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DISTRIBUTION PROPERTIES OF DISCRETE CHARACTERISTICS IN M/G/1 AND GI/M/1 QUEUES

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Abstract It has been shown by many authors that distribution properties of some characteristics in queues are inherited from those of the service times. For instances, Keilson (1978) showed that the length of a busy period in an M/G/1 queue has a completely monotone density if so does the service time, Shanthikumar (1988) proved that the waiting time distribution in a GI/G/1 queue is DFR if so is the service time distribution, etc. This paper studies such inheritance of distribution properties of discrete characteristics in M/G/1 and GI/M/1 queues. To do so, uniformly monotone discrete time Markov chains are first investigated. Various first passage times which are of independent interest for such Markov chains are considered. By showing that some characteristics in M/G/1 and GI/M/1 queues are expressible as the first passage times, distribution properties of those characteristics are studied.

§0. Introduction

Consider a standard GI/G/1 queueing system having the inter-arrival time distribution function $H(x)$ and the service time distribution function $S(x)$. We assume that the queue under consideration is stable. In the context of queueing theory, such descriptive characteristics as the number of customers in the system (denoted by N), the number served during a busy period (R), the waiting time of a customer (W) and the length of a busy period (B) are often of interest. Many papers have been devoted to studying extensively to obtain ergodic distributions of such random variables.

Among those, the most notable results may be ones for GI/M/1 queues, i.e. $S(x)$ is exponential (write $S \in EXP$). For an arbitrary $H(x)$ ($H \in ARB$), this system results in that the conditional distribution function of W given $W > 0$ is exponential and that of the number of customers found by an arrival ($N_{|\text{arrival}}$) is geometric. These results are of some interest in the sense that the exponentiality of $S(x)$ is inherited to the distribution functions of $W_{|W>0}$ and $N_{|\text{arrival}}$. This kind of inheritance has also been reported in Keilson [5] where he proved that the densities of both B and $W_{|W>0}$ are completely monotone if the density of $S(x)$ is completely monotone ($S \in CM$) and $H \in EXP$ (see e.g. [1,3,4] for the definitions of various distribution functions appearing hereafter in this paper).

In a recent paper by Shanthikumar [14], it is shown that geometric compounding of *i.i.d.* DFR (decreasing failure rate) distibution functions is DFR. This striking result proves that the distribution of W is DFR when $H \in EXP$ and $S \in IMRL$ (increasing mean residual lifetime) or when $S \in DFR$ and $H \in ARB$. As a byproduct, he also showed that the distribution of R is DFR when $S \in IFR$ in M/G/1 queues. This paper studies such inheritance of distribution properties of discrete characteristics in M/G/1 and GI/M/1 queues, thereby extending Shanthikumar's results [14]. To do so, we first investigate discrete time Markov chains having monotone transition matrices with respect to the uniform stochastic orderings $\prec_{\ell}$, $\prec_{(-)}$ and $\prec_{(+)}$ introduced in Keilson and Sumita [9] and Whitt [15]. Various first passage times which are of independent interest for such Markov chains are then considered. By showing that some discrete characteristics in M/G/1 and GI/M/1 queues are expressible as the first passage times of Markov chains, distribution properties of those characteristics are studied.

The construction of this paper is as follows. In Section 1, definitions and notation needed for the subsequent developments are given. In Section 2, Theorem 3.1 of Shanthikumar [14] is recaptured and extended accordingly from the point of view of preservation of the uniform orderings mentioned above. Based on this extention, we then study distributions of three first passage times of a uniformly monotone Markov chain. Let $N(t)$ be the number of customers in the system at time t. For an M/G/1 queue, the embedded process $N_k = N(\tau_k+)$ where τ_k is the departure epoch of the kth customer in the system constitutes a Markov chain on the state space $\{0, 1, 2, \cdots\}$ while so does $N_k = N(\tau_k-)$ for a GI/M/1 queue where τ_k is in turn the arrival epoch of the kth customer. Some sufficient conditions under which the embedded Markov chains preserve the uniform orderings are derived in Section 3. The distribution of the number served during a busy period is studied in Section 4. Section 5 is devoted to M/G/1 and GI/M/1 queues with finite waiting rooms.

§1. Some Preliminaries

In this section, we give some definitions and notations needed for the subsequent studies. Let $f(x, y)$ be a non-negative function defined on $\mathcal{E} \times \mathcal{E}$ where $\mathcal{E}$ is either $\mathcal{R} = (-\infty, \infty)$

or $\mathcal{I} = \{\cdots, -1, 0, 1 \cdots\}$. For $x_n \in \mathcal{E}$ and $y_n \in \mathcal{E}$, $1 \le n \le p$, we write

$$f\begin{pmatrix} x_1 & \cdots & x_p \\ y_1 & \cdots & y_p \end{pmatrix} = \begin{vmatrix} f(x_1, y_1) & \cdots & f(x_1, y_p) \\ \vdots & \ddots & \vdots \\ f(x_p, y_1) & \cdots & f(x_p, y_p) \end{vmatrix}. \tag{1}$$

The next definitions are due to Karlin [4]. The reader is referred to the monograph [4] for more detailed discussions.

Definition 1.1.

(a) A function $f(x, y)$ is sign regular of order r (write $f \in SR_r$) if, for every $x_1 < \cdots < x_p$, $y_1 < \cdots < y_p$ and $1 \le p \le r$, there exists a sequence of numbers each either +1 or −1 such that

$$\varepsilon_p f\begin{pmatrix} x_1 & \cdots & x_p \\ y_1 & \cdots & y_p \end{pmatrix} \ge 0.$$

The number ε_p is called the sign of $f\begin{pmatrix} x_1 & \cdots & x_p \\ y_1 & \cdots & y_p \end{pmatrix}$. In the case where ε_p can be chosen as $(-1)^{p(p-1)/2}$, we say that f is sign reverse rule of order r ($f \in RR_r$). If $\varepsilon_p = +1$ for every $1 \le p \le r$, i.e. $f\begin{pmatrix} x_1 & \cdots & x_p \\ y_1 & \cdots & y_p \end{pmatrix} \ge 0$ for every $1 \le p \le r$, then f is called totally positive of order r ($f \in TP_r$).

(b) A function $g(x)$ of a single variable on $\mathcal{E}$ is said to be Pólya frequency of order r ($g \in PF_r$) if $f(x, y) = g(x - y)$ is TP_r.

For a non-negative function $g(x)$ defined on $\mathcal{E}$ having $g(x) = 0$ for $x < 0$, the following proposition holds. The proof is found in p159 of Karlin [4]. In what follows, we write, for example, $g(x + y) \in TP_r$ if the function f of two variables defined by $f(x, y) = g(x + y)$ is TP_r. Other conventions should be understood in the same manner.

Proposition 1.1. Let $g(x)$ be such that $g(x) = 0$ for $x < 0$ where x runs over $\mathcal{E}$. Then:

(a) $g(x + y) \in RR_2$ in $x, y \ge 0$ if and only if $g(x) \in PF_2$.

(b) $g(x) \in PF_r$ implies that $g(x + y) \in RR_r$ in $x, y \ge 0$.

If $g(x)$ is defined on $\mathcal{I}$, the content of Proposition 1.1 can be understood via a matrix representation. Let (a_n) be a non-negative sequence defined on $\mathcal{I}$ such that $a_n = 0$ for $n < 0$. For $(a_n)_{n=0}^{\infty}$, define the two matrices $\boldsymbol{A}^-$ and $\boldsymbol{A}^+$ by

$$\boldsymbol{A}^- = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots \\ 0 & a_0 & a_1 & \cdots \\ 0 & 0 & a_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}; \quad \boldsymbol{A}^+ = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots \\ a_1 & a_2 & a_3 & \cdots \\ a_2 & a_3 & a_4 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{2}$$

These matrices play an important role in the sequel. A matrix $A = (a_{ij})$ is called totally positive of order r (denote $A \in TP_r$) if each $p \times p$ minor, $p \leq r$, of A is non-negative. If each $p \times p$ minor, $p \leq r$, of A has the sign $(-1)^{p(p-1)/2}$, A is called sign reverse rule of order r ($A \in RR_r$). It is not hard to see that $a_m \in PF_r$ and $a_{m+n} \in TP_r$ in $m, n \geq 0$ if and only if $A^- \in TP_r$ and $A^+ \in TP_r$, respectively. Also $a_{m+n} \in RR_r$ in $m, n \geq 0$ if and only if $A^+ \in RR_r$. The statements in Proposition 1.1 are readily verified for sequences via A^- and A^+.

Remark 1.1. Let $g(x)$ be given as in Proposition 1.1.
(a) $g(x+y) \in TP_r$, $r \geq 2$, in $x, y \geq 0$ does not imply $g(x-y) \in RR_r$ in $-\infty < x, y < \infty$. To see this, consider a sequence $(a_n)_{n=0}^{\infty}$. Suppose $a_{m+n} \in TP_2$ in $m, n \geq 0$ so that $A^+ \in TP_2$. However, A^- can not be RR_2 except the trivial case that $a_n = 0$ for all n.
(b) $g(x+y) \in RR_r$, $r \geq 3$, in $x, y \geq 0$ does not imply $g(x) \in PF_r$. For, suppose $a_{m+n} \in RR_3$ in $m, n \geq 0$, i.e. $A^+ \in RR_3$. This does not imply $A^- \in TP_3$, since the non-negativity of the minor $\begin{vmatrix} a_1 & a_2 & a_3 \\ a_0 & a_1 & a_2 \\ 0 & a_0 & a_1 \end{vmatrix}$ is not guaranteed in general. Hence Corollary 3.7 of [2] is incorrect.
□

We next define important classes of distribution functions in applied probability by the aid of the total positivity, see [1,3,4]. Let X be a non-negative random variable having the distribution function $F(x)$. If $F(x)$ belongs to a class $\mathcal{H}$ of distributions, then we write either $X \in \mathcal{H}$ or $F \in \mathcal{H}$ for convenience. The survival function $\overline{F}(x)$ of $F(x)$ is defined by $\overline{F}(x) = 1 - F(x)$ for $x \geq 0$ and $\overline{F}(x) = 0$ for $x < 0$.

Definition 1.2.
(a) A distribution function $F(x)$ is said to be strongly unimodal (write $F \in SU$), if the density $f(x) = \frac{d}{dx}F(x)$ exists and $f(x) \in PF_2$.
(b) A distribution function $F(x)$ is said to be PF_2, IFR, or DFR ($F \in PF_2$, $F \in IFR$, or $F \in DFR$) if $F(x) \in PF_2$, $\overline{F}(x) \in PF_2$, or $\overline{F}(x+y) \in TP_2$ in $x, y \geq 0$, respectively.
(c) A distribution function $F(x)$ is said to have a completely monotone density $f(x)$ (write $F \in CM$) if $f(x+y) \in TP_\infty$ in $x, y \geq 0$.

Remark 1.2.
(a) $F \in SU$ implies both $F \in PF_2$ and $F \in IFR$. $F \in CM$ implies $F \in DFR$.
(b) If $F(x)$ has the density $f(x)$, the failure rate function $r_F(x) = f(x)/\overline{F}(x)$ is defined whenever $\overline{F}(x) > 0$. For this case, $F \in IFR$ if and only if $r_F(x)$ is increasing, and $F \in DFR$ if and only if $r_F(x)$ is decreasing (in this paper, "increasing" and "decreasing" are used in the weak sense). Also, $F \in PF_2$ if and only if $h_F(x) = f(x)/F(x)$ is decreasing

in x whenever $F(x) > 0$.
(c) The ordinary definition for $F \in CM$ is to require that $(-1)^n f^{(n)}(x) \geq 0$ for $x \geq 0$ and $n \geq 0$, where $f^{(n)}(x)$ denotes the nth derivative of $f(x)$ with $f^{(0)}(x) = f(x)$. In [4], it is shown that $F \in CM$ if and only if $f(x)$ is expressible as $f(x) = \int_0^\infty \mu e^{-\mu x} dG(\mu)$ for some distribution function $G(\mu)$. □

For a non-negative lattice random variable X, let $f_n = \Pr[X = n]$, $n \geq 0$. Define $F_n = \Pr[X \leq n] = \sum_{k=0}^n f_k$ and $\overline{F}_n = \Pr[X \geq n] = \sum_{k=n}^\infty f_k = 1 - F_{n-1}$, $n \geq 0$, with $F_{-1} = 0$. The lattice counterpart of the classes of distributions described in Definition 1.2 can be given using (f_n), (F_n) and $(\overline{F}_n)$ in an obvious manner, see also [1,6,12]. A further definition is given for lattice distributions.

Definition 1.3. The lattice distribution $(f_n)_{n=0}^\infty$ is called a Kaluza sequence (write $f_n \in KS$) if the corresponding $\boldsymbol{F}^+$ in (2) is TP_2.

Remark 1.3.
(a) By definition, it is obvious that $CM \subset KS \subset DFR$.
(b) For the lattice distribution $(f_n)_{n=0}^\infty$, the failure rate function is defined as $r_F(n) = f_n/\overline{F}_n$. The terms "$IFR$" and "$DFR$" are understood through $r_F(n)$ for lattice distributions.
(c) The content of Remark 1.2(c) holds also for the lattice case. Here the kth difference $\Delta^{(k)} f_n$ is used instead of the kth derivative $f^{(k)}(x)$. f_n can be expressed as a mixture of geometric distributions. □

Let X and Y be non-negative lattice random variables having the distributions $(a_n)_{n=0}^\infty$ and $(b_n)_{n=0}^\infty$ respectively. The ordinary stochastic ordering $\prec_d$ is defined by $X \prec_d Y$ if and only if $\sum_{k=n}^\infty a_k \leq \sum_{k=n}^\infty b_k$ for all $n \geq 0$. In Keilson and Kester [7], the following matrix is introduced to study the ordering $\prec_d$ for a Markov chain on $\{0, 1, \cdots\}$. Define $\boldsymbol{T} = (t_{ij})$ with $t_{ij} = 1$, $i \geq j$, and $t_{ij} = 0$, $i < j$, i.e.,

$$\boldsymbol{T} = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 1 & 1 & 0 & \cdots \\ 1 & 1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{3}$$

In the aid of $\boldsymbol{T}$, one easily sees that $X \prec_d Y$ if and only if $\boldsymbol{aT} \leq \boldsymbol{bT}$, where $\boldsymbol{a} = (a_0, a_1, \cdots)$

and $\boldsymbol{b} = (b_0, b_1, \cdots)$. Define the matrix $\boldsymbol{S}$ of the transpose of $\boldsymbol{T}$, i.e.

$$\boldsymbol{S} = \begin{pmatrix} 1 & 1 & 1 & \cdots \\ 0 & 1 & 1 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{4}$$

It is also easy to see that $X \prec_d Y$ if and only if $\boldsymbol{aS} \geq \boldsymbol{bS}$. Hence as far as concerned with the ordering $\prec_d$, $\boldsymbol{T}$ and $\boldsymbol{S}$ play the same role. The inverse matrices of $\boldsymbol{T}$ and $\boldsymbol{S}$ are obtained respectively as

$$\boldsymbol{T}^{-1} = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ -1 & 1 & 0 & \cdots \\ 0 & -1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \text{ and } \boldsymbol{S}^{-1} = \begin{pmatrix} 1 & -1 & 0 & \cdots \\ 0 & 1 & -1 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{5}$$

Keilson and Sumita [9] studied the uniform stochastic orderings $\prec_\ell$, $\prec_{(-)}$ and $\prec_{(+)}$. These orderings are defined in terms of $\boldsymbol{T}$ and $\boldsymbol{S}$. Here the difference between the two matrices gets significant.

Definition 1.4. Let X and Y be non-negative lattice random variables having the distributions $\boldsymbol{a} = (a_n)_{n=0}^{\infty}$ and $\boldsymbol{b} = (b_n)_{n=0}^{\infty}$ respectively.
(a) X and Y are ordered in the sense of likelihood ratio ordering (write $X \prec_\ell Y$) if $\begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{pmatrix} \in TP_2$.

(b) X is uniformly smaller than Y in the negative direction ($X \prec_{(-)} Y$) if $\begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{pmatrix} \boldsymbol{S} \in TP_2$.

(b) X is uniformly smaller than Y in the positive direction ($X \prec_{(+)} Y$) if $\begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{pmatrix} \boldsymbol{T} \in TP_2$.

It should be noted that $\prec_\ell$ is stronger than $\prec_{(-)}$ and $\prec_{(+)}$, i.e. $X \prec_\ell Y$ implies that both $X \prec_{(-)} Y$ and $X \prec_{(+)} Y$, and $\prec_d$ is weaker than the three orderings.

The next theorem plays a key role in the following developments. In the theorem, Part (a) is essentially shown in Lemma 2.3 of [13] and Part (b) is given in Lemma 2.1 of [14]. Part (c) seems new but can be proved in a similar manner to the proof of lemma 2.1 of [14] and the proof is omitted.

Theorem 1.1. Let $\boldsymbol{A}$ and $\boldsymbol{R}$ be non-negative matrices.
(a) Let $\boldsymbol{RS} \in TP_2$ and $\boldsymbol{S}^{-1}\boldsymbol{RS} \geq \boldsymbol{O}$, where $\boldsymbol{O}$ denotes the zero matrix. If $\boldsymbol{AS} \in TP_2$ then $\boldsymbol{ARS} \in TP_2$.
(b) Let $\boldsymbol{RT} \in TP_2$ and $\boldsymbol{T}^{-1}\boldsymbol{RT} \geq \boldsymbol{O}$. If $\boldsymbol{AT} \in TP_2$ then $\boldsymbol{ART} \in TP_2$.

(c) Under the same conditions as in (a), if $\boldsymbol{AS} \in RR_2$ then $\boldsymbol{ARS} \in RR_2$. If $\boldsymbol{AT} \in RR_2$ then $\boldsymbol{ART} \in RR_2$ under the conditions of (b).

§2. Uniformly Monotone Markov Chains and Their First Passage Times

Let $\{X_n;\ n \geq 0\}$ be a discrete time Markov chain on the state space $\mathcal{N} = \{0, 1, \cdots\}$ governed by the transition probability matrix $\boldsymbol{P}$. The state probability vector for X_n is designated by $\boldsymbol{\pi}_n = (\pi_n(k))_{k \in \mathcal{N}}$ where $\pi_n(k) = \Pr[X_n = k]$. If $X_n \prec_i X_{n+1}$ for all $n \geq 0$, the process $\{X_n;\ n \geq 0\}$ is called uniformly increasing in the sense of $\prec_i$, where $\prec_i$ is either $\prec_\ell$, $\prec_{(-)}$ or $\prec_{(+)}$. If in turn $X_{n+1} \prec_i X_n$ then it is called uniformly decreasing. The process is uniformly monotone when it is either uniformly increasing or decreasing. The aim of this section is to obtain sufficient conditions on the transition probability matrix $\boldsymbol{P}$ under which $\{X_n;\ n \geq 0\}$ is uniformly monotone. As we shall see later, uniformly monotone Markov chains have interesting first passage time properties.

Our first theorem shows that the TP_2-ness of $\boldsymbol{P}$, $\boldsymbol{PS}$ and $\boldsymbol{PT}$ preserves the uniform ordering $\prec_\ell$, $\prec_{(-)}$ and $\prec_{(+)}$, respectively.

Theorem 2.1. Suppose $X_0 = 0$ *a.s.*
(a) If $\boldsymbol{P} \in TP_2$ then $X_n \prec_\ell X_{n+1}$ for all $n \geq 0$.
(b) If $\boldsymbol{PS} \in TP_2$ then $X_n \prec_{(-)} X_{n+1}$ for all $n \geq 0$.
(c) If $\boldsymbol{PT} \in TP_2$ then $X_n \prec_{(+)} X_{n+1}$ for all $n \geq 0$.

Proof. Since $\boldsymbol{\pi}_0 = (1, 0, \cdots)$, one easily sees that $X_0 \prec_i X_1$ where $\prec_i = \prec_\ell$, $\prec_{(-)}$ and $\prec_{(+)}$. Hence Part (a) follows from Theorem 4.2 of [9]. For (b), suppose $X_{n-1} \prec_{(-)} X_n$, i.e., $\begin{pmatrix} \boldsymbol{\pi}_{n-1} \\ \boldsymbol{\pi}_n \end{pmatrix} \boldsymbol{S} \in TP_2$. Note that $\boldsymbol{Q} = (q_{ij}) = \boldsymbol{PS} \in TP_2$ implies $\boldsymbol{S}^{-1}\boldsymbol{Q} \geq \boldsymbol{O}$ since $q_{ij}q_{kn} - q_{kj}q_{in} \geq 0$ for all $i < k$ and $j < n$ so that $q_{ij} \geq q_{kj}$ as $n \to \infty$. Hence, from Theorem 1.1(a),

$$\begin{pmatrix} \boldsymbol{\pi}_n \\ \boldsymbol{\pi}_{n+1} \end{pmatrix} \boldsymbol{S} = \begin{pmatrix} \boldsymbol{\pi}_{n-1} \\ \boldsymbol{\pi}_n \end{pmatrix} \boldsymbol{S}\boldsymbol{S}^{-1}\boldsymbol{PS} \in TP_2.$$

It follows that $X_n \prec_{(-)} X_{n+1}$, proving (b). Statement (c) can be proved similarly. □

Let H be a non-empty subset of $\mathcal{N}$ and let $\{{}_HX_n;\ n \geq 0\}$ be a sequence of random variables defined as ${}_HX_n \stackrel{\mathrm{d}}{=} \{X_n \mid X_m \in H;\ 0 \leq m \leq n\}$, $n \geq 0$. Here $\stackrel{\mathrm{d}}{=}$ denotes equality in law. Let $\boldsymbol{P}_H$ be a substochastic matrix obtained by deleting the rows and columns corresponding to the states in $\mathcal{N} - H$. The distribution of ${}_HX_n$ is denoted by ${}_H\boldsymbol{\pi}_n = ({}_H\pi_n(k))_{k \in \mathcal{N}-H}$. The matrix $\boldsymbol{P}_H$ governs the sequence $\{{}_HX_n; n \geq 0\}$ in the following way. For a given ${}_H\boldsymbol{\pi}_n$, the distribution ${}_H\boldsymbol{\pi}_{n+1}$ is obtained by

$$ {}_H\boldsymbol{\pi}_{n+1} = \frac{{}_H\boldsymbol{\pi}_n \boldsymbol{P}_H}{e_n}; \qquad e_n = {}_H\boldsymbol{\pi}_n \boldsymbol{P}_H \boldsymbol{1}, \tag{6}$$

where $\mathbf{1} = (1, 1, \cdots)^T$ (T denotes the transpose).

Theorem 2.2. Let H be an arbitrary subset of $\mathcal{N}$.

(a) If $X_0 = \min\{i : i \in H\}$ *a.s.* then $\boldsymbol{P} \in TP_2$ implies ${}_HX_n \prec_{\ell H} X_{n+1}$ for all $n \geq 0$.

(b) If H has the maximal element, say C, i.e. $C \geq i$ for all $i \in H$, and if $X_0 = C$ *a.s.* then $\boldsymbol{P} \in TP_2$ implies ${}_HX_{n+1} \prec_{\ell H} X_n$ for all $n \geq 0$.

Proof. For (a), since ${}_H\boldsymbol{\pi}_0 = (1, 0, \cdots)$ one has $\begin{pmatrix} {}_H\boldsymbol{\pi}_0 \\ {}_H\boldsymbol{\pi}_1 \end{pmatrix} \in TP_2$. Suppose $\begin{pmatrix} {}_H\boldsymbol{\pi}_{n-1} \\ {}_H\boldsymbol{\pi}_n \end{pmatrix} \in TP_2$. Then from (6),

$$\begin{pmatrix} {}_H\boldsymbol{\pi}_n \\ {}_H\boldsymbol{\pi}_{n+1} \end{pmatrix} = \begin{pmatrix} e_{n-1} & 0 \\ 0 & e_n \end{pmatrix}^{-1} \begin{pmatrix} {}_H\boldsymbol{\pi}_{n-1} \\ {}_H\boldsymbol{\pi}_n \end{pmatrix} \boldsymbol{P}_H. \tag{7}$$

Since $\boldsymbol{P} \in TP_2$, so is $\boldsymbol{P}_H$. Hence $\begin{pmatrix} {}_H\boldsymbol{\pi}_n \\ {}_H\boldsymbol{\pi}_{n+1} \end{pmatrix} \in TP_2$. The inductive argument then proves Part (a). For Statement (b), we note that ${}_H\boldsymbol{\pi}_0 = (0, \cdots, 0, 1)$ so that $\begin{pmatrix} {}_H\boldsymbol{\pi}_0 \\ {}_H\boldsymbol{\pi}_1 \end{pmatrix} \in RR_2$. A similar argument to the proof for (a) then leads to $\begin{pmatrix} {}_H\boldsymbol{\pi}_n \\ {}_H\boldsymbol{\pi}_{n+1} \end{pmatrix} \in RR_2$ for all $n \geq 0$. Hence ${}_HX_{n+1} \prec_{\ell H} X_n$. □

For an arbitrary $H \subseteq \mathcal{N}$, $\boldsymbol{PT} \in TP_2$ does not in general imply either $\boldsymbol{P}_H\boldsymbol{T} \in TP_2$ or $\boldsymbol{T}^{-1}\boldsymbol{P}_H\boldsymbol{T} \geq \boldsymbol{O}$. For example, consider

$$\boldsymbol{A} = \begin{pmatrix} 0.4 & 0.3 & 0.3 \\ 0.3 & 0.2 & 0.5 \end{pmatrix}; \qquad \boldsymbol{AT} = \begin{pmatrix} 1 & 0.6 & 0.3 \\ 1 & 0.7 & 0.5 \end{pmatrix}.$$

Here, though $\boldsymbol{A}$ is not TP_2, $\boldsymbol{AT} \in TP_2$. If the third column of $\boldsymbol{A}$ is deleted, then the resulting $\boldsymbol{A}_3\boldsymbol{T}$ is not TP_2 and $\boldsymbol{T}^{-1}\boldsymbol{A}_3\boldsymbol{T}$ is not positive. A similar example is readily constructed for that $\boldsymbol{PS} \in TP_2$ implies neither $\boldsymbol{P}_H\boldsymbol{S} \in TP_2$ nor $\boldsymbol{S}^{-1}\boldsymbol{P}_H\boldsymbol{S} \geq \boldsymbol{O}$. Hence the same argument as in Theorem 2.2 can not be applied for the orderings $\prec_{(-)}$ and $\prec_{(+)}$. For a particular H, however, one has the next theorem.

Theorem 2.3.

(a) For $H = \{0, 1, \cdots, C\}$, $C \geq 1$, $\boldsymbol{PS} \in TP_2$ implies ${}_HX_n \prec_{(-)_H} X_{n+1}$ for all $n \geq 0$ provided that ${}_HX_0 = 0$ *a.s.* If ${}_HX_0 = C$ *a.s.* then $\boldsymbol{PS} \in TP_2$ implies ${}_HX_{n+1} \prec_{(-)_H} X_n$ for all $n \geq 0$.

(b) For $H = \{j, j+1, \cdots\}$, $j \geq 1$, if ${}_HX_0 = j$ *a.s.* then $\boldsymbol{PT} \in TP_2$ implies ${}_HX_n \prec_{(+)_H} X_{n+1}$ for all $n \geq 0$.

Proof. Let $H = \{0, 1, \cdots, C\}$ and let ${}_HX_0 = 0$ *a.s.* so that ${}_H\boldsymbol{\pi}_0 = (1, 0, \cdots, 0)$. This implies that $\begin{pmatrix} {}_H\boldsymbol{\pi}_0 \\ {}_H\boldsymbol{\pi}_1 \end{pmatrix} \boldsymbol{S} \in TP_2$. Suppose ${}_HX_{n-1} \prec_{(-)} {}_HX_n$, i.e. $\begin{pmatrix} {}_H\boldsymbol{\pi}_{n-1} \\ {}_H\boldsymbol{\pi}_n \end{pmatrix} \boldsymbol{S} \in TP_2$.

We will show that $\begin{pmatrix} {}_H\boldsymbol{\pi}_n \\ {}_H\boldsymbol{\pi}_{n+1} \end{pmatrix} \boldsymbol{S} \in TP_2$. For ${}_H\boldsymbol{\pi}_n$, one has from (6) that $({}_H\boldsymbol{\pi}_n, \mathbf{o})\boldsymbol{P} = (e_n {}_H\boldsymbol{\pi}_{n+1}, \boldsymbol{a}_n)$ for some $\boldsymbol{a}_n$, where $\mathbf{o} = (0, 0, \cdots)$. Hence

$$\begin{pmatrix} e_{n-1} {}_H\boldsymbol{\pi}_n & \boldsymbol{a}_{n-1} \\ e_n {}_H\boldsymbol{\pi}_{n+1} & \boldsymbol{a}_n \end{pmatrix} \boldsymbol{S} = \begin{pmatrix} {}_H\boldsymbol{\pi}_{n-1} & \mathbf{o} \\ {}_H\boldsymbol{\pi}_n & \mathbf{o} \end{pmatrix} \boldsymbol{S}\boldsymbol{S}^{-1}\boldsymbol{P}\boldsymbol{S}. \tag{8}$$

$\begin{pmatrix} {}_H\boldsymbol{\pi}_{n-1} \\ {}_H\boldsymbol{\pi}_n \end{pmatrix} \boldsymbol{S} \in TP_2$ implies $\begin{pmatrix} {}_H\boldsymbol{\pi}_{n-1} & \mathbf{o} \\ {}_H\boldsymbol{\pi}_n & \mathbf{o} \end{pmatrix} \boldsymbol{S} \in TP_2$. Thus, from Theorem 1.1(a), the left hand side of (8) is TP_2 so that $\begin{pmatrix} e_{n-1} {}_H\boldsymbol{\pi}_n \\ e_n {}_H\boldsymbol{\pi}_{n+1} \end{pmatrix} \boldsymbol{S} \in TP_2$, which in turn implies the desired conclusion. If ${}_H X_0 = C$ *a.s.*, it is readily seen that $\begin{pmatrix} {}_H\boldsymbol{\pi}_0 \\ {}_H\boldsymbol{\pi}_1 \end{pmatrix} \boldsymbol{S} \in RR_2$. Suppose $\begin{pmatrix} {}_H\boldsymbol{\pi}_{n-1} \\ {}_H\boldsymbol{\pi}_n \end{pmatrix} \boldsymbol{S} \in RR_2$. From (8) and using Theorem 1.1(c), one has $\begin{pmatrix} {}_H\boldsymbol{\pi}_n \\ {}_H\boldsymbol{\pi}_{n+1} \end{pmatrix} \boldsymbol{S} \in RR_2$. Thus ${}_H X_{n+1} \prec_{(-)} {}_H X_n$, proving Part (a).

Next let $H = \{j, j+1, \cdots\}$ and suppose ${}_H X_0 = j$ *a.s.* Consider $(\mathbf{o}, {}_H\boldsymbol{\pi}_n)\boldsymbol{P} = (\boldsymbol{a}_n, e_n {}_H\boldsymbol{\pi}_{n+1})$ for some $\boldsymbol{a}_n$. Here $\mathbf{o}$ represents the zero vector $(0, \cdots, 0)$. Then

$$\begin{pmatrix} \boldsymbol{a}_{n-1} & e_{n-1} {}_H\boldsymbol{\pi}_n \\ \boldsymbol{a}_n & e_n {}_H\boldsymbol{\pi}_{n+1} \end{pmatrix} \boldsymbol{T} = \begin{pmatrix} \mathbf{o} & {}_H\boldsymbol{\pi}_{n-1} \\ \mathbf{o} & {}_H\boldsymbol{\pi}_n \end{pmatrix} \boldsymbol{T}\boldsymbol{T}^{-1}\boldsymbol{P}\boldsymbol{T}. \tag{9}$$

Following the same arguments as in the proof for (a), one can show that ${}_H X_n \prec_{(+)} {}_H X_{n+1}$, $n \geq 0$. This completes the proof of the theorem. □

The following three first passage times of the Markov chain $\{X_n\}$ are of interest to us. Define

$$T_0 = \inf\{n \geq 1 : X_n = 0 \mid X_0 = 0\}, \tag{10}$$

$$T_C = \inf\{n \geq 1 : X_n > C \mid X_0 = 0\}, \quad C \geq 0, \tag{11}$$

and

$$T_C^+ = \inf\{n \geq 1 : X_n > C \mid X_0 = C\}, \quad C \geq 0. \tag{12}$$

Theorem 2.4.
(a) If $\boldsymbol{P} \in TP_2$ then T_0, $T_C^+ \in KS$ and $T_C \in SU$.
(b) If $\boldsymbol{PS} \in TP_2$ then $T_C \in IFR$ and $T_C^+ \in DFR$.
(c) If $\boldsymbol{PT} \in TP_2$ then $T_0 \in DFR$.

Proof. Statement (a) follows from Karlin [4]. (c) has been proved in Shanthikumar [14]. For (b), let $H = \{0, 1, \cdots, C\}$ and $\{{}_H X_n\}$ be the sequence of random variables obtained from $\{X_n\}$ as before. Let $r(n)$ be the failure rate function of T_C, i.e.,

$$r(n) = \Pr[T_C = n \mid T_C \geq n] = E\left[\sum_{k \geq C+1} p({}_HX_{n-1}, k)\right] \tag{13}$$

where $\boldsymbol{P} = (p(i,j))$. Let $\eta_n(k) = \Pr[{}_HX_n \leq k]$ and write $\boldsymbol{\eta}_n = (\eta_n(0), \cdots, \eta_n(C))$. It is not hard to see from (13) that

$$r(n+1) - r(n) = \sum_{i=0}^{C}(q_{iC} - q_{i+1,C})(\eta_{n-1}(i) - \eta_n(i)) \tag{14}$$

where $\boldsymbol{Q} = (q_{ij}) = \boldsymbol{PS}$. Here $\eta_{n-1}(C) = \eta_n(C) = 1$ is used. From Theorem 2.3(a), one sees that ${}_HX_{n-1} \prec_{(-)} {}_HX_n$, provided that ${}_HX_0 = 0$ *a.s.* This implies that ${}_HX_{n-1} \prec_d {}_HX_n$ so that $\boldsymbol{\eta}_{n-1} \geq \boldsymbol{\eta}_n$, see [9]. On the other hand, $\boldsymbol{Q} \in TP_2$ implies that $q_{iC} \geq q_{kC}$ for $i < k$. Thus $r(n)$ is increasing in n (see also Shaked and Shanthikumar [13]). To prove $T_C^+ \in DFR$, let $r^+(n)$ be the failure rate function of T_C^+. It is evident that $r^+(n+1) - r^+(n)$ has the same expression as the right hand side of (14). The only difference is that ${}_HX_0 = C$ *a.s.*, which in turn yields ${}_HX_n \prec_{(-)} {}_HX_{n-1}$ so that $\boldsymbol{\eta}_{n-1} \leq \boldsymbol{\eta}_n$. Hence $r^+(n)$ is decreasing, completing the proof. □

Remark 2.1. From (14), it is sufficient for $T_C \in IFR$ that

$$q_{iC} \text{ decreases in } i \text{ and } {}_HX_{n-1} \prec_d {}_HX_n. \tag{15}$$

As we have already seen, even though the original process $\{X_n\}$ is stochastically increasing, the sequence of random variables $\{{}_HX_n\}$ needs not be so. The condition in (b) of Theorem 2.4 simply ensures (15). □

§3. Embedded Markov Chains Associated with M/G/1 and GI/M/1 Queues

In this section, we derive a condition for N_k, the number of customers in an M/G/1 queue left by the kth departing customer, to be uniformly monotone. For a GI/M/1 queue, N_k is defined as the number of customers seen by the kth arrival. A simple condition for uniform monotonicity of this case is also obtained.

Let $p_\lambda(n,t)$ be the Poisson kernel defined by

$$p_\lambda(n,t) = \begin{cases} \lambda\frac{(\lambda t)^{n-1}}{(n-1)!}\exp\{-\lambda t\}, & n = 1, 2, \cdots, \\ \delta(t), & n = 0, \end{cases} \tag{16}$$

for λ, $t \geq 0$. Here $\delta(t)$ is the delta function, meaning that $\delta(t) = 0$ for $t > 0$ and $\int_0^\infty \delta(t)dt = 1$. Consider then the integral transformation

$$g(n) = \int_0^\infty p_\lambda(n,t)f(t)dt, \qquad n \geq 0 \tag{17}$$

for $f(t)$ defined on $[0,\infty)$. We first see properties of the integral kernel $p_\lambda(n,t)$ and the transformation (17).

Proposition 3.1.
(a) $p_\lambda(n,t)$ is TP_∞ in $n \geq 0$ and $t \geq 0$.
(b) $p_\lambda(m,t) * p_\lambda(n,t) = p_\lambda(m+n,t)$, $m,n \geq 0$, where $*$ denotes the convolution with respect to t.
(c) $f(x+y) \in SR_r$ in $x,y \geq 0$ if and only if $g(m+n) \in SR_r$ in $m,n \geq 0$ having the same sign as for $f(x+y)$.

Proof. Let $\delta_\epsilon(t) = \epsilon e^{-\epsilon t}$ so that $\delta_\epsilon(t) \to \delta(t)$ as $\epsilon \to \infty$. It is obvious that $p_\lambda(n,t) \in TP_\infty$ in $n \geq 1$ and $t \geq 0$ (see Karlin [4]). It is aslo easy to see that $p_\lambda^\epsilon(n,t)$ defined by $p_\lambda(n,t)$ for $n \geq 1$ and $\delta_\epsilon(t)$ for $n=0$ is TP_∞ in $n \geq 0$ and $t \geq 0$ for sufficiently large ϵ. Since the total positivity is preserved under limiting operations, one has Part (a). Statement (b) is immediately derived by taking the Laplace transform on the both sides. (c) is obtained by applying Theorem 5.4 in p.130 of [4] (see also Theorem 3.6 of [2]). □

For a distribution function $A(x)$ with a positive support, define the three vectors $\boldsymbol{a}(\lambda) = (a_n(\lambda))$, $\boldsymbol{a}^-(\lambda) = (a_n^-(\lambda))$ and $\boldsymbol{a}^+(\lambda) = (a_n^+(\lambda))$ by

$$a_n(\lambda) = \frac{1}{\lambda}\int_0^\infty p_\lambda(n+1,t)dA(t), \quad n \geq 0, \tag{18}$$

$$a_n^-(\lambda) = \frac{1}{\lambda}\int_0^\infty p_\lambda(n+1,t)A(t)dt, \quad n \geq 0, \tag{19}$$

and

$$a_n^+(\lambda) = \int_0^\infty p_\lambda(n,t)\overline{A}(t)dt, \quad n \geq 0, \tag{20}$$

respectively. When the meaning is clear, the parameter λ is dropped in the above notation. It should be noted that, if a distribution function belongs to IFR or PF_2, it necessarily has a positive support. It is readily checked that

$$a_n^- = \frac{1}{\lambda}\sum_{k=0}^n a_k, \quad n \geq 0; \qquad \boldsymbol{a}^- = \frac{1}{\lambda}\boldsymbol{a}\boldsymbol{S}, \tag{21}$$

and

$$a_n^+ = \sum_{k=n}^\infty a_k, \quad n \geq 0; \qquad \boldsymbol{a}^+ = \boldsymbol{a}\boldsymbol{T}. \tag{22}$$

Let λ be the arrival rate in an M/G/1 queue and $A(x)$ be the service time distribution function. Let $N(t)$ be the number of customers in the system at time t and let τ_k be the departure epoch of the kth customer from the system. It is well known that the embedded process $N_k = N(\tau_k+)$ constitutes a Markov chain on $\mathcal{N}$. For a GI/M/1 queue, λ is considered as the service rate and $A(x)$ the inter-arrival time distribution. The process

$N_k = N(\tau_k-)$ for the GI/M/1 queue also forms a Markov chain on $\mathcal{N}$ where τ_k is in turn the arrival epoch of the kth customer at the system. Define the matrices

$$\boldsymbol{U} = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots \\ a_0 & a_1 & a_2 & \cdots \\ 0 & a_0 & a_1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \text{ and } \boldsymbol{L} = \begin{pmatrix} a_1^+ & a_0 & 0 & \cdots \\ a_2^+ & a_1 & a_0 & \cdots \\ a_3^+ & a_2 & a_1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{23}$$

It is obvious that the matrix $\boldsymbol{U}$ governs the Markov process $\{N_k\}$ associated with the M/G/1 queue whereas $\boldsymbol{L}$ governs that of the GI/M/1 queue.

Lemma 3.1.
(a) If $A \in SU$ then both $\boldsymbol{U}$ and $\boldsymbol{L}$ are TP_2.
(b) If either $A \in PF_2$ or $A \in IFR$ with a_n decreasing in n then $\boldsymbol{US}$, $\boldsymbol{LT} \in TP_2$.
(c) If $A \in IFR$ then $\boldsymbol{UT}$, $\boldsymbol{LS} \in TP_2$.

Proof. If $a(x) = \frac{d}{dx}A(x)$ exists and $a(x) \in PF_2$ then $a(x+y) \in RR_2$ in $x, y \geq 0$ from Proposition 1.1(a). Proposition 3.1(c) then shows that $a_{m+n} \in RR_2$ in $m, n \geq 0$ so that $a_n \in PF_2$. Hence, noting that $\boldsymbol{A}^-$ in (2) is TP_2, $\boldsymbol{U}$, $\boldsymbol{L} \in TP_2$. Similarly, Part (c) can be proved by noting the relation (22). For Part (b), let $A \in IFR$. Then $a_n^+ \in PF_2$. This is equivalent to saying that $0 \leq (a_n^+)^2 - a_{n-1}^+ a_{n+1}^+ = (a_{n-1}^-)^2 - a_{n-2}^- a_n^- + \frac{1}{\lambda}(a_n - a_{n-1})$ from (21). If $a_n \leq a_{n-1}$ then $(a_{n-1}^-)^2 \geq a_{n-2}^- a_n^-$ must hold. Since the TP_2 property is a local one (see Theorem 2.2 of [8]), one has $a_n^- \in PF_2$. $A \in PF_2$ evidently implies $a_n^- \in PF_2$. The conclusions in (b) are now immediate. □

It should be noted that the stability conditions $\rho = \lambda \int_0^\infty \overline{A}(x)dx < 1$ for the M/G/1 queue and $\rho^{-1} < 1$ for the GI/M/1 queue respectively are not required in Lemma 3.1. Accordingly, for any ρ, one has the following theorem.

Theorem 3.1. Suppose $N_0 = 0$ *a.s.*
(a) If $A \in SU$ then $N_k \prec_\ell N_{k+1}$ for all $k \geq 0$ in both M/G/1 and GI/M/1 queues.
(b) If either $A \in PF_2$ or $A \in IFR$ with a_n decreasing in n then $N_k \prec_{(-)} N_{k+1}$ for all $k \geq 0$ in M/G/1 queues and $N_k \prec_{(+)} N_{k+1}$ for all $k \geq 0$ in GI/M/1 queues.
(c) If $A \in IFR$ then $N_k \prec_{(+)} N_{k+1}$ for all $k \geq 0$ in M/G/1 queues and $N_k \prec_{(-)} N_{k+1}$ for all $k \geq 1$ in GI/M/1 queues.

Proof. The theorem is immediate by combining Theorem 2.1 and Lemma 3.1. □

For a GI/M/1 queue, let $\rho^{-1} < 1$ so that $N_k \to N_\infty$ in distribution. It is well known that N_∞ is geometrically distributed for any inter-arrival time distribution.

Consider next an M/G/1 queue. For the stability condition, we require $\rho < 1$. Then, $N_k \to N_\infty$ in distribution and $N_\infty = V + K$ where V and K are independent of each other and V is the number of arrivals during a service. Let $\gamma(z) = \sum_{n=0}^{\infty} \Pr[K = n]z^n$. $\gamma(z)$ is given by

$$\gamma(z) = \frac{(1-\rho)(1-z)}{\alpha(\lambda - \lambda z) - z} = (1-\rho)\left[1 - \rho\frac{1-\alpha(\lambda-\lambda z)}{\lambda(1-z)ES}\right]^{-1} \tag{24}$$

where $\alpha(s) = \int_0^\infty e^{-sx}dA(x)$ and $ES = \int_0^\infty \overline{A}(x)dx$. The Laplace-Stieltjes transform of the residual lifetime distribution $A_R(x)$ of $A(x)$ is given by $(1-\alpha(s))/sES$. Define

$$a_n^R = \frac{1}{\lambda}\int_0^\infty p_\lambda(n+1, x)dA_R(x), \quad n \geq 0, \tag{25}$$

as in (18). It is readily seen that

$$\sum_{n=0}^{\infty} a_n^R z^n = \frac{1-\alpha(\lambda-\lambda z)}{(\lambda-\lambda z)ES}. \tag{26}$$

Let K_m^R, $m \geq 1$, be independently and identically distributed by $(a_n^R)_0^\infty$ and let L be geometrically distributed with the parameter ρ. Assume that $\{K_m^R\}$ and L are independent. Equation (24) then implies that $K \stackrel{\mathrm{d}}{=} \sum_{m=1}^{L} K_m^R$.

Theorem 3.2. Let K be defined as above for the M/G/1 queue.

(a) If $A \in IMRL$, i.e. its residual lifetime distribution $A_R(x)$ is DFR, then $K \in DFR$.

(b) If $A \in CM$ then $K_{|K>0} \in CM$.

Proof. Part (a) follows from Theorem 3.5 of [14]. Next, let $A \in CM$. Then $A_R(x)$ is also CM. If we write the Laplace-Stieltjes transform of $A_R(x)$ by $\alpha_R(s)$ and $\beta(z) = \alpha_R(\lambda - \lambda z)$ then

$$\gamma(z) = \frac{1-\rho}{1-\rho\beta(z)} = 1 - \rho + \frac{\rho(1-\rho)\beta(z)}{1-\rho\beta(z)}.$$

Since $\alpha_R(s)$ is of the form $\int_0^\infty \frac{\mu}{\mu+s}d\hat{G}(\mu)$ for some distribution function $\hat{G}(\mu)$ on $[0,\infty)$, $\beta(z) = \int_0^1 \frac{1-\theta}{1-\theta z}dG(\theta)$ for some distribution $G(\theta)$ on $(0,1)$, where $\theta = \frac{\lambda}{\lambda+\mu} < 1$ ($A(0+) = 0$ implies that $\hat{G}(0+) = 0$). Note that $\beta(z)$ is the generating function of a completely monotone sequence if and only if $\beta(z)$ has simple poles located at $(1,\infty)$ and $\beta(z)$ is increasing in z between these poles. A similar argument as in Theorem 3.1 of [5] then proves (b), completing the proof. □

§4. The Number Served during a Busy Period

Let R be the number served during a busy period for a queue and denote $g_n = \Pr[R = n]$, $n \geq 1$. Define the generating function of R by $\sigma(z) = \sum_{n=1}^{\infty} g_n z^n$. For the M/G/1 queue having the arrival rate λ and the service time distribution $A(x)$, it is well known that $\sigma(z) = z\alpha(\lambda - \lambda\sigma(z))$ where $\alpha(s) = \int_0^\infty e^{-sx}dA(x)$.

Theorem 4.1. For an M/G/1 queue, if $A \in CM$ then $R \in CM$.

Proof. The theorem can be shown by mimicking the arguments in the proof of Theorem 2.2 in Keilson [5]. □

Consider the Markov chain having the transition matrix of either $\boldsymbol{U}$ or $\boldsymbol{L}$ in (23). Here we require the stability condition to ensure that the Markov chain is positive recurrent. It is easy to see that T_0 defined in (10) for this Markov chain represents the number of customers served during a busy period. Combining the results in Theorem 2.4 and Lemma 3.1, one has:

Theorem 4.2.
(a) If $A \in SU$ then $R \in KS$ for both M/G/1 and GI/M/1 queues.
(b) If either $A \in PF_2$ or $A \in IFR$ with a_n decreasing in n then $R \in DFR$ for GI/M/1 queues.
(c) If $A \in IFR$ then $R \in DFR$ for M/G/1 queues.

We note that (c) of Theorem 4.2 is proved in Shanthikumar [14].

In Kijima [10], the Markov transition matrix $\boldsymbol{A} = (a_{ij})$ such that $a_{ij} = 0$ for $i > j+1$ or $j > i+1$ is called a semi-triangular matrix and first passage times of the associated Markov chain are studied. Because of the skip-free nature of the Markov chain governed by such matrices, more about the first passage time T_0 may be derived. For example, let $H = \{1, 2, \cdots\}$ and let $\boldsymbol{\pi}_n = (\pi_n(1), \pi_n(2), \cdots)$ be defined by

$$\boldsymbol{\pi}_{n+1} = \boldsymbol{\pi}_n \boldsymbol{U}_H, \quad n \geq 0, \tag{27}$$

starting with $\boldsymbol{\pi}_0 = (1, 0, \cdots)$. It is then easy to see that

$$g_n = \Pr[X_n = 0;\ X_m \geq 1,\ 1 \leq m \leq n-1 \mid X_0 = 0] = a_0 \pi_{n-1}(1), \tag{28}$$

where $g_n = \Pr[T_0 = n]$, $n \geq 1$. Let $e_n = \boldsymbol{\pi}_n \mathbf{1}$, $n \geq 0$, so that $e_n = \boldsymbol{\pi}_0 \boldsymbol{U}_H^n \mathbf{1}$. Since $(0, \boldsymbol{\pi}_n)\boldsymbol{U} = (a_0 \pi_n(1), \boldsymbol{\pi}_{n+1})$, one has

$$g_{n+1} = e_n - e_{n+1}. \tag{29}$$

Note that $e_n \to 0$ as $n \to \infty$ since $\boldsymbol{U}$ governs a positive recurrent Markov chain. Let $r_0(n) = \Pr[T_0 = n \mid T_0 \geq n]$. It then follows from (29) that

$$r_0(n) = 1 - \frac{e_n}{e_{n-1}}; \quad r_0(1) = a_0. \tag{30}$$

Hence $r_0(n)$ is decreasing in n if and only if $e_n \in KS$.

For a matrix $\boldsymbol{P} = (p_{ij})_{i,j=1}^{\infty,\infty}$, define the $m \times n$ submatrix $[\boldsymbol{P}]_{mn}$ of $\boldsymbol{P}$ by $[\boldsymbol{P}]_{mn} = (p_{ij})_{i=1,j=1}^{m,n}$. Consider the matrix $[\boldsymbol{U}_H]_{n+2,n+1}$ and suppose $[\boldsymbol{U}_H]_{n+2,n+1} \in TP_2$. It is easy to see that this is equivalent to that a_k/a_{k-1} decreases in $1 \le k \le n+1$. Since $\boldsymbol{\pi}_0 = (1, 0, \cdots)$, $\left[\begin{pmatrix} \boldsymbol{\pi}_0 \\ \boldsymbol{\pi}_1 \end{pmatrix}\right]_{2,n+2} \in TP_2$. From (27) and the upper semi-triangularity of $\boldsymbol{U}_H$, it then follows that

$$\left[\begin{pmatrix} \boldsymbol{\pi}_1 \\ \boldsymbol{\pi}_2 \end{pmatrix}\right]_{2,n+1} = \left[\begin{pmatrix} \boldsymbol{\pi}_0 \\ \boldsymbol{\pi}_1 \end{pmatrix}\right]_{2,n+2} [\boldsymbol{U}_H]_{n+2,n+1} \in TP_2. \tag{31}$$

Repeating the argument, one finally has $\left[\begin{pmatrix} \boldsymbol{\pi}_n \\ \boldsymbol{\pi}_{n+1} \end{pmatrix}\right]_{22} \in TP_2$. Equivalently,

$$\pi_n(1)/\pi_{n+1}(1) \ge \pi_n(2)/\pi_{n+1}(2).$$

From (27) and (28), $g_{n+2} = a_1 g_{n+1} + a_0^2 \pi_n(2)$. Hence

$$\frac{g_{n+1}}{g_{n+2}} \ge \frac{g_{n+2} - a_1 g_{n+1}}{g_{n+3} - a_1 g_{n+2}} \text{ and } \frac{g_{n+1}}{g_{n+2}} \ge \frac{g_{n+2}}{g_{n+3}}. \tag{32}$$

Since $[\boldsymbol{U}_H]_{k+2,k+1} \in TP_2$, $1 \le k \le n$, it can be proved that g_{k+1}/g_k increases in $2 \le k \le n+2$. Note that $g_2 = a_0 a_1$ and $g_1 = a_0$ so that $g_2 - a_1 g_1 = 0$. Thus (32) holds also for $n = 0$. In summary, if a_k/a_{k-1} decreases in $1 \le k \le n$ then $\Pr[T_0 = k+1]/\Pr[T_0 = k]$ increases in $1 \le k \le n+1$.

§5. Queues with Finite Waiting Capacity

In this section, we consider an M/G/1 queue with finite waiting capacity, say C. That is, the maximum number of customers in the system including the customer being served is $(C+1)$. Customers who arrive when the system is full are lost. Of interest for this system are the number served until a customer is first lost when the system starts with no customer (denote L_0), and that of when it starts from being full (L). Note that $L_0 = T_C - 1$ and $L = T_C^+ - 1$, where T_C and T_C^+ are defined in (11) and (12) respectively. As for Theorem 4.2, one easily has, from Theorem 2.4 and Lemma 3.1:

Theorem 5.1. For an M/G/1 queue with finite waiting capacity,
(a) If $A \in SU$ then $L_0 \in SU$ and $L \in KS$.
(b) If either $A \in PF_2$ or $A \in IFR$ with a_n decreasing in n, then $L_0 \in IFR$ and $L \in DFR$.

Other quantities of interest are e.g. the number served during a busy period given that no customers are lost (${}^{C+1}T_{10}$), the number served until a customer is first lost in a busy period (${}^{0}T_{1,C+1}$), etc. Here ${}^{k}T_{ij}$, $k < i < j$, denotes the upward conditional passage time from i to j and ${}^{j}T_{ik}$ is the downward conditional passage time from i to k given no visits to $\mathcal{N} - \{k+1, \cdots, j-1\}$. Symmetry among these conditional passage times has been discussed in Kijima [11]. Distribution properties of such quantities as described above can be studied using the results in [10,11].

For a GI/M/1 queue with waiting capacity $(C-1)$, similar results to above hold, if variables are defined in terms of arrivals. Moreover, let $H = \{0,1,\cdots,C\}$ and let $\sigma_0(z) = \sum_{n=1}^{\infty} \Pr[L_0 = n]z^n$. From Theorem 1.5 of Kijima [10], it can be readily seen that, for a skip-free positive Markov chain,

$$\sigma_0(z) = \prod_{j=0}^{C} \frac{1-u_j}{1-u_j z}, \tag{33}$$

where u_j are the eigenvalues of the strictly substochastic matrix $\boldsymbol{L}_H$ so that $|u_j| < 1$. This means that if u_j are all real and positive, L_0 is represented as a sum of independent geometric random variables, which is PF_∞. A sufficient condition for all u_j to be real and positive is that $\boldsymbol{L}_H \in TP_\infty$. For, if so, one has $L_0 \in PF_\infty$ whose associated generating function is as in (33), see Karlin [4]. However, $\boldsymbol{L}_H \in TP_\infty$ is not necessary. Indeed, one can easily construct such lower semi-triangular matrices that are not TP_∞ but have positive eigenvalues only.

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