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CONTINUOUS REVIEW (s,S) INVENTORY MODEL WITH LIMITED BACKLOGGING LEVELS AND STOCHASTIC LEAD TIMES

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Abstract A (s,S) inventory policy is studied for a continuous review inventory model in which backloggings are restricted at limited levels and stochastic lead times are allowed. The model assumes that at most one order is outstanding and demands occur in a Poisson process. The steady-state probability distributions of the inventory levels are derived so as to determine the long-run expected average cost. Then, an optimal solution is characterized and its computational procedure is presented.

1. Introduction

This paper considers a (s,S) inventory problem with limited backlogging levels and stochastic lead times in which demands occur in a Poisson process with parameter α, where s is the reorder point and S is the order-up-to level. Lead times, defined as the time from an order placement until its shipment and denoted by L, are exponentially distributed with parameter β. If a demand occurs when the system is out of stock, the demand is backlogged. However, if backlogging sales exceed a limited level b, the exceeding sales is lost. It is further assumed that at most one order is outstanding.

If b=0, the limited backlogging model becomes the ordinary (s,S) inventory model only with lost sales allowed, while it represents the backlogging case when $b=\infty$ (infinite backlogging). In fact, if the backlogging level is 0, the model represents a stochastic extension of the work of Archibald [1] where lead time is constant and $S-s > s$. However, if the backlogging level is infinite, the model is the same as that of Dirickx et al [4].

Many other authors (Montgomery et al. [8], Rosenberg [10] and Whitin [13]) have treated lot-size models with mixtures of backlogging and lost sales allowed where a fixed partial fraction of whole stockout demand (i.e., proportion to backlogging demand) is backlogged. Nahmias [9] has studied a similar problem in a periodic review case where random lead time and proportional backlogging are permitted, and searched an approximate solution. However, such a proportional backlogging allowance may cause impracticable backlogging burden to inventory systems due to uncertain huge demands.

Aucamp and Fogaty [2] have examined a similar problem but with allowing a limited backlog delay. Doshi et al. [5] have worked on a production-inventory problem with a given backlogging level for an optimal inventory decision so as to determine an optimal production rate. Dirickx and Koevoets [4] have applied the Markov renewal theory (referring to Cinlar [3]) to analyze an inventory model with complete backlogging and stochastic lead time allowed.

The objective of this paper is to determine a limited (fixed) backlogging level along with s and S that minimize the long-run overall expected average cost. The backlogging level provides an information of whether each instantaneous demand will be backlogged or lost.

The problem will be analyzed by applying the Markov renewal theory to derive the steady-state probabilities of inventory levels upon which the long-run overall expected average inventory system cost function is determined. The cost function is further investigated to characterize the optimal solution, for which the computational procedure is exploited.

2. Semi-Regenerative Inventory Process

This section will show that some results from the Markov renewal theory can be applied to analyze the given inventory problem. For the rest of the paper, the followings are defined first.

(1) $Z_t (t \geqq 0)$ denotes the random inventory level at time point t taking values on $F=\{S,S-1,S-2,\ldots,-b\}$, and

(2) $X_n (n=0,1,2,\ldots)$ denotes the random inventory level at time period $T_n (T_0=0)$ (i.e., $X_n=Z_{T_n}$), where $T_n (n=1,2,3,\ldots)$ denotes the first ordering time point after T_{n-1} if $X_{n-1}=i$ for $i>s$, or the arrival time piont of an order placed at T_{n-1} if $X_{n-1} \leqq s$.

Lemma 1. The stochastic process $(X,T) = \{(X_n,T_n): n=0,1,2,\ldots\}$ is a Markov renewal process with state space E, where $E = \{S, S-1, \ldots, S-s-b, s\}$ for $S-s-b > s$,

$= \{S, S-1, \ldots, s\}$ for $S-s-b = s$,

$= \{S, S-1, \ldots, s, \ldots, S-s-b\}$ for $S-s-b < s$.

Proof: Let $D(T_{n+1}-T_n)$ be the cumulative demand during the time interval $(T_n, T_{n+1}]$. According to the definition of X_n and T_n, if $X_n=i>s$, T_{n+1} is the first ordering point after T_n and so $X_{n+1}=s$ where $D(T_{n+1}-T_n)=X_n-s$; if $X_n=i\leqq s$, T_{n+1} is the arrival time point of an order placed at T_n where $X_{n+1}=S-(X_n+b)$ for $D(T_{n+1}-T_n)\geqq X_n+b$, but $X_{n+1}=S-D(T_{n+1}-T_n)$ for $D(T_{n+1}-T_n)<X_n+b$ and the time interval $T_{n+1}-T_n$ is lead time.

Since Poisson demand process is memoryless, the state transition from $X_n=i$ to $X_{n+1}=j$ is Markovian where $X_n(n=0,1,2,\ldots)$ takes values on $\{S, S-1, \ldots, S-s-b\} \cup \{s\}$. This result along with the $D(T_{n+1}-T_n)$ function implies that the distribution of the time interval $T_{n+1}-T_n$ depends only on X_n. Thus,

$$\Pr[X_{n+1}=j,\ T_{n+1}-T_n \leqq t \mid X_0,\ldots,X_n;\ T_0,\ldots,T_n]$$

$$= \Pr[X_{n+1}=j,\ T_{n+1}-T_n \leqq t \mid X_n].$$

This completes the proof.

The following lemma is then immediate.

Lemma 2. $Z=\{Z_t : t\geqq 0\}$ is a semi-regenerative process with respect to the Markov renewal process (X,T).

Since the Markov renewal process (X,T) is time-homogeneous, the semi-Markov kernel of the Markov renewal process (X,T) is defined as the family of probabilities $\{Q(i,j,t);\ i,j\in E,\ t\geqq 0\}$

$$(3)\quad Q(i,j,t) = \Pr[X_{n+1}=j,\ T_{n+1}-T_n \leqq t \mid X_n=i] = \Pr[X_1=j, T_1 \leqq t \mid X_0=i].$$

Hence, for any $i,j\in E$ and $t \geqq 0$,

$$(4)\quad Q(i,j,t) = \begin{cases} \Pr[D(T_1)=i-j,\ T_1 \leqq t], & \text{for } i>s \text{ and } j=s,\\ \Pr[D(L)=S-j,\ L \leqq t], & \text{for } i\leqq s \text{ and } j>S-i-b,\\ \Pr[D(L) \geqq i+b,\ L \leqq t], & \text{for } i\leqq s \text{ and } j=S-i-b,\\ 0, & \text{otherwise.} \end{cases}$$

$$= \begin{cases} \int_0^t \alpha(\alpha x)^{i-j-1} e^{-\alpha x}/(i-j-1)!\,dx, & \text{for } i>s \text{ and } j=s,\\ \int_0^t e^{-\alpha x}(\alpha x)^{S-j}/(S-j)!\,\beta e^{-\beta x}dx, & \text{for } i\leqq s \text{ and } j>S-i-b,\\ \int_0^t \sum_{\ell=i+b}^{\infty} e^{-\alpha x}(\alpha x)^{\ell}/\ell!\,\beta e^{-\beta x}dx, & \text{for } i\leqq s \text{ and } j=S-i-b,\\ 0, & \text{otherwise.} \end{cases}$$

3. Steady-state Probability of Inventory Levels

Using the properties of the Markov renewal process and the semi-regenerative process described in the preceding section, the steady-state probability of inventory levels will be derived. In order to characterize the transition structure of the imbedded Markov chain $X=\{X_n: n=0,1,2,\dots\}$, let p_{ij} be the transition probability of moving from state i to j. Then, from the results of Cinlar [3], $p_{ij} = \Pr[X_{n+1}=j \mid X_n=i] = \lim_{t\to\infty} Q(i,j,t)$.
Hence, for any $i,j \in E$,

$$(5) \qquad p_{ij} = \begin{cases} 1, & \text{for } i>s \text{ and } j=s, \\ (1-r)r^{S-j}, & \text{for } i\leqq s \text{ and } j>S-i-b, \\ r^{S-j}, & \text{for } i\leqq s \text{ and } j=S-i-b, \\ 0, & \text{otherwise,} \end{cases}$$

where $r=\{\alpha/(\alpha+\beta)\}$.

Lemma 3. If b is infinitely large, then the Markov chain X is irreducible, aperiodic and non-null recurrent.

Proof: Aperiodicity and irreducibility follow from (5). In order to show recurrency, let π_i be defined as follows:

$$(6) \qquad \pi_i = \begin{cases} (1-r)r^{S-i}A, & \text{for } i\leqq S \text{ and } i\neq s, \\ (1-r^{S-s+1})A, & \text{for } i=s, \end{cases}$$

where $A=(2-r^{S-s})^{-1}$.

Then, π_i's satisfy the following relations:

$$\pi_i = \sum_{j\in E} p_{ji}\pi_j,$$

$$\sum_{i\in E} \pi_i = 1,$$

$$\pi_i \geqq 0 \qquad \text{for any } i \in E.$$

From Theorem (6.2.1) of Cinlar [3], the Markov chain X is non-null recurrent.

Lemma 4. If b is finite, then the Markov chain X is irreducible and non-null recurrent.

Proof: Irreducibility follows from (5). Since b is finite, the Markov chain X has a finite state space. Thus the Markov chain X is non-null recurrent according to Corollary (5.3.22) of Cinlar [3] and Lemma III.2.1. of Isaacson and Madsen [7].

Now, let π_i be defined as follows:

for $S-s-b > s$;

$$(7) \qquad \pi_i = \begin{cases} (1-r)r^{S-i}/2, & \text{for } S-s-b+1 \leqq i \leqq S, \\ r^{s+b}/2, & \text{for } i = S-s-b, \\ 1/2, & \text{for } i = s, \end{cases}$$

for $S-s-b = s$;

$$(8) \qquad \pi_i = \begin{cases} (1-r)r^{S-i}A, & \text{for } s+1 \leqq i \leqq S, \\ A, & \text{for } i = s, \end{cases}$$

for $S-s-b < s$;

$$(9) \qquad \pi_i = \begin{cases} (1-r)r^{S-i}A, & \text{for } s+1 \leqq i \leqq S, \\ A(1-r^{S-s+1})/(1-r^{S+b+1}), & \text{for } i = s, \\ A(1-r)r^{S-i}/(1-r^{S+b+1}), & \text{for } S-s-b+1 \leqq i \leqq s-1, \\ A(1-r^{S-s+1})r^{s+b}/(1-r^{S+b+1}), & \text{for } i = S-s-b. \end{cases}$$

Then, π_i's satisfy the following relations:

$$\pi_i = \sum_{j \in E} p_{ji}\pi_j,$$

$$\sum_{i \in E} \pi_i = 1,$$

$$\pi_i \geqq 0 \qquad \text{for any } i \in E.$$

Hence, π_i's are the invariant probability measure of the Markov chain X.

Lemma 5. The Markov renewal process (X,T) is aperiodic.

Proof: It has been shown that Markov chain X is irreducible. We can also see from (4) that $Q(i,j,t)$ are not all step functions. Thus, it follows from Corollary (10.2.24) and Corollary (10.2.27) of Cinlar [3] that the Markov renewal process (X,T) is an aperiodic process.

Lemmas 3, 4 and 5 lead to the next theorem.

Theorem 1. The Markov renewal process (X,T) is irreducible, aperiodic and non-null recurrent.

Now, we want to define a conditional probability function $K_t(\cdot,\cdot)$;

$K_t(i,j) = \Pr[Z_t=j,\ T_1 > t \mid X_0=i]$, for any $i \in E$, $j \in F$, $t \geqq 0$, so that

$$(10)\qquad K_t(i,j) = \begin{cases} e^{-\alpha t}(\alpha t)^{i-j}/(i-j)!, & \text{for } i>s \text{ and } i\geqq j>s\ , \\ e^{-\beta t}e^{-\alpha t}(\alpha t)^{i-j}/(i-j)!, & \text{for } i\leqq s \text{ and } -b<j\leqq i, \\ e^{-\beta t}\sum_{\ell=i+b}^{\infty} e^{-\alpha t}(\alpha t)^{\ell}/\ell!, & \text{for } i\leqq s \text{ and } j=-b, \\ 0, & \text{otherwise.} \end{cases}$$

The expected regeneration interval with initial state i is also defined, denoted by $m(i)$

$$m(i) = E[T_1|X_0=i] = \int_0^{\infty}[1-\sum_{j\in E} Q(i,j,t)]dt$$

$$(11)\qquad = \begin{cases} 1/\beta, & \text{for } i\leqq s, \\ (i-s)/\alpha, & \text{for } i>s. \end{cases}$$

Let P_j^* be the steady-state probability of inventory level j under the (s,S) inventory policy and given b.

Theorem 2. For any $j \in F$,

$$(12)\qquad P_j^* = \lim_{t\to\infty} \Pr[Z_t=j|X_0=i] = \{\sum_{i\in E}\pi_i\int_0^{\infty}K_t(i,j)dt\}/\{\sum_{i\in E}\pi_i m(i)\}.$$

Proof: In view of Theorem (10.6.12) of Cinlar [3], it suffices to observe that (X,T) is an irreducible, aperiodic and recurrent process and that $K_t(i,j)$ is Riemann-integrable.

It follows from Theorem 2 that

for $S-s-b \geqq s$;

$$(13)\qquad P_j^* = \begin{cases} \{(1-r^{S-j+1})/\alpha\}/q(s,S,b), & \text{for } S-s-b+1 \leqq j \leqq S, \\ (1/\alpha)/q(s,S,b), & \text{for } s+1 \leqq j \leqq S-s-b, \\ \{r^{s-j}/(\alpha+\beta)\}/q(s,S,b), & \text{for } -b+1 \leqq j \leqq s, \\ (r^{s+b}/\beta)/q(s,S,b), & \text{for } j = -b, \end{cases}$$

for $S-s-b < s$;

$$(14)\qquad P_j^* = \begin{cases} \{(1-r^{S-j+1})/\alpha\}/q(s,S), & \text{for } s+1 \leqq j \leqq S, \\ \{r^{s-j}-r^{S-j+1}+(s-j)(1-r)r^{S-j}\}/\{(\alpha+\beta)q(s,S)AA\}, & \\ & \text{for } S-s-b+1 \leqq j \leqq s, \\ \{r^{s-j}-r^{2S-s-j+1}+(2s+b-S)(1-r)r^{S-j}\}/\{(\alpha+\beta)q(s,S)AA\}, & \\ & \text{for } -b+1 \leqq j \leqq S-s-b, \\ \{r^{s-j}-r^{2S-s+b+1}+(2s+b-S)(1-r)r^{S+b}\}/\{\beta q(s,S)AA\}, & \\ & \text{for } j = -b, \end{cases}$$

where $q(s,S,b) = (S-s)/\alpha + r^{s+b}/\beta$,

$q(s,S) = (S-s)/\alpha + r^{S-s}/\beta$,

$AA = 1-r^{S+b+1}$.

4. Cost Structure

The fixed cost of ordering and the holding cost of on-hand inventory per unit per unit time are, respectively, denoted by c_0 and c_1. The backlogging cost per unit backlogging sale per unit time is denoted by c_2 and a fixed cost of lost sale is denoted by c_3.

The expected cost associated with the (s,S) inventory policy having limited backlogging levels depends on the steady-state probabilities of inventory levels and the amount of lost sales. Therefore, it is necessary to figure out the expected amount of lost sales. Let $E_i(LS)$ be the expected amount of lost sales during a regeneration interval with initial state $i \leqq s$, where LS stands for lost sales. Then it holds that

$$\begin{aligned} E_i(LS) &= \sum_{k=1}^{\infty} k \Pr[ALS(T_n, T_{n+1})=k \mid X_n = i \leqq s] \\ &= \sum_{k=1}^{\infty} k \Pr[ALS(0, T_1)=k \mid X_0 = i \leqq s] \\ &= \sum_{k=1}^{\infty} k \Pr[D(L)=i+k+b] \\ &= \sum_{k=1}^{\infty} k \int_0^{\infty} e^{-\alpha x}(\alpha x)^{i+k+b}/(i+k+b)!\,\beta e^{-\beta x} dx \\ &= (\alpha/\beta) r^{i+b}, \end{aligned} \tag{15}$$

where $ALS(T_n, T_{n+1})$ is the amount of lost sales during (T_n, T_{n+1}).

Let $EC(s,S,b)$ be the overall expected average cost under the (s,S) inventory policy and given b. Fron the results of Ross [11] and Schellhaas [12],

$$\begin{aligned} EC(s,S,b) = &\sum_{j \leqq s} \pi_j \{c_0 + c_3 E_j(LS)\} / \{\sum_{k \in E} \pi_k m(k)\} \\ &+ c_1 \sum_{j=0}^{S} j P_j^{*} - c_2 \sum_{j=-b}^{0} j P_j^{*}. \end{aligned} \tag{16}$$

Then, equations (11), (13), (14) and (15) give the following $EC(s,S,b)$ functions specified over the distinct ranges of b.

For $b \leqq S-2s$,

$$EC(s,S,b) = [c_0+c_1\{(S-s)(S+s+1)/2\alpha-(S-s)/\beta+r^{s+b}(S-s-b)/\beta\} \tag{17}$$
$$+(c_1+c_2)(\alpha r^s/\beta^2-\alpha r^{s+b}/\beta^2)+c_3\alpha r^{s+b}/\beta]/q(s,S,b).$$

For $S-2s < b < S-s$,

$$EC(s,S,b) = [c_0+c_1\{(S-s)(S+s+1)/2\alpha-(\beta S-\alpha)/\beta^2+r^{S-s}(\beta s-\alpha)/\beta^2\}$$
$$+\{c_1(\beta s-\alpha)/\beta-\alpha c_1 r^{S-s}/\beta+\alpha(c_1+c_2)(r^s-r^{2S-s+1})/\beta \tag{18}$$
$$+\alpha(c_1-c_2+\beta c_3)r^{s+b}/\beta+(2s+b-S)(c_1+c_2)r^{S+1}$$
$$+\alpha(c_2-\beta c_3)r^{2S-s+b+1}/\beta+c_1(\alpha/\beta-S+b+s)r^{S+b+1}$$
$$+(\beta c_3-c_2)(2s+b-S)r^{S+b+1}\}/(AA\beta)]/q(s,S).$$

For $S-s \leqq b$,

$$EC(s,S,b) = [c_0+c_1\{(S-s)(S+s+1)/2\alpha-(\beta S-\alpha)/\beta^2+r^{S-s}(\beta s-\alpha)/\beta^2\}$$
$$+\{c_1(\beta s-\alpha)/\beta-\alpha c_1 r^{S-s}/\beta+\alpha(c_1+c_2)r^s/\beta \tag{19}$$
$$+\alpha(c_3-2c_2/\beta)r^{s+b}+(s+1+\alpha/\beta)(c_1+c_2)r^{S+1}$$
$$+\alpha(c_2-\beta c_3)r^{2S-s+b+1}/\beta+\beta(2s+b-S)c_3 r^{S+b+1}$$
$$+(2S-3s-b-\alpha/\beta)c_2 r^{S+b+1}\}/(AA\beta)]/q(s,S).$$

Now, a solution search procedure will be exploited for the optimal values of s, S and b with which the overall expected average cost $EC(s,S,b)$ is minimized. In fact, the function $EC(s,S,b)$ is too complicated to solve for the optimal values of s, S and b. However, $EC(s,S,b)$ can be characterized to follow unimodal trends over each possible range of b.

Lemma 6. For $b \geqq S-s$, $EC(s,S,\cdot)$ is either unimodal or monotonic.

Proof: Let $g(b)=EC(s,S,b)$ for $b \geqq S-s$ where s and S are fixed values. Then from equation (19), it can be written,

$$g(b) = bW_1+W_2+(-bW_1+W_3)/(1-W_0 r^b), \tag{20}$$

where $W_0 = r^{S+1}$,

$$W_1 = 2c_2/\beta-c_3,$$
$$W_2 = [c_0+c_1\{(S-s)(S+s+1)/2\alpha-(\beta S-\alpha)/\beta^2+r^{S-s}(\beta s-\alpha)/\beta^2\}]/q(s,S),$$
$$W_3 = \{\alpha(c_2-\beta c_3-c_1)+(\beta s+\beta+\alpha)(c_1+c_2)r^{S+1}+\alpha(\beta c_3-2c_2)r^{s-S-1}$$
$$+\alpha(c_1+c_2)r^s+(\beta s-\alpha)c_1-(3\beta s-2\beta S+\alpha)\}/\beta^2+(2s-S)c_3.$$

It follows that

$$g(b+1)-g(b) = h(b)W_0r^b/(1-W_0r^{b+1})(1-W_0r^b),$$

where $h(b) = W_1r(W_0r^b-1)+(1-r)(bW_1-W_3)$.

Now we get

$$h(b+1)-h(b) = (1-r)W_1(1-W_0r^{b+1}).$$

Since $0 < W_0 < 1$ and $0 < r < 1$, $W_0r^b/(1-W_0r^{b+1})(1-W_0r^b) > 0$ and $h(b)$ is monotonic. Thus there exists at most one finite b such that $g(b+1)-g(b) = 0$. This completes the proof.

Corollary 1. If $2c_2/\beta-c_3 > 0$, the optimal value of b is finite.

Proof: It is seen from the proof of Lemma 6 that if $W_1 = 2c_2/\beta-c_3 > 0$, $g(b+1)-g(b) > 0$ for large b. Thus, the proof is completed.

Likewise, Lemma 7 holds.

Lemma 7. For $S-2s < b < S-s$, $EC(s,S,\cdot)$ is either unimodal or monotonic.

Lemma 8. For $b \leqq S-2s$, $EC(s,S,\cdot)$ is either unimodal or monotonic.

Proof: Let $g(b) = EC(s,S,b)$ for $b \leqq S-2s$ where s and S are fixed. Then from equation (17), it can be written,

(21) $$g(b) = g_1(b)+g_2(b),$$

where $g_1(b) = -c_1b+W_4$,

$g_2(b) = [\{c_1(S-s)/\alpha\}b+W_5]/\{r^{s+b}/\beta+(S-s)/\alpha\}$,

$W_4 = c_1(S-s)-\alpha(c_1+c_2)/\beta+\alpha c_3$,

$W_5 = c_0+(S-s)(3s-S+1)c_1/2\alpha+\alpha(c_1+c_2)r^s/\beta^2+(S-s)(c_2-\beta c_3)/\beta$.

This shows that $g_1(b)$ is a linearly decreasing function and $g_2(b)$ is a monotonic increasing function. Thus, $g(b)$ is either monotonic or unimodal.

Lemmas 6, 7 and 8 provide the ranges of b and the cost relations under which there exists a finite b with which the long-run overall expected average cost is minimized for fixed s and S. The existence of such a finite b implies that the given (s,S) inventory model may be better than other (s,S) inventory models with either complete backlogging or lost sales allowed.

From equations (17), (18) and (19), it is further observed that the optimal value of S is finite, since $EC(s,S,b)$ increases as S increases. Based on this observation along with our computational experience in Section 5, the optimal value of S is upper bounded by

(22) $$\bar{S} = \sqrt{2\alpha(c_0+\alpha c_3/\beta)/c_1} \ ,$$

where $\bar{S}$ corresponds to Hadley and Whitin [6]'s upper bound of Q in (r,Q) inventory model. Moreover, since s is less than S, it may be easy to search both the optimal values of s and S by simple grid method when b is fixed. Let $s^*(b)$ and $S^*(b)$ denote the local optimal values of s and S, respectively, for a given problem with a fixed b. Then, it can not be analytically verified but experienced in our various numerical problems test that $EC(s^*(b),S^*(b),b)$ is unimodal in b. If the unimodality holds and if $2c_2/\beta-c_3 \leqq 0$, then it follows from Corollary 1 that the optimal value b^* is infinite.

Based on the above solution characteristics, a solution search procedure for problems satisfying unimodality in b is described below:

Step 1. Compute $W = 2c_2/\beta-c_3$. If $W \leqq 0$, go to Step 2. Otherwise, go to Step 3.

Step 2. Let $b^* = \infty$, and search $s^*(\infty)$ and $S^*(\infty)$. Then, stop the search procedure.

Step 3. Select an arbitrary initial value for b, less than $\bar{S}$ computed from equation (22).

Step 4. Search $s^*(b)$, $S^*(b)$ and compute $EC(s^*(b),S^*(b),b)$

Step 5. Continue Steps 3 and 4 over b in a standard local descent search procedure by adjusting b in the direction of descent until any further change in b value increases the total cost.

5. A Numerical Example

The behavior of the optimal policy with respect to input parameters α, β, or $c_i (i=0,1,2,3)$ are examined. The corresponding optimal values of s, S and b are summarized in Tables 1 through 6, where the examination starts with a base problem, where $\alpha=1.0$, $\beta=.05$, $c_0=100.0$, $c_1=1.0$, $c_2=2.0$ and $c_3=50.0$. The problem test is repeated with varying the given base problem parameters one at a time. Each problem is numerically searched for optimal solutions of s, S and b on an IBM/PC-XT.

Table 1. The Optimal Values of s, S and b with the Variations of α.

α	s	S	b	α	s	S	b
.5	2	22	10	3.0	30	121	42
1.0	8	42	16	3.5	35	140	48
1.5	13	62	23	4.0	41	160	55
2.0	19	81	28	4.5	36	180	62
2.5	24	101	36	5.0	52	199	67

Table 2. The Optimal Values of s, S and b with the Variations of β.

β	s	S	b	β	s	S	b
.01	18	58	0	.06	5	39	25
.02	15	49	0	.07	3	36	31
.03	14	44	0	.08	2	33	39
.04	12	40	0	.09	1	30	∞
.05	8	42	16	.10	0	27	∞

Table 3. The Optimal Values of s, S and b with the Variations of c_0.

c_0	s	S	b	c_0	s	S	b
50.	9	41	15	300.	4	46	19
100.	8	42	16	350.	3	47	20
150.	6	43	18	400.	2	48	21
200.	5	44	19	450.	2	49	22
250.	5	45	19	500.	1	50	23

Table 4. The Optimal Values of s, S and b with the Variations of c_1.

c_1	s	S	b	c_1	s	S	b
.25	24	78	32	1.50	4	33	14
.50	15	59	22	1.75	3	31	14
.75	10	48	18	2.00	2	28	14
1.00	8	42	16	2.25	1	26	13
1.25	6	37	15	2.50	1	24	12

Table 5. The Optimal Values of s, S and b with the Variations of c_2.

c_2	s	S	b	c_2	s	S	b
.25	0	21	∞	1.50	3	42	34
.50	0	28	∞	1.75	5	42	25
.75	0	33	∞	2.00	8	42	16
1.00	0	38	∞	2.25	10	39	5
1.25	0	41	∞	2.50	11	37	0

Table 6. The Optimal Values of s, S and b with the Variations of c_3.

c_3	s	S	b	c_3	s	S	b
1.0	0	5	0	40.0	9	33	0
5.0	0	8	0	50.0	8	42	16
10.0	0	11	0	60.0	6	46	31
20.0	2	19	0	70.0	6	48	39
30.0	6	27	0	80.0	6	49	∞

From Tables 1 and 2, as the ratio α/β increases, policy parameters s and S also increase as expected. This is interpreted as more inventory holdings are required. Tables 3 and 4 show that $S-s$ increases as the ratio c_0/c_1 increases. This implies that the expected amount per each order increases, as the ratio c_0/c_1 increases. Tables 5 and 6 show that as the ratio c_3/c_2 increases, b increases. This is because larger backlogging levels are required to compensate for lost sales cost.

6. Conclusion

This paper has examined a (s,S) inventory model with limited (fixed) backlogging levels and stochastic lead times, where demand follows a Poisson process. Using the Markov renewal theory, the steady-state probabilities of inventory levels are derived, upon which a long-run expected average cost function is described. The cost function is then characterized as satisfying unimodality over certain ranges of b when s and S are fixed. On the basis of characteristic, a solution search procedure is suggested and illustrated with a numerical example.

As an immediately following topic, the limited backlogging level inventory model is under study for demands occurring in a compound Poisson process.

References

[1] Archibald, B. C.: Continuous Review (s,S) Policies with Lost Sales. *Management Science*, Vol.27 (1981), 1171-1177.

[2] Aucamp, D. C. and Fogaty, D. W.: *EOQ* with Limited Backorder Delays. *Computers and Operations Research*, Vol.11 (1984), 333-335.

[3] Cinlar, E.: Introduction to Stochastic process. Prentice-Hall, Englewood Cliffs, New Jersey, 1975.

[4] Dirickx, Y. M. I. and Koevoets, D.: A Continuous Review Inventory Model with Compound Poisson Demand Process and Stochastic Lead Times. *Naval Research Logistics Quarterly*, Vol.24 (1977), 577-585.

[5] Doshi, B. T., van der Duyn Schouten, F. A. and Talman, A. J. J.: A Production-Inventory Control Model with a Mixture of Backorders and Lostsales. *Management Science*, Vol.24 (1978), 1078-1086.

[6] Hadley, G. and Whitin, T. M.: Analysis of Inventory Systems. Prentice-Hall, Englewood Cliffs, New Jersey, 1963.

[7] Isaacson, D. L. and Madsen, R. W.: Markov Chains Theory and Applications. John Wiley & Sons, New York, 1976.

[8] Montgomery, D. C., Bazaraa, M. S. and Keswani, A. K.: Inventory Models with a Mixture of Backorder and Lost Sales. *Naval Research Logistics Quarterly*, Vol.20 (1973), 255-263.

[9] Nahmias, S.: Simple Approximations for a Variety of Dynamic Leadtime Lost-Sales Inventory Models. *Operations Research*, Vol.27 (1979), 904-924.

[10] Rosenberg, D.: A New Analysis of a Lost-Size Model with Partial Back-logging. *Naval Research Logistics Quarterly*, Vol.26 (1979), 349-353.

[11] Ross, S. M.: Applied Probability Models with Optimization Applications. Holden-day, San Francisco, 1970.

[12] Schellhaas, H.: Bewertete Regenerative Prozesse mit Anwendung auf Lagerhaltungsmodelle mit Zustandsabhangigen Paramentern. Preprint Nr. 136, Technische Hochschule Darmstadt, (1974).

[13] Whitin, T. M.: Recent Articles on Partial Backorders: Comment. *Naval Research Logistics Quarterly*, Vol.32 (1985), 361-362.

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