^* to be optimal.

Letting $P(\phi) = -f(\phi)$ given by (7), we can apply Lemmas 1 and 2 to our problem. The following theorem provides necessary and sufficient conditions for ϕ^* to be optimal.

Theorem 1. A necessary conditoin for the T-optimal search plan ϕ_T^* is that there exists a non-negative function $\lambda(t)$, $\{\lambda(t),\ t \in \boldsymbol{T}\} \neq \{0\}$, such that for a.e. y and t

$$(12)\qquad \begin{aligned} -A_{yt}^{T}(\phi^*) &= \lambda(t)c_0(y,t) \quad \text{for } \phi^*(y,t) > 0,\\ -A_{yt}^{T}(\phi^*) &\leq \lambda(t)c_0(y,t) \quad \text{for } \phi^*(y,t) = 0, \end{aligned}$$

where

$$(13)\qquad A_{yt}^{T}(\phi) = \int_t^T (R'(\tau)-C'(\tau,\phi))D_{yt}^{\tau}(\phi)d\tau + c_0(y,t)(1-P_t(\phi)) - R(T)D_{yt}^{T}(\phi),$$

$$(14)\qquad D_{yt}^{T}(\phi) = \alpha(y,t)E_{yt}[b'(\int_{t_0}^{T}\alpha(X_\tau,\tau)\phi(X_\tau,\tau)d\tau)]q_t(y),$$

and $E_{yt}[X]$ denotes the expectation over the sample paths of X conditional on $X_t = y$.

If $\lambda(t) > 0$, $t \in \boldsymbol{T}$, then

$$(15)\qquad \int_{Z_t} c_0(y,t)\phi(y,t)d\eta(y) = m(t).$$

If $f(\phi)$ is convex and $\phi^* \in \Psi_0$, the necessary conditoins mentioned above are also sufficient for ϕ^* to be optimal.

Proof: The functional $f(\phi_T)$ given by (7) has a Gateaux differential at ϕ in any direction with a kernel $A_{yt}^{T}(\phi)$ given by (13). To derive it, we calculate the Gateaux differential of each term of (7). Since the detection function $b(\psi)$ is regular by the assumption, $b(\psi)$ is differentiable and has a bounded positive derivative $b'(\psi)$. Hence the Gateaux differential of $P_t(\phi)$ exists and obtained as

$$(16)\qquad \begin{aligned} dP_t(\phi,h) &= \int_{t_0}^{t}\int_{Z_\tau}\alpha(y,\tau)E_{y\tau}[b'(\int_{t_0}^{t}\alpha(X_\xi,\xi)\phi(X_\xi,\xi)d\xi)]q_\tau(y)h(y,\tau)d\eta(y)d\tau\\ &= \int_{t_0}^{t}\int_{Z_\tau}D_{y\tau}^{t}(\phi)h(y,\tau)d\eta(y)d\tau. \end{aligned}$$

Since $C(t,\phi)$ and $C'(t,\phi)$ are linear with respect to ϕ, we have

$$(17)\qquad dC(t,\phi,h) = \int_{t_0}^{t}\int_{Z_\tau}c_0(y,\tau)h(y,\tau)d\eta(y)d\tau,$$

$$(18)\qquad dC'(t,\phi,h) = \int_{Z_t}c_0(y,t)h(y,t)d\eta(y).$$

Therefore, the Gateaux differential of $f(\phi)$ is obtained from (16), (17), (18) and (7) as

$$df(\phi,h) = \int_{t_0}^{T}\int_{Z_\tau}\{\int_{\tau}^{T}(R'(\xi)-C'(\xi,\phi))D_{y\tau}^{\xi}(\phi)d\xi + c_0(\tau)(1-P_\tau(\phi))$$

$$(19) \qquad -R(T)D_{y\tau}^{T}(\phi)\}h(y,\tau)d\eta(y)d\tau$$

$$= \int_{t_0}^{T}\int_{Z_\tau} A_{y\tau}^{T}(\phi)h(y,\tau)d\eta(y)d\tau.$$

If we set $P(\phi) = -f(\phi)$, Lemma 1 is applicable to our problem since the objective functional $-f(\phi)$ has the Gateaux differential which is a linear functional defined by integration with the kernel function $-A_{yt}^{T}(\phi)$ given by (13). Letting $d(\phi^*,y,t) = -A_{yt}^{T}(\phi^*)$ in (A), the relation (12) is derived.

The non-negativity of $\lambda(t)$ is proved as follows. We assume $\phi^*(y,t) > 0$ and $\lambda(t) < 0$ in the neighborhood of (y_1,t_1). Consider a search plan $\tilde{\phi}$ which differs from the optimal plan ϕ^* only in the neighborhood of (y_1,t_1),

$$\tilde{\phi}(y,t) = \begin{cases} \phi^*(y,t) - \delta, & \text{in } \Delta Z\text{-neighborhood of } (y_1,t_1), \\ \phi^*(y,t), & \text{otherwise.} \end{cases}$$

Then it is easily proved that $\{\tilde{\phi}(y,t)\}$ is feasible, and we have

$$f(\tilde{\phi}) - f(\phi^*) = -A_{y_1t_1}^{T}(\phi^*)\delta\Delta Z$$

for sufficiently small δ (> 0). Since $\phi^*(y,t) > 0$, the right-hand side of the above equation is denoted as $\lambda(t_1)\delta\Delta Z$ and is negative by the assumption. Hence we have $f(\tilde{\phi}) < f(\phi^*)$. This result contradicts the optimality of ϕ^*, and therefore, we can conclude $\lambda(t) \geq 0$.

Equation (15) is proved similarly. Here we assume

$$0 < \int_{Z_t} c_0(y,t)\phi^*(y,t)d\eta(y) < m(t) \text{ for } \lambda(t) > 0.$$

Then there exists (y_1,t_1) such that $\phi^*(y,t) > 0$ in the neighborhood of (y_1,t_1). We consider a search plan $\tilde{\phi}$,

$$\tilde{\phi}(y,t) = \begin{cases} \phi^*(y,t) + \delta, & \text{in } \Delta Z\text{-neighborhood of } (y_1,t_1), \\ \phi^*(y,t), & \text{otherwise.} \end{cases}$$

Then $\tilde{\phi}(y,t) \in \Psi$ is also proved and the following is derived.

$$f(\tilde{\phi}) - f(\phi^*) = A_{y_1t_1}^{T}(\phi^*)\delta\Delta Z = -\lambda(t)\delta\Delta Z < 0$$

for a sufficiently small δ(> 0). This contradicts the optimality of ϕ^*, and therefore, Equation (15) is obtained.

As stated in Lemma 2, if $P(\phi)$ is concave (in their maximization problem) and $\phi^* \in \Psi_0$, the necessary condition is also sufficient. However, in this statement, the concavity of the objective functional must be changed to the convexity of the expected risk function $f(\phi_T)$ in our minimization problem.

(Q.E.D.)

The following lemma gives a sufficient condition for $f(\phi_T)$ to be convex.

Lemma 3. The expected risk function $f(\phi_T)$ given by (7) is convex, if

$$\int_{t_0}^{T}(C'(\tau,\phi^1)-C'(\tau,\phi^2))(P_\tau(\phi^2)-P_\tau(\phi^1))d\tau \geq 0 \tag{20}$$

for any ϕ^1 and $\phi^2 \in \Psi$. (A sufficient condition.)

Proof: Let $\theta\phi = \{\theta\phi(y,t),\ (y,t) \in \mathbf{Z}\}$ and $\phi = (1-\theta)\phi^1+\theta\phi^2$, $0 \leq \theta \leq 1$, ϕ^1 and $\phi^2 \in \Psi$. $\phi \in \Psi$ is easily comfirmed by

$$\int_{Z_t} c_0(y,t)\phi(y,t)d\eta(y) = (1-\theta)\int_{Z_t} c_0(y,t)\phi^1(y,t)d\eta(y) + \theta\int_{Z_t} c_0(y,t)\phi^2(y,t)d\eta(y) \leq m(t). \tag{21}$$

By substituting $\phi = (1-\theta)\phi^1 + \theta\phi^2$ into (7), we have

$$f(\phi) \leq (1-\theta)f(\phi^1) + \theta f(\phi^2) + \theta(1-\theta)\int_{t_0}^{T}(C'(\tau,\phi^1)-C'(\tau,\phi^2))(P_\tau(\phi^1)-P_\tau(\phi^2))d\tau, \tag{22}$$

because of the concavity of $b(\psi)$. Hence, if the last term of the right-hand side of the above inequality is non-positive, $f(\phi)$ is a convex function of $\phi \in \Psi$.

(Q.E.D.)

3.2 Optimal stopping time

In this section, we derive necessary and sufficient conditions for the optimal stopping time T^* defined by (9) when the search time is not restricted. For the derivation of the conditions, hereafter we deal with T as a variable defined on $[t_0,\infty)$ instead of the definition as the limit time of the search in the previous section. T is called stopping time of the search hereafter.

Lemma 4.

1. If $T > T^*$, the T-optimal search plan ϕ_T^* which is obtained from Theorem 1 is identical with the (unconditional) optimal search plan $\phi_{T^*}^*$.

$$\phi_T^*(y,t) = \begin{cases} \phi_{T^*}^*(y,t), & \text{if } 0 \leq t \leq T^*, \\ 0, & \text{if } T^* < t \leq T. \end{cases} \tag{23}$$

2. The expected risk $f(\phi_T^*)$ is a non-increasing function of T and is a constant $f(\phi_{T^*}^*)$ for $T > T^*$.

Proof: 1. From the definition of $\phi_{T^*}^*$, $\phi_{T^*}^*$ is not influenced by the limit time T of the search if $T \geq T^*$. Therefore, $\phi_{T^*}^*$ is identical with the conditionally optimal search plan ϕ_T^* for T ($\geq T^*$).

2. In order to prove 2 of the lemma, we consider two arbitrary stopping times T_1 and T_2, $t_0 \leq T_1 < T_2$, and a search plan $\phi_{T_2} \in \Psi$ as follows,

$$\phi_{T_2}(y,t) = \begin{cases} \phi_{T_1}^*(y,t), & t_0 \leq t \leq T_1, \\ 0, & T_1 < t \leq T_2. \end{cases} \tag{24}$$

Then, since ϕ_{T_2} does not necessary be optimal for T_2, $f(\phi_{T_1}^*) = f(\phi_{T_2}) \geq f(\phi_{T_2}^*)$ holds. Therefore, $f(\phi_T^*)$ is a non-increasing function of T through ϕ_T^*, $t_0 \leq T \leq T^*$. The statement, $f(\phi_T^*) = f(\phi_{T^*}^*)$ for $T \geq T^*$, is obvious from (23).

Theorem 2. A necessary and sufficient condition for T^* to be optimal is

$$\begin{aligned} &\lambda_T(T) > 0 \quad \text{for } T^*-\Delta T < T < T^*, \\ &\lambda_T(T) = 0 \quad \text{for } T^* \leq T, \end{aligned} \tag{25}$$

for a sufficiently small $\Delta T > 0$.

Proof: From Lemma 4, the optimal stopping time T^* is given an alternative definition irrespective of (9):

The optimal stopping time T^* is the time such that

$$\begin{aligned} &\frac{\partial f(\phi_T^*)}{\partial T} < 0 \qquad \text{for } T^*-\Delta T < T < T^*, \\ &\frac{\partial f(\phi_T^*)}{\partial T} = 0 \qquad \text{for } T^* \leq T, \end{aligned} \tag{26}$$

for a sufficiently small ΔT.

By considering the variations of $\phi^*(y,t)$ and $P_t(\phi_T^*)$ when the stopping time T is prolonged to $T+\Delta T$, the following expression is obtained.

$$\begin{aligned} f(\phi_{T+\Delta T}^*) = f(\phi_T^*) &+ \Delta T\{C'(T,\phi_T^*)(1-P_T(\phi_T^*))-R(T)\int_{Z_T} D_{yT}^T(\phi_T^*)\phi_T^*(y,T)d\eta(y)\} \\ &- \int_{t_0}^T\int_{Z_\tau} A_{y\tau}^T(\phi_T^*)\Delta\phi_T^*(y,\tau)d\eta(y)d\tau + o(\Delta T), \end{aligned} \tag{27}$$

where

$$\Delta\phi_T^*(y,t) = \phi_{T+\Delta T}^*(y,t) - \phi_T^*(y,t)$$

$$\Delta P_t(\phi_T^*) = P_t(\phi_{T+\Delta T}^*)-P_t(\phi_T^*) = \int_{t_0}^t\int_{Z_\tau} D_{y\tau}^t(\phi_T^*)\Delta\phi_T^*(y,\tau)d\eta(y)d\tau + o(\Delta T) \tag{28}$$

(29)
$$\Delta P_{T+\Delta T}(\phi^*_{T+\Delta T}) = P_{T+\Delta T}(\phi^*_{T+\Delta T}) - P_T(\phi^*_T) = \int_{t_0}^{T}\int_{Z_\tau} D^T_{y\tau}(\phi^*_T)\Delta\phi^*_T(y,\tau)d\eta(y)d\tau + \Delta T\int_{Z_T} D^T_{yT}(\phi^*_T)\phi^*_T(y,T)d\eta(y) + o(\Delta T).$$

It is easily derived by Theorem 1 that the second term of the right-hand side of (27) is rewritten as $-\Delta T\lambda_T(T)m(T)$ and the third term is zero. Hence we have

(30)
$$\Delta f(\phi^*_T) = f(\phi^*_{T+\Delta T}) - f(\phi^*_T) = -\Delta T\lambda_T(T)m(T) + o(\Delta T).$$

Theorem is directly obtained from the definition of T^*, (26), and (30).

(Q.E.D.)

It is important to distinguish the limit time T of the search in Section 3.1 and the stopping time T^* in Theorem 2. As mentioned in Theorem 2, the stopping time T^* implies that the search is conducted at $T^*-\Delta T$ and is stopped at T^* eventually. On the other hand, the limit time T means merely an upper limit of the available search time and if $T > T^*$ the search is not to be conducted in the interval $(T^*,T]$. Therefore, the result, $\phi^*_T(y,t) = 0$ for all (y,t), $t \in (T^*,T]$ if $T > T^*$, is also obtained from Theorem 1; Theorem 1 includes Theorem 2 in this case, notwithstanding we discuss Theorem 2 assuming $T \to \infty$.

We define $S(T)$ by

(31)
$$S(T) = \{y \in Z_T \mid \phi^*_T(y,T) > 0\}.$$

$S(T)$ is called a search region at T. The following corollary states a property of the search region $S(T^*)$ at T^*.

Corollary 1. A necessary condition for $S(T^*)$ to be optimal is

(32)
$$(1-P_{T^*}(\phi^*_{T^*}))\int_{S(T^*)} c_0(y,T^*)d\eta(y) = R(T^*)E_{S(T^*)}[b'(\int_{t_0}^{T^*}\alpha(X_\tau,\tau)\phi^*_{T^*}(X_\tau,\tau)d\tau)].$$

If the stopping time T is less than T^*, we have

(33)
$$\lambda_T(T)\int_{S(T)} c_0(y,T)d\eta(y) + (1-P_T(\phi^*_T))\int_{S(T)} c_0(y,T)d\eta(y) = R(T)E_{S(T)}[b'(\int_{t_0}^{T}\alpha(X_\tau,\tau)\phi^*_T(X_\tau,\tau)d\tau)].$$

Proof: On $S(T)$, the next relation is obvious by (12) and the difinition of $A^T_{yT}(\phi^*_T)$ given by (13).

$$\lambda_T(T)c_0(y,T) = R(T)D^T_{yT}(\phi^*_T) - c_0(y,T)(1-P_T(\phi^*_T)).$$

Integrating $\lambda_T(T)$ over $S(T)$, we have

$$\lambda_T(T)\int_{S(T)} c_0(y,T)d\eta(y) = R(T)\int_{S(T)} D^T_{yT}(\phi^*_T)d\eta(y) - (1-P_T(\phi^*_T))\int_{S(T)} c_0(y,T)d\eta(y).$$

The first term of the right-hand side of the above equation is rewritten as $R(T)E_{S(T)}[b'(\int_{t_0}^T \alpha(X_\tau,\tau)\phi(X_\tau,\tau)d\tau)]$ from the definition of $D^T_{yt}(\phi^*_T)$ given by (14), and therefore, (33) is proved. If we substitute T^* for T in (33), Equation (32) is derived since $\lambda_{T^*}(T^*) = 0$ by Theorem 2.

(Q.E.D.)

4. FAB Algorithm and Examples

The T-optimal allocation of search effort given by Theorem 1 usually cannot be obtained in an analytical form; often we must calculate it numerically. An algorithm for an iterative approximation called FAB algorithm (the forward and backward algorithm), proposed by Iida [6], Brown [1] and later by Washburn [21] in a more general form, can be applied to our model.

4.1 FAB algorithm

In this section, we consider the case where the time space $\boldsymbol{T}$ and the target space $\boldsymbol{Y}$ are both discrete, $\boldsymbol{T} = \{t_i,\ i=1,\cdots,n\}$ and $\boldsymbol{Y} = \{j,\ j=1,\cdots,m\}$, and η is counting measure.

We shall give another expression of the condition (12) for the convenience of numerical calculation. When $\boldsymbol{T}$ and $\boldsymbol{Y}$ are discrete, $q_t(y)$ is defined as probability mass function and $A^T_{yt}(\phi)$ given by (13) is rewritten as follows.

$$A^T_{yt}(\phi) = \sum_{\tau=t}^{T-1}(\Delta R(\tau)-\Delta C(\tau,\phi))D^\tau_{yt}(\phi) + c_0(y,t)(1-P_{t-1}(\phi)) - R(T)D^T_{yt}(\phi), \tag{34}$$

where

$$D^\tau_{yt}(\phi) = \alpha(y,t)E_{yt}[b'(\sum_{\xi=t_0}^{\tau}\alpha(X_\xi,\xi)\phi(X_\xi,\xi))]q_t(y),$$

$$\Delta R(t) = R(t+1) - R(t),$$

$$\Delta C(t,\phi) = C(t+1,\phi) - C(t,\phi) = \sum_{Z_t} c_0(y,t+1)\phi(y,t+1),$$

$$P_t(\phi) = E[b(\sum_{\tau=t_0}^{t}\alpha(X_\tau,\tau)\phi(X_\tau,\tau))].$$

Suppose the T-optimal allocation of search effort ϕ^* is given except for the allocation at t. In (34), $\phi(y,t)$ is related to $D_{yt}^{\tau}(\phi)$, $\tau \geq t$, and $D_{yt}^{\tau}(\phi)$ is a continuous and strictly decreasing function of $\phi(y,t)$ by Assumption 5. Since $(\Delta R(t)-\Delta C(t,\phi)) < 0$, $A_{yt}^{T}(\phi)$ is a continuous and strictly increasing function of $\phi(y,t)$, and therefore, the left-hand side of (12), $-A_{yt}^{T}(\phi^*)$, is a continuous and strictly decreasing function of $\phi^*(y,t)$. We denote the function $-A_{yt}^{T}(\phi^*)$ by $\rho_{yt}(\phi^*(y,t))$ in order to emphasize the argument $\phi^*(y,t)$. Then there exists an inverse function $\rho_{yt}^{-1}(x)$ of $\rho_{yt}(\cdot)$ which is defined on a domain

(35) $$\rho_{yt}(0) \geq x > \lim_{\psi\to\infty} \rho_{yt}(\psi) = -c_0(y,t)(1-P_t(\phi_T^*)).$$

Here we define a function $[\rho_{yt}^{-1}(x)]^+$ as

(36) $$[\rho_{yt}^{-1}(x)]^+ = \begin{cases} \rho_{yt}^{-1}(x), & -c_0(y,t)(1-P_t(\phi^*)) < x \leq \rho_{yt}(0), \\ 0, & \rho_{yt}(0) < x. \end{cases}$$

The summation of $c_0(y,t)[\rho_{yt}^{-1}(x)]^+$ over Z_t, $\sum_{Z_t} c_0(y,t)[\rho_{yt}^{-1}(x)]^+$, gives the total search cost at t when $\phi^*(y,t)$ is determined by (12) for $x = c_0(y,t)\lambda(t)$. Since the total cost at t, $m(t)$, is given, the situation is either of the two, $\sum_{Z_t} c_0(y,t)[\rho_{yt}^{-1}(0)]^+ > m(t)$ or $\sum_{Z_t} c_0(y,t)[\rho_{yt}^{-1}(0)]^+ \leq m(t)$. For the former case, since $\rho_{yt}^{-1}(\cdot)$ is a continuous strictly decreasing function and $\sum_{Z_t} c_0(y,t) \times [\rho_{yt}^{-1}(\bar{\lambda}c_0(y,t))]^+ = 0$ for $\bar{\lambda} = \max_{y\in Z_t} \rho_{yt}(0)$, there exists a positive constant $\lambda(t)\in(0,\bar{\lambda})$ which satisfies

(37) $$\sum_{Z_t} c_0(y,t)[\rho_{yt}^{-1}(\lambda(t)c_0(y,t))]^+ = m(t).$$

Then, the optimal allocation of search effort at t is given by

(38) $$\phi^*(y,t) = [\rho_{yt}^{-1}(\lambda(t)c_0(y,t))]^+.$$

The optimality of $\phi^*(y,t)$ given by (38) is easily proved as follows. If $\phi^*(y,t)$ is positive, $\phi^*(y,t) = \rho_{yt}^{-1}(\lambda(t)c_0(y,t))$ from (38) and definition of $[\rho_{ty}^{-1}(\lambda(t)c_0(y,t))]^+$. Hence equation $\lambda(t)c_0(y,t) = \rho_{yt}(\phi^*(y,t)) = -A_{yt}^{T}(\phi^*)$ is derived. Similarly, $\rho_{yt}(0) = -A_{yt}^{T}(\phi^*) \leq \lambda(t)c_0(y,t)$ hold for $\phi^*(y,t) = 0$. Therefore relation (12) is satisfied and Equation (15) is obvious from (37). For the latter case, $\sum_{Z_t} c_0(y,t)[\rho_{yt}^{-1}(0)]^+ \leq m(t)$, the optimal allocation at t is given by

(39) $$\phi^*(y,t) = [\rho_{yt}^{-1}(0)]^+.$$

Since $\rho_{yt}(\phi^*(y,t)) = 0$ for $\phi^*(y,t) > 0$ and $\rho_{yt}(0) \leq 0$ for $\phi^*(y,t) = 0$, $\lambda(t) = 0$ is concluded and the allocation $\phi^*(y,t)$ given by (39) is proved to be optimal from Theorem 1. The results mentioned above are summarized as the following corollary.

Corollary 2. $\Theta(t)$ is defined by

$$\Theta(t) = \sum_{Z_t} c_0(y,t)[\rho_{yt}^{-1}(0)]^+. \tag{40}$$

The optimal allocation of search effort $\phi^*(y,t)$ is given by

$$\begin{aligned} \phi^*(y,t) &= [\rho_{yt}^{-1}(\lambda(t)c_0(y,t))]^+ \quad \text{for } \Theta(t) > m(t), \\ \phi^*(y,t) &= [\rho_{yt}^{-1}(0)]^+ \quad \text{for } \Theta(t) \leq m(t), \end{aligned} \tag{41}$$

where $\lambda(t)$ is a positive number which is uniquely determined by Equation $\sum_{Z_t} c_0(y,t)[\rho_{yt}^{-1}(\lambda(t)c_0(y,t))]^+ = m(t)$.

The above corollary states the relation between the positivity of $\lambda(t)$ and the exhaustive employment of the search cost $m(t)$ as well as the way how to obtain the optimal allocation. Corollary 2 implies that if the total search cost $m(t)$ at t is relatively small and $\Theta(t) > m(t)$, $\lambda(t) > 0$ hold and the whole $m(t)$ is to be allocated in the target space exhaustively. If $\Theta(t) < m(t)$, then $\lambda(t) = 0$ and we must save some $m(t)$ to avoid overemployment of search cost at t. We shall call the former case the complete search and the latter case the partial search. When $\Theta(t) = m(t)$, the situation lies just between them, namely $\lambda(t) = 0$ and the complete search is optimal.

Applying Corollary 2, the FAB algorithm to obtain the T-optimal allocation of search effort is described as follows:

1. Let δ be a small positive number and set $k = 0$.
2. Set $\phi^0(j,t_i) = 0$ for all j and t_i.
3. Perform Step 4 and 5 for all t_i from t_0 to t_n sequentially.
4. Calculate $\phi^*(j,t_i)$ for all j by Corollary 2 and set $\phi^k(j,t_i) = \phi^*(j,t_i)$.
5. Go to the next t_i.
6. If $|f(\phi^k)-f(\phi^{k-1})| > \delta$, increase k by 1 and go back to Step 3.
7. If $|f(\phi^k)-f(\phi^{k-1})| \leq \delta$, stop. $\{\phi^k(j,t_i)\}$ is an approximation of the T-optimal allocation within error δ.

That the algorithm mentioned above gives a T-optimal allocation of search effort if $f(\phi)$ is convex is verified as follows. Consider the improvement of

$f(\phi)$ at t_i in some iteration k of Step 4. According to the FAB algorithm, $\phi^{k-1}(j,t_i)$ is replaced by ϕ^k given by Corollary 2 at the iteration (k,t_i) and the search plan $\phi = \{\phi^k(j,t),\ t_1 \le t \le t_{i-1},\ \phi^{k-1}(j,t),\ t_i \le t \le t_n\}$ is renewed to

$$\{\tilde{\phi}(j,t)\} = \begin{cases} \phi^k(j,t), & t_1 \le t \le t_i, \\ \phi^{k-1}(j,t), & t_{i+1} \le t \le t_n. \end{cases} \tag{42}$$

We denote the above transformation by $\tilde{\phi} = \Lambda^k_{t_i}\phi$. If $\Lambda^k_t\phi = \phi$ for all t, $t_1 \le t \le t_n$, ϕ is a T-optimal search plan, because ϕ satisfies the condition of Theorem 1 for all t and j. Suppose $\tilde{\phi}(j,t) \ne \phi(j,t)$ for some (j,t). Then, it is easily proved that the transformation $\Lambda^k_t\phi$ improves $f(\phi)$ as follows. When $f(\phi)$ is a convex function of (y,t), the following is deduced.

$$\begin{aligned} f(\phi) - f(\Lambda^k_t\phi) > df(\Lambda^k_t\phi,\phi-\Lambda^k_t\phi) &= \sum_{\tau=t_1}^{t_n}\sum_{j=1}^{m} A^T_{j\tau}(\tilde{\phi})(\phi(j,\tau) - \tilde{\phi}(j,\tau)) \\ &= \sum_{j=1}^{m} A^T_{jt}(\phi^k(j,t))\{\phi(j,t)-\phi^k(j,t)\} \\ &\ge -\lambda(t)\sum_{j=1}^{m} c_0(j,t)\phi(j,t) + \lambda(t)\sum_{j=1}^{m} c_0(j,t)\phi^k(j,t) = 0. \end{aligned} \tag{43}$$

Therefore, $f(\phi) > f(\Lambda^k_t\phi)$ is concluded and this means that the transformation $\Lambda^k_t\phi$ improves $f(\phi)$. In FAB algorithm, since this operation Λ^k_t is applied sequentially for t and k, $f(\phi^k)$ decreases strictly. (ϕ^k is the solution of the kth iteration, $\phi^k = \{\phi^k(j,t)\}$.) This also implies that $f(\phi^k)$ approaches $f(\phi^{**})$ by the sequential application of Λ^k_t, since $f(\phi) > f(\phi^k_t)$ and $f(\Lambda^k_t\phi) \ge f(\phi^*)$. Therefore, we have $f(\phi^{**}) = f(\Lambda^k_t\phi^{**})$. Furthermore, $f(\phi^{**}) = f(\Lambda^k_t\phi^{**})$ means $\phi^{**} = \Lambda^k_t\phi^{**}$, because if $\phi^{**} \ne \Lambda^k_t\phi^{**}$, then $f(\Lambda^k_t\phi^{**}) < f(\phi^{**})$ is deduced from the above discussion and it contradicts $f(\phi^{**}) = f(\Lambda^k_t\phi^{**})$. When $\phi^{**} = \Lambda^k_t\phi^{**}$ for all t, ϕ^{**} satisfies Theorem 1 and if $f(\phi)$ is convex, $\phi^* = \phi^{**}$ by the theorem.

When $f(\phi)$ is convex, the next inequality is derived

$$\begin{aligned} f(\phi^*) - f(\phi) > df(\phi,\phi^*-\phi) &= \sum_{\tau=t_1}^{t_n}\sum_{j=1}^{m} A^T_{y\tau}(\phi)\{\phi^*(j,\tau)-\phi(j,\tau)\} \\ &\ge \sum_{\tau=t_1}^{t_n}\{\underline{A}^T_\tau(\phi)m(\tau) - \sum_{j=1}^{m} A^T_{j\tau}(\phi)\phi(j,\tau)\} \end{aligned} \tag{44}$$

where

$$\underline{A}_{\tau}^{T}(\phi) = \operatorname*{ess\ inf}_{j\varepsilon Y} \{ A_{j\tau}^{T}(\phi) \mid A_{j\tau}^{T}(\phi) \le 0 \}. \tag{45}$$

Therefore, an upper bound of the difference between the expected risk of ϕ^k and that of the optimal plan ϕ^* is evaluated by

$$f(\phi^k) - f(\phi^*) \le \sum_{\tau=t_1}^{t_n} [\sum_{j=1}^{m} A_{j\tau}^{T}(\phi^k)\phi^k(j,\tau) - \underline{A}_{\tau}^{T}(\phi^*)m(\tau)]. \tag{46}$$

4.2 A numerical example

Suppose the target space consists of discrete regions named cells, numbered $j = 1,2,\cdots,5$ from the left to the right, and the time space is discrete time points t, $t = 1,2,\cdots\cdots$. We assume that if the target is in Cell j ($j \le 4$) at t, he selects either j or $j+1$ with probability (0.5, 0.5) for his next position and if he enters Cell 5, he stays there forever. Namely the transition probability matrix $[p(j,k)]$ is given by

$$[p(j,k)] = \begin{bmatrix} 0.5 & 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{47}$$

The initial distribution of the target is assumed to be uniform over j, i.e., $q_1(j) = 0.2$, $j = 1,\cdots,5$. It is also assumed that the search time is limited to $T = 10$ and a total search cost $m(t) = 7.5$ is available to the searcher at each time. The conditional detection function $b(\psi)$ is assumed to be an exponential function, $b(\psi) = 1 - \exp(-\psi)$. The unit cost of the search effort is $c_0(j,t) = 5$ and the reward given when the target is detected successfully is $R(t) = 100$. For simplicity, these values, $p(j,k)$, $m(t)$, $c_0(j,t)$ and $R(t)$, are assumed to be constant during the whole search time.

Applying the FAB algorithm, we obtain the T-optimal allocation of search effort as shown in Table 1.

Table 1. $\phi_{10}^{*}(j,t)$

j \ t	1	2	3	4	5	6	7	8	9	10
1	0.415	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0.212	0.027	0
4	0	0	0	0	0	0	0	0.290	0.454	0.265
5	0	0	0	0	0.165	0.420	0.776	0.998	1.018	1.235
Total	0.415	0	0	0	0.165	0.420	0.776	1.500	1.500	1.500
$\lambda(t)$	0	0	0	0	0	0	0	0.071	0.716	1.270
$f(\phi_{10}^{*})$	-82.822									

In Table 1, we observe that the T-optimal search plan is started with a partial search, followed by breaks at t = 2, 3 and 4. After that the partial search resumes, and finally at t = 8, 9 and 10 the complete search in which the effort is concentrated to the right-most cells takes place. This result is explained intuitively as follows. Because of the Markov chain-like motion of the target, the target is more and more likely in the right-most cell as time passes by. Therefore if the limit time of the search is long enough, the searcher would wait all the time, and put his full effort onto Cell 5 at the last time points. If the limit time of the search is shortened a little, its effect might be as this: First, even at the last time point, there is some probability that the target is still in Cell 4 or 3, and thus the searcher has to divide his effort to these cells besides the most probable Cell 5. The effort distribution at t = 8, 9 and 10 in Table 1 is the case. The second effect is on the starting partial search. Calculation shows that if the target was in Cell 1 at t = 1, the probability of being still in Cell 1 through 4 at t = 10 is about 0.254 (= $1-p^{(10)}(1,5)$, where $p^{(n)}(i,j)$ is the n-step transition probability given by $[p^{(n)}(i,j)] = [p(i,j)]^n$.) To reduce the probability, search in Cell 1 at the beginning might be effective, which explains the partial search at t = 1. If the limit time of the search is shortened further, the two effect grow and the break of search shrinks and disappears as is seen in Table 2 and 3.

Table 2. $\phi_6^*(j,t)$

j \ t	1	2	3	4	5	6
1	0.995	0	0	0	0	0
2	0.364	0.327	0.233	0.144	0	0
3	0.144	0.332	0.258	0.353	0.329	0
4	0	0.090	0.197	0.291	0.362	0.265
5	0	0.229	0.667	0.712	0.809	1.235
Total	1.463	0.978	1.355	1.500	1.500	1.500
$\lambda(t)$	0	0	0	0.522	1.085	1.712
$f(\phi_6^*)$	-69.689					

Table 3. $\phi_3^*(j,t)$

j \ t	1	2	3
1	0.769	0	0
2	0.475	0.378	0
3	0.256	0.514	0
4	0	0.292	0.265
5	0	0.316	1.235
Total	1.500	1.500	1.500
$\lambda(t)$	4.089	4.142	5.508
$f(\phi_3^*)$	-49.725		

It is interesting to note that the allocation of search effort at the end of the search is identical irrespective of T; $\phi_T^*(4,T) = 0.265$, $\phi_T^*(5,T) = 1.235$ for $T = 3$, 6 and 10.

The expected risk $f(\phi_T^*)$ of the T-optimal search plan is shown in Fig. 1. As mentioned before, $f(\phi_T^*)$ is a strictly decreasing function of T since the target density concentrates to Cell 5 as time goes by. Therefore, no stopping is optimal in this case. This is also suggested by the fact that $\lambda_T(t)$'s in Table 1 ∿ 3 increase with t.

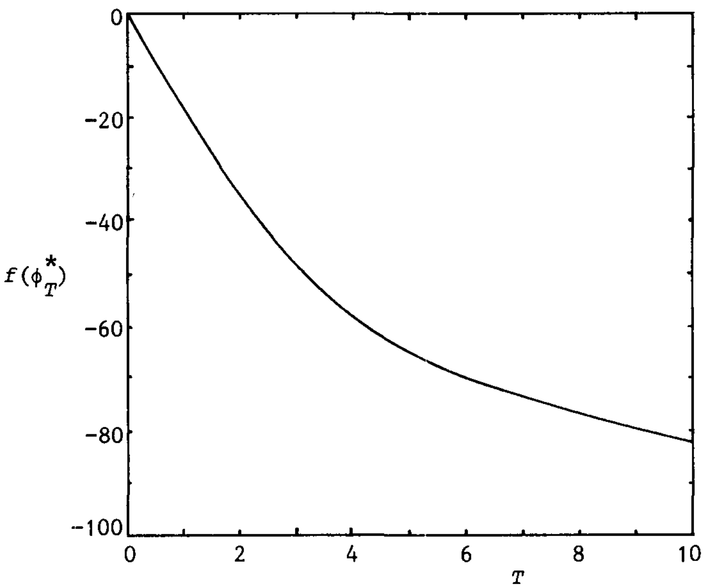

Fig. 1 The optimal expected risk $f(\phi_T^*)$

$$D_{rt}^{T}(\phi) = \begin{cases} \dfrac{1}{\pi v_0^2 t^2} \exp\left(-\int_{t_0}^{T} \phi\left(\frac{r}{t}\tau, \tau\right)d\tau\right), & 0 \le r \le v_0 t, \\ 0, & v_0 t \le r. \end{cases}$$

From (13) and (50), the kernel $A_{rt}^{T}(\phi)$ of the Gateaux differential of $f(\phi)$ is

$$(51) \qquad -A_{rt}^{T}(\phi) = \begin{cases} \dfrac{R}{\pi v_0^2 t^2} \exp\left(-\int_{t_0}^{T} \phi\left(\frac{r}{t}\tau, \tau\right)d\tau\right) + \int_t^T \left\{c_0 \int_0^{2\pi}\int_0^{\infty} \phi(r,\tau) r dr d\theta\right\} \\ \qquad \times \dfrac{1}{\pi v_0^2 t^2} \exp\left(-\int_{t_0}^{\tau} \phi\left(\frac{r}{t}\zeta, \zeta\right)d\zeta\right)d\tau \\ \qquad - c_0 \int_0^{v_0} \exp\left(-\int_{t_0}^{t} \phi(v\tau,\tau)d\tau\right) g(v)dv, & 0 \le r \le v_0 t, \\ -c_0 \int_0^{v_0} \exp\left(-\int_{t_0}^{t} \phi(v\tau,\tau)d\tau\right) g(v)dv, & v_0 t < r. \end{cases}$$

Therefore, if $r > v_0 t$, $\phi_T^*(r,t) = 0$ is obvious and the following allocation of search effort is easily verified to satisfy the conditions of Theorem 1.

$$(52) \qquad \phi_T^*(r,t) = \begin{cases} \dfrac{\Phi}{\pi v_0^2 t^2}, & 0 \le r \le v_0 t,\ t \le T^*, \\ 0, & \text{otherwise}. \end{cases}$$

Then $\lambda(t)$ is calculated by

$$(53) \qquad \lambda(t) = \frac{R}{\pi c_0 v_0^2 t^2} \exp\left(-\frac{\Phi}{\pi v_0^2}\left(\frac{1}{t_0} - \frac{1}{T}\right)\right) - \exp\left(-\frac{\Phi}{\pi v_0^2}\left(\frac{1}{t_0} - \frac{1}{t}\right)\right) + \frac{1}{\pi v_0^2 t^2} \int_t^T \exp\left(-\frac{\Phi}{\pi v_0^2}\left(\frac{1}{t_0} - \frac{1}{\tau}\right)\right)d\tau ,$$

if the right-hand side of (53) is non-negative; otherwise $\lambda(t) = \lambda(T) = 0$. From (53), it is obvious that $\lambda(T)$ is strictly decreasing and positive function of T, if $R / \pi c_0 v_0^2 T^2 > 1$. Hence the following is concluded.

$$(54) \qquad \begin{aligned} &\lambda(T) > 0 \quad \text{for } t_0 < T < \sqrt{\frac{R}{\pi v_0^2 c_0}}, \\ &\lambda(T) = 0 \quad \text{for } \sqrt{\frac{R}{\pi v_0^2 c_0}} \le T. \end{aligned}$$

4.3 Optimal search for a target with a conditionally deterministic motion

We consider a search problem in which the target space is a continuous 2-dimensional space and the time space is also continuous. A target is assumed to move straight from the origin of the target space with (θ, v) selected randomly, where the course θ is chosen from the uniform probability distribution in $[0, 2\pi]$ and the speed v is selected according to the probability density function $g(v)$,

$$(48) \qquad g(v) = \begin{cases} \dfrac{2v}{v_0^2}, & 0 \le v \le v_0,\ v_0 > 0, \\ 0, & \text{otherwise.} \end{cases}$$

(This target motion belongs to the conditionally deterministic motion which was investigated by Stone [13,14].) An amount of total search effort Φ is available to the searcher at any time t, $t \in [t_0, \infty)$, $t_0 > 0$. The exponential function $b(\psi) = 1-\exp(-\psi)$ is assumed for the conditional detection function and a constant reward R is given when the target is detected. A constant unit cost $c_0 (> 0)$ of the search effort is also assumed. (Similar problem was investigated by Danskin [2] from the standpoint of game theory. In his paper, the detection probability of the target was assumed as the measure of effectiveness of the search and $g(v)$ given by (48) was considered as the strategy of the target instead of a given function.)

Since the distribution of θ is uniform in $[0, 2\pi]$, the distribution of the target and the search effort are both circularly symmetric about the origin of the target space and these are denoted by $q_t(r)$ and $\phi(r,t)$, where r is the distance from the origin. It is easily derived that the density function $q_t(r)$ of the target position at t is a uniform distribution in the circular area with a radius $v_0 t$,

$$(49) \qquad q_t(r) = \begin{cases} \dfrac{1}{\pi v_0^2 t^2}, & 0 \le r \le v_0 t, \\ 0, & v_0 t < r. \end{cases}$$

Then $P_t(\phi)$ and $D_{rt}^T(\phi)$ in (3) and (14) are calculated as,

$$(50) \qquad P_t(\phi) = 1 - \int_0^{v_0} \exp\left(-\int_{t_0}^{t} \phi(v\tau, \tau)\,d\tau\right) \frac{2v}{v_0^2}\,dv,$$

Therefore, $T^* = \sqrt{R / \pi v_0^2 c_0}$ is the optimal stopping time by Theorem 2 if $t_0 < \sqrt{R / \pi v_0^2 c_0}$ and $T^* = 0$ if $t_0 \geq \sqrt{R / \pi v_0^2 c_0}$. This result is also obtained from Corollary 1. If ϕ_T^* given by (52) and $b'(\psi) = \exp(-\psi)$ are substituted into (32), the equation for T^*, $\pi c_0 v_0^2 T^{*2} = R$, is easily derived. Hence $T^* = \sqrt{R / \pi v_0^2 c_0}$ is obtained. Summarizing the above, we have

$$(55) \qquad \phi^*(r,t) = \begin{cases} \dfrac{\Phi}{\pi v_0^2 t^2}, & 0 \leq r \leq v_0 t, \\ 0, & \text{otherwise}, \end{cases}$$

for $t_0 \leq t \leq T^*$, where $T^* = \sqrt{R / \pi v_0^2 c_0}$. If $t_0 > T^*$ the search should not be begun.

5. Discussions

In this section, discussions on the results obtained in the previous sections are presented.

(1) The physical interpretation of the main theorem

Let us consider the physical meaning of Theorem 1. According to the theorem, the T-optimal search plan is to allocate the search effort by, $-A_{yt}^T(\phi^*) = (\leq)\lambda(t)c_0(y,t)$, if $\phi^*(y,t) > (=)0$, for a.e. $(y,t) \in Z$. Integrating (13) by parts, we obtain

$$(56) \qquad \begin{aligned} -A_{yt}^T(\phi^*) = {} & (R(t) - C(t,\phi^*))D_{yt}^t(\phi^*) - c_0(y,t)(1 - P_t(\phi^*)) \\ & + \int_{\tau \in [t,T]} (R(\tau) - C(\tau,\phi^*))dD_{yt}^{\tau}(\phi^*) + C(T,\phi^*)D_{yt}^T(\phi^*). \end{aligned}$$

$D_{yt}^T(\phi^*)$, the kernel of the Gateaux differential of $P_t(\phi^*)$ given by (14), is the increment of detection probability density when unit search effort density is added to $\phi^*(y,t)$ at (y,t). Therefore, the first term in the right-hand side of (56), $(R(t) - C(t,\phi^*))D_{yt}^t(\phi^*)$, is interpreted as the expected net reward gained by the additional unit search effort at (y,t). The second term, $-c_0(y,t)(1-P_t(\phi^*))$, is the expected cost of the additional unit search effort and the third term, $\int_{\tau \in [t,T]} (R(\tau) - C(\tau,\phi^*))dD_{yt}^{\tau}(\phi^*)$, represents the variation of the net reward in the interval $[t,T]$ caused by the additional unit search effort at (y,t). The last term, $C(T,\phi^*)D_{yt}^T(\phi^*)$, is the expected search cost in $[t_0,T]$ which is saved by the increment of detection probability density at (y,t). One should note that the sign of the last term is positive since the saved cost is considered as profit instead of cost. Therefore, $-A^T_{yt}(\phi)$ means the expected net reward which is earned by the searcher when he allocates additional unit search effort to $\phi^*(y,t)$ at (y,t). We shall call it the marginal expected net reward of ϕ^* at (y,t) hereafter. By this reasoning, Equation (12) of Theorem 1 is interpreted as follows. If search effort is to be allocated to the neighborhood of $(y,t) \in Z$, the amount of search effort should be balanced in such a way that the marginal expected net reward versus cost ratio, $-A^T_{yt}(\phi^*)/c_0(y,t)$, is equal to $\lambda(t)$ in the region of the target space being searched at t. If search effort should not be allocated to (y,t), the point does not have a larger marginal expected net reward versus cost ratio than $\lambda(t)$.

As stated above, since $-A^T_{yt}(\phi)$ is the marginal expected net reward at t, if $\lambda(t) > 0$ in Equation (12), the expected reward increases (namely, the expected risk decreases) as the search effort increases. Therefore, if $\lambda(t) > 0$, the search cost rate $m(t)$ should be used exhaustively; hence, the complete search is optimal in this case. This is the meaning of Equation (15).

Theorem 2 is also explained by the meaning of $\lambda(T^*)$ mentioned above. By the definition, the optimal stopping time T^* is such a time that the search is conducted in $[T^*-\Delta t, T^*]$ and is stopped eventually in (T^*,∞). If the marginal expected reward at T is positive, the search should be continued because the search is motivated by the positive increment of the expected net reward, and the reverse is also true. Therefore, if the search should be stopped at T, $\lambda(T)$ must not be positive. But since $\lambda(t)$, $t \in T$, is non-negative from Theorem 1, $\lambda(T) = 0$ for $T \geq T^*$.

(2) Generalization of the reward function

If the reward earned by the searcher when he detects the target successfully depends on the position y of the target as well as the time t, the problem is considerably complicated. We assume the searcher earns reward $R(y, t) \geq 0$ when the target is detected at (y,t) and other system parameters are identical with the model of Section 2. The conditional detection probability at t is defined by

$$\Delta b(t,\phi) = \frac{db(\psi)}{dt}\,\Delta t \ .$$

The conditional expected risk at t is given by $(C(t,\phi)-R(X_t,t))\Delta b(t,\phi)$. Hence the expected risk $f(\phi)$ is obtained as follows.

$$f(\phi) = \int_{t_0}^{T} E[(C(t,\phi)-R(X_t,t))\Delta b(t,\phi)]dt + C(T,\phi)(1-P_T(\phi))$$

(57) $$= \int_{t_0}^{T} E[(R'(X_t,t)-C'(t,\phi))b(\int_{t_0}^{t} \alpha(X_\tau,\tau)\phi(X_\tau,\tau)d\tau)]dt$$

$$+ C(T,\phi) - E[R(X_T,T)b(\int_{t_0}^{T} \alpha(X_t,t)\phi(X_t,t)dt)].$$

The theorem for the optimal search plan minimizing $f(\phi)$ given by (57) is presented as follows.

Theorem 3. A necessary condition for the T-optimal allocation of search effort is that there exists a non-negative function $\lambda(t)$ such that

(58) $$-A_{yt}^{T}(\phi^*) =(\le) \lambda(t)c_0(y,t) \quad \text{for} \quad \phi^*(y,t) >(=) 0,$$

where

$$A_{yt}^{T}(\phi) = \int_{t}^{T} E_{yt}[(R'(X_\tau,\tau)-C'(\tau,\phi))b'(\int_{t_0}^{\tau} \alpha(X_\xi,\xi)\phi(X_\xi,\xi)d\xi)]$$

(59) $$\times q_t(y)\alpha(y,t)d\tau + c_0(y,t)(1-P_t(\phi))$$

$$- E_{yt}[R(X_T,T)b'(\int_{t_0}^{T} \alpha(X_\tau,\tau)\phi(X_\tau,\tau)d\tau)]\alpha(y,t)q_t(y).$$

If $\lambda(t) > 0$, $t \in \boldsymbol{T}$, then

(60) $$\int_{Zt} c_0(y,t)\phi^*(y,t)d\eta(y) = m(t).$$

If $f(\phi)$ is convex and $\phi^* \in \Psi_0$, the above condition is also a sufficient condition for ϕ^* to be optimal.

Proof: The kernel of the Gateaux differential of $f(\phi)$ is calculated as $A_{yt}^{T}(\phi)$ given by (59). Applying Lemma 1, we have the necessary condition. The sufficiency is derived directly from Lemma 2. (Q.E.D.)

(3) Relations between our model and the previous studies

i. Stromquist and Stone discussed in their paper [17] five examples which were solved by their theorems. The fourth example is the case of minimizing the expected return for a moving target and our model mentioned in this paper corresponds to a modefication of this example. Neglecting the search cost, they adopted the expected return as the measure of effectiveness, on the other hand, we employ the expected risk by considering the reward and the search cost. The crucial difference between their model and our model is that they limit the search plan to Ψ_0 and deal with the T-optimal search plan, on the other hand, we investigate the optimal plan in Ψ and derive the optimal stopping time T^* as well as the optimal distribution of search effort.

ii. Tierney and Kadane [18] investigated an optimal surveillanse search for a moving target. They consider a finite set of discete time points for search time and finite cells for the target space. They assume that a target moves from cell to cell according to a Markov transition probability (not necessarily time-homogeneous) and the search terminates at a time and location (called the stopping set) that satisfy certain specified conditions. The objective is to maximize the expected value of a payoff received during and the end of the search. In this model, if we define an appropreate payoff and a stopping set, we can derive a similar detection search model for a moving target to our model. However, this model is not identical with our model at the next points.

(i) In our model, the target space and the time space are not necessary discrete.

(ii) Tierney and kadane limit their model to a Markovian motion target, however we deal with a more general moving target. The example shown in Section 4.2 is the case of the Markovian motion target and the example in Section 4.3 is the target with the conditionally deterministic motion.

(iii) Tierney and Kadane deal with the model in a finite time points and under a stopping set. Therefore, the stopping of the search is definitely prescribed. This problem is same as the problem dealt with in Section 3.1; the T-optimal allocation of search effort. However in our model, we consider a case in which the search time is not limited in advance, and therefore, when to stop the search is an important problem to be investigated. The optimal stopping time such as $T^* = \sqrt{R/\pi v_0^2 c_0}$ for the target with the conditionally deterministic motion is a main result of our model as well as the T-optimal allocation of search effort.

(iv) Tierney and Kadane concider a general payoff function and propose an improved FAB algorithm which avoids the trouble when the objective function has any flat spots. Meanwhile, our algorithm described in Section 4.1 is identical with the FAB algorithm proposed by Brown [1] and later by Washburn [21] in principle.

iii. In Section 4.3, we derive the optimal search plan for a target with the conditionally deterministic motion. As mentioned before, Stone [14] investigated the optimal search for a target of this type which maximized the detection probability. In spite of the difference between the measure of effectiveness, the optimal search plan obtained by (52) is identical with the uniformly P-optimal search plan given by Stone. However, this agreement is not surprising because of the homogeneity of both the reward and the search cost for all $(y,t) \in Z$. If the reward or the search cost varies in the target space, the optimal search plan of our model is not the uniformly P-optimal plan.

iv. Let us apply the theorems obtained in Section 3 to a stationary target. The discrete target space and the continuous time are assumed. Here we specify the system parameters as follows,

$$R(y,t) = R_y, \quad \alpha(y,t) = \alpha_y, \quad c_0(y,t) = c_y, \quad m(t) = m, \tag{61}$$
$$b(\psi) = 1-\exp(-\alpha_y \psi), \quad q_t(y) = p_y.$$

In this case, $A^T_{yt}(\phi)$ is given by the next equation

$$A^T_{ty}(\phi) = -p_y\alpha_y\{R_y\exp(-\alpha_y\psi(y,T)) + m\int_t^T\exp(-\alpha_y\psi(y,\tau))d\tau\} \tag{62}$$
$$+ c_y\sum_{Z_t} p_y\exp(-\alpha_y\psi(y,t)),$$

where $\psi(y,t) = \int_{t_0}^{t} \alpha_y\phi(y,\tau)d\tau$. Therefore, a necessary condition for ϕ^* is derived from Theorem 1;

$$\frac{\alpha_y p_y}{c_y}\{R_y\exp(-\alpha_y\psi^*(y,T)) + m\int_t^T\exp(-\alpha_y\psi^*(y,\tau))d\tau\} \tag{63}$$
$$=(\leq)\ \lambda(t) + Q(t) \qquad \text{for } \phi^*(y,t) >(=)\ 0$$

where $Q(t) = \sum_{Z_t} p_y\exp(-\alpha_y\psi(y,t))$. Since ϕ^* can be restricted to Ψ_0 by the assumption of the stationary target, $f(\phi)$ is convex and the necessary condition mentioned above is also sufficient. Since $\lambda_T(T^*) = 0$ by Theorem 2, from (63), we have

$$\frac{\alpha_y p_y(T^*)R_y}{c_y} = 1 \tag{64}$$

at $t = T^*$, if $\phi^*_{T^*}(y,T^*) > 0$, where $p_y(T^*)$ is the posterior probability distribution of the target at T^*, $p_y(T^*) = p_y\exp(-\alpha_y\psi^*(y,T^*))/Q(T^*)$. The conditions (63) and (64) are identical with the conditions which was derived by Iida [8].

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