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OPTIMAL STOPPING PROBLEM WITH UNCERTAIN RECALL

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Abstract Consider a discrete-time optimal stopping problem with a finite planning horizon in which an offer passed up $j \geqq 0$ periods ago becomes unavailable at the next time with a known probability p_j, provided that it remains available at present. The objective is to maximize the expected discounted gain where the term gain means the value of the offer accepted less the total search cost paid up to the termination of the process with its acceptance. The main results obtained are the next four. (1) Let a and b be, respectively, the lower bound and the upper bound in the distribution of offer w. Then the optimal stopping rule has the following property. For at least one set consisting of past offers available at present, there exists the following two critical numbers ξ and ξ' such as $a < \xi < \xi' < b$. For a given present offer w, if $\xi \leqq w \leqq \xi'$, pass up it and continue the search; otherwise, stop the search with accepting the best of offers available at present. The necessary and sufficient condition for the optimal stopping rule to have the property is $\beta\mu - c > a$ where β, μ, and c are, respectively, a discount factor, an expectation of offer w, and a cost per search. The property is called a double reservation value property or DRV-property for short. (2) The property gradually disappears as a planning horizon tends to infinity, with totally vanishing in its limit. (3) In the limit of a planning horizon, it suffices to memorize only the present offer with neglecting all past offers; in other words, the problem is eventually reduced to an infinite horizon optimal stopping problem with no recall. (4) Under the optimal stopping rule, each of the maximum expected discounted gain attained, the expected number of searches made, and the expectation of the offer accepted is less than or equal to one in an optimal stopping problem with recall, and both become the same in the limit of a planning horizon.

1. Introduction

In almost all models of an optimal stopping problem presented so far [1-15], the assumption has been made that an offer once inspected and passed up either becomes forever unavailable or remains forever available. The former case is called *with-no-recall,* the ratter *with-recall.* In applying these models to actual economic or managerial decision problems, however, it is rather realistic to postulate that the future availability of an offer once passed up is uncertain. An optimal stopping problem defined on such an assumption is said to be *with-uncertain-recall* [4],[6]. Here let us show three examples of with-uncertain-recall; the auxiliary verbs in italics in the sentences below imply the uncertainty of recall.

a. House purchasing problem

Suppose you are searching for a house in which you live, and suppose you have just found one for sale. If it does not seem to be desirable enough, you will avoid a ready answer about whether to purchase it or not and continue the search in the attempt to find a more desirable one. Such suspension, however, will involve the risk that, even if you want to buy it later on, it *will* have already been purchased by any other person; what is worse, the misfortune may follow that more desirable ones than it will not appear within the remaining planning horizon. Taking such risk into consideration, you must decide either to stop the search with accepting one of the most desirable of houses which were founded so far and are available at present or to continue the search in the attempt to find a more desirable one.

b. Job search problem

Suppose you, unemployed at present, have just now received an employment notice from one of the companies to which you applied for a position. Then, if you postpone your decision of whether to join the company or not, it *will* employ other applicant in place of you. In such a situation, as being concerned about the possibility of missing the presently available employment opportunity on the one hand and as expecting the possibility of encountering more desirable employment opportunities afterward on the other, you must decide a company to join out of ones sending you employment notices one after the other before your unemployment insurance expires.

c. R&D problem

In a manufacturing company, it is a crucial management problem to decide which product to be marketed as a new product among ones developed so far by its R&D department before other companies *will* put products on the idea similar to or identical with it on the market.

2. Model

Consider the following version of the standard discrete-time stopping problem with a finite planning horizon [13]. First, for convenience, let points in time be numbered backward from the final point in time of the horizon, termed *time 0*, as time 0, time 1, ⋯ and so on. If some fixed cost $c \geqq 0$, *search cost,* is paid over a period (interval between two successive times), then you can get an offer, say such as wanting to buy one of your assets, at the end of the period. Each offer has a value; values of successive offers, w, w', ⋯, are assumed to be independent identically distributed random variables with a known continuous distribution function $F(w)$ having a finite expectation μ; for given numbers a and b with $0 < a < b$ ($< \infty$), let $F(w) = 0$ for $w \leq a$, $0 < F(w) < 1$ for $a < w < b$, and $F(w) = 1$ for $b \leq w$ (so $a < \mu < b$). Let p_j, $j = 0,1,\cdots$, be the probability that an offer inspected and passed up j periods ago becomes unavailable at the next time, provided that it remains available at present (so p_0 is the probability that an offer made at present becomes unavailable at the next time). Postulate that there exists a fixed non-negative integer N such that $0 < p_j < 1$ for $j < N$ and $p_j = 1$ for $j \geq N$; in other words, every offer has at most N periods of age. Assume that, for any past offer, it can be known instantly without paying any additional cost whether it is available at present or not. Finally, throughout the paper, let

$$(2.1) \qquad \beta\mu - c > 0$$

where $\beta (0 < \beta \leq 1)$ represents a per-period discount factor. This is the assumption made in order not to render the problem meaningless, implying that the present value of the expectation of an offer w obtained by an additional search, $\beta\mu$, always makes up for its search cost c.

Below, by the term *expected discounted gain*, we shall mean the expectation of the present value of the offer w accepted less the present value of the total search cost paid over all times up to the termination of the process with its acceptance. The *objective* in the model is to find the optimal stopping rule, maximizing the expected discounted gain, provided that an offer must be accepted up to time 0. We shall call the problem an *optimal stopping problem with uncertain recall*. Here note that the optimal stopping problem with no recall is the special case with $N = 0$ of the optimal stopping problem with uncertain recall defined above. The optimal stopping problem with recall, however, is not so; it is the case of $N \to \infty$ in $p_j = 0$ for $N > j \geq 0$.

The pioneering works on this subject were made by Landsberger and Peled [6] and by Karni and Schwartz [4]. In the former, it is assumed that the best offer available at present becomes unavailable at the next time with a known probability, with all offers except the best being neglected. They pointed out that the model in which the probability of future unavailability is defined for every past offer will become remarkably intractable in its mathematical treatment. The latter literature dealt with the case with such difficulties, succeeding in drawing some interesting conclusions, however, on some rather severe assumptions: the probability is strictly decreasing in age, the search cost is strictly increasing with the number of searches, and so on.

In Section 3, the fundamental equations of the model is given, and Section 4 exemplifies the structure of the optimal stopping rule by using the simple case of $N = 1$. In Section 5, the conclusions deduced analytically in Section 6 that follows are summarized with some considerations and numerical examples.

3. Functional Equation

Suppose the search process starts from time t, and let w_j denote an offer of time

$t+j$ (j periods ago). If the offer w_j is available at present (time t), let $k_j = w_j$; otherwise, $k_j = 0$, so always $k_0 = w_0$. Without loss in generality, the k_j can be defined on $0 \leq k_j \leq b$ where $k_j = 0$ means that offer w_j has already been unavailable. Let $K = (k_0, k_1, \cdots, k_N)$ and $G = (k_1, k_2, \cdots, k_N)$, vectors. For the convenience of later discussions, we shall define vectors K and G at time $t+1$ (previous time) by, respectively, $K'' = (k''_0, k''_1, \cdots, k''_N)$ and $G'' = (k''_1, k''_2, \cdots, k''_N)$ and those at time $t-1$ (next time) by, respectively, $K' = (k'_0, k'_1, \cdots, k'_N)$ and $G' = (k'_1, k'_2, \cdots, k'_N)$. The relationship among K'', K, and K' can be illustrated as Figure 1. Let the maximum elements in vectors K'', K', K, G'', G', and G be denoted by, respectively, k'', k', k, g'', g', and g.

random sample ⟶ ↘
$K'' = (k''_0, k''_1, \cdots, k''_{j-1}, k''_j, k''_{j+1}, \cdots, k''_{N-1}, k''_N)$ time t+1 (previous)
random sample ⟶ ↘ ↘ ↘ ↘ ↘ ↘ ↘ ↘ ↘ ↘
$K = (k_0, k_1, \cdots, k_{j-1}, k_j, k_{j+1}, \cdots, k_{N-1}, k_N)$ time t (present)
random sample ⟶ ↘ ↘ ↘ ↘ ↘ ↘ ↘ ↘ ↘
$K' = (k'_0, k'_1, \cdots, k'_{j-1}, k'_j, k'_{j+1}, \cdots, k'_{N-1}, k'_N)$ time t-1 (next)

Fig. 1 Relationship among the previous state K'', the present state K, and the next state K'

A *state* of the search process at each time is described by the vector K, so a *state space* of each time is given by $I = \{K | 0 \leq k_j \leq b,\ j = 0,1,\cdots,N\}$. Let a_0 and a_1 denote, respectively, an *action of continuing* the search to find an offer and an *action of stopping* the search with accepting the best offer k, and let A_t denote an *action space* of time t; clearly $A_0 = \{a_1\}$ and $A_t = \{a_0, a_1\}$ for $t \geq 1$. Then a *stopping strategy* is provided by the time sequence of history-dependent, randomized *stopping rule* where a stopping rule of time $t \geq 1$ when in state K is defined by the vector $P_t(K) = (p_{t,0}(K), p_{t,1}(K))$ in which $p_{t,0}(K)(p_{t,1}(K))$ represents the probability of taking action $a_0(a_1)$ at time t, dependent on the entire history of the process up to the previous time. Now define

(3.1) $v_t(K)$ = the maximum expected discounted gain attainable over all possible stopping strategies, starting from time t when in state $K(\in I)$,

(3.2) $V_t(G) = \int v_t(w_0, G)\, dF(w_0)$,

(3.3) $V_t = V_t(0,0,\cdots,0)$.

From the definition of the problem, clearly

(3.4) $v_0(K) = k, \quad K \in I$.

We shall call the stopping strategy attaining the $v_t(K)$ for all $K \in I$ and all $t \geq 1$ an *optimal stopping strategy,* proved to be history-independent as well as nonrandomized [3], and each of the stopping rules composing the optimal stopping strategy an *optimal stopping rule.*

Now suppose the process starts from time $t \geqq 1$ when in state K, using the optimal stopping strategy. Then, if action a_1 is taken, the gain obtained is k. If action a_0 is taken, then the expected discounted gain obtained $U_t(K)$ is given by the present value of the maximum expected discounted gain attained by starting from time $t-1$ minus the search cost c, paid to get an offer at time $t-1$. Here note that the $U_t(K)$ is independent of offer k_N due to the assumption of $p_N = 1$; it is expressed as follows.

$$(3.5) \qquad U_t(K) = \beta \sum_{G' \in \Gamma(K)} P(K,G') \int v_{t-1}(w,G')dF(w) - c$$
$$= \beta \sum_{G' \in \Gamma(K)} P(K,G')V_{t-1}(G') - c$$

where

$$(3.6) \qquad \Gamma(K) = \{(k'_1,k'_2,\cdots,k'_N) | \text{for } j=0,1,\cdots,N-1, \text{ if } k_j>0, \text{ then } k'_{j+1}=0 \text{ or } k_j; \text{ otherwise, } k'_{j+1} = 0\}$$

$$(3.7) \qquad P(K,G') = \prod_{j=0}^{N-1} p_j(k_j,k'_{j+1})$$

with

$$(3.8) \qquad p_j(k_j,k'_{j+1}) = \begin{cases} 1-p_j & \text{if } k_j = k'_{j+1} > 0, \\ p_j & \text{if } k_j > 0 \text{ and } k'_{j+1} = 0, \\ 1 & \text{if } k_j = k'_{j+1} = 0. \end{cases}$$

Here

$$(3.9) \qquad U_t(0,0,\cdots,0,k_N) = \beta V_{t-1} - c.$$

For the convenience of mathematical analysis in Section 6, we shall transform (3.5), (3.6), and (3.8) as follows;

$$(3.10) \qquad U_t(K) = \beta \sum_{G' \in \Delta(K)} P(K,G')V_{t-1}(G') - c$$

where

$$(3.11) \qquad \Delta(K) = \{(k'_1,k'_2,\cdots,k'_N) | \ k'_{j+1} = k_j \text{ or } 0 \text{ for } j = 0,1,\cdots,N-1\},$$

$$(3.12) \qquad p_j(k_j,k'_{j+1}) = \begin{cases} 1-p_j & \text{if } k'_{j+1} = k_j, \\ p_j & \text{if } k'_{j+1} = 0. \end{cases}$$

The equivalence of (3.5) and (3.10) can be verified as follows. When $K > 0$, the equivalence is obvious. Let $K \geqq 0$. In the case, for example, consider the case of $N = 2$ and $K = (w_0,0,w_2)$. Then, using (3.5), we have

$$U_t(w_0,0,w_2) = \beta(p_0V_{t-1}(0,0) + (1-p_0)V_{t-1}(w_0,0)) - c.$$

This is transformed into

$$U_t(w_0,0,w_2) = \beta(p_0p_1V_{t-1}(0,0) + p_0(1-p_1)V_{t-1}(0,0) + (1-p_0)p_1V_{t-1}(w_0,0) + (1-p_0)(1-p_1)V_{t-1}(w_0,0)) - c,$$

which is identical with what is expressed by using (3.10). From the simple example, it is immediately realized that not only $U_t(k_0,k_1,k_2)$ for any $(k_0,k_1,k_2) \geqq 0$ but also, in general, $U_t(K)$ for any $K \geqq 0$ can be expressed by using (3.10).

Here notice that $p_j(k_j,k'_{j+1})$ defined by (3.12) is independent of the value of k_j; it only depends on whether or not k_j is replaced by 0 at the next time where, for k_j already with value 0, the replacement is regarded as that of *the 0* by *new 0.* Accordingly, it follows that we may write (3.10), (3.7), and (3.12) as follows.

$$(3.13) \qquad U_t(K) = \beta \sum_{G'\in\Delta(K)} P(G')V_{t-1}(G') - c,$$

$$(3.14) \qquad P(G') = \prod_{j=0}^{N-1} p_j(k'_{j+1}),$$

$$(3.15) \qquad p_j(k'_{j+1}) = \begin{cases} 1-p_j & \text{if } k'_{j+1} = k_j, \\ p_j & \text{if } k'_{j+1} = 0. \end{cases}$$

Remark 1. (a). The sum of $P(G')$ over $\Delta(K)$ equals 1. (b). $P(G')$ is independent of K, only depending on whether or not each element of K is replaced by value 0.

From the principle of optimality in dynamic programming, we have

$$(3.16) \qquad v_t(K) = \max\{k, U_t(K)\}, \quad t \geqq 0.$$

Then the optimal stopping rule of time t is provided as follows: if $U_t(K) - k < 0$, stop the search with accepting the best offer k; otherwise, continue the search. Define

$$(3.17) \qquad Q_t(K,j) = U_t(K) - k_j, \qquad j = 0,1,\cdots,N,$$

$$(3.18) \qquad S_t(j) = \{K|Q_t(K,j) < 0\}, \ j = 0,1,\cdots,N,$$

$$(3.19) \qquad S_t = \bigcup_{j=0}^{N} S_t(j).$$

Suppose $K \in S_t$. Then, for at least one $j\in\{0,1,\cdots,N\}$, $K \in S_t(j)$; i.e., $Q_t(K,j) < 0$. Consequently, since $U_t(K) - k \leqq U_t(K) - k_j = Q_t(k,j) < 0$, it follows that stopping is optimal. On the contrary, suppose $K \notin S_t$. Then since $K \notin S_t(j)$ for all j; i.e., $Q_t(K,j) \geqq 0$ for all j, we have $U_t(K) - k \geqq 0$, implying that continuing is optimal. Accordingly, we may call the S_t a *stop region* and its complement, denoted by C_t, a *continuation region.*

The successive sections except Section 6.5. are exclusively devoted to characterizing the structure of the continuation region C_t and its sequence $C_0, C_1, \cdots$ as well as proving that the inequality $\beta\mu - c > a$ is the necessary and sufficient condition for the optimal stopping rule to have DRV-property defined below.

DEFINITION 0. The optimal stopping rule is said to have a *double reservation value property* or *DRV-property* for short when, for at least one $\dot{G}$, there exist such two critical values ξ and ξ' with $a < \xi < \xi' < b$ that, for a present offer w_0, if $\xi \leqq w_0 \leqq \xi'$, then continuing is optimal; otherwise, stopping is optimal.

Finally, we shall provide a lemma used in the subsequent sections. Let

(3.20) $T(x) = \int_{x+}^{\infty} (w-x)\,dF(w), \quad -\infty < w < \infty,$

(3.21) $H(x) = \beta(x + T(x)) - x - c,$

which are continuous functions of x, and let the smallest solution of $H(x) = 0$, if exists, be denoted by h^*; i.e.,

(3.22) $H(h^*) = 0.$

Throughout the paper, a function $\phi(x)$ is said to be increasing (decreasing) in x if $\phi(x) \geq (\leq)\ \phi(y)$ for any $x > y$ and strictly increasing (strictly decreasing) in x if $\phi(x) > (<)\ \phi(y)$ for any $x > y$. Furthermore, a function $\phi(x)$ of a vector x is said to be increasing (decreasing) in x if it is increasing (decreasing) in each element of x.

LAMMA 0. We have

(a) $T(x)$ is decreasing and convex on $-\infty < x < +\infty$, strictly decreasing on $x \leq b$, and equal to $\mu - x$ on $x \leq a$ and to 0 on $b \leq x$.

(b) $x + T(x)$ is increasing and convex on $-\infty < x < +\infty$, strictly increasing on $a \leq x$, and equal to μ on $x \leq a$ and to x on $b \leq x$.

(c) If $(1-\beta)^2 + c^2 \neq 0$, then h^* exists. In addition, if $a < \beta\mu - c$, then $a < h^* < b$; otherwise, $h^* = \beta\mu - c < b$.

(d) If $(1-\beta)^2 + c^2 = 0$, then $h^* = b$ where $H(x) > 0$ for $x < h^*$ and $H(x) = 0$ for $h^* \leq x$.

Proof: Easy. □

4. Simple Case

Let $N = 1$, and suppose the search process starts from time 1 when in state $K = (k_0, k_1)$. Then $v_1(k_0,k_1) = \max\{\max\{k_0,k_1\},\ U_1(k_0,k_1)\}$ where

(4.1) $U_1(k_0,k_1) = \beta(p_0 \int w\,dF(w) + (1-p_0)\int \max\{w,k_0\}\,dF(w)) - c$

$$= \beta(p_0\mu + (1-p_0)(k_0 + T(k_0))) - c$$

in which w represents an offer of time 0. In the case, the stop region of time 1 is given by $S_1 = S_1(0) \cup S_1(1)$ where

$$S_1(0) = \{(k_0,k_1) \mid Q_1((k_0,k_1),0) < 0\}, \quad Q_1((k_0,k_1),0) = U_1(k_0,k_1) - k_0,$$
$$S_1(1) = \{(k_0,k_1) \mid Q_1((k_0,k_1),1) < 0\}, \quad Q_1((k_0,k_1),1) = U_1(k_0,k_1) - k_1.$$

In the discussions below, note that $U_1(h,k_1) = \beta\mu - c$ for $h \leq a$, is strictly increasing in $h \geq a$, and is independent of k_1 and that $U_1(h,k_1) - h$ is strictly decreasing in h.

The continuation region C_1 is given by the domain enclosed by the bold lines in

Figure 2 where the straight line $\overline{fe}$ and the curved line $\overline{de}$ are the loci of points (k_0,k_1) satisfying, respectively, $U_1(k_0,k_1) - k_0 = 0$ and $U_1(k_0,k_1) - k_1 = 0$; the curved line is an increasing function of k_0. In the figure, $h_1(1) = U_1(0,h_1(1)) = \beta\mu - c > 0$, and h_1 is the solution of $U_1(h,k_1) - h = 0$, positive and unique, where $h_1 = U_1(h_1,k_1) \geq U_1(0,k_1) = h_1(1)$.

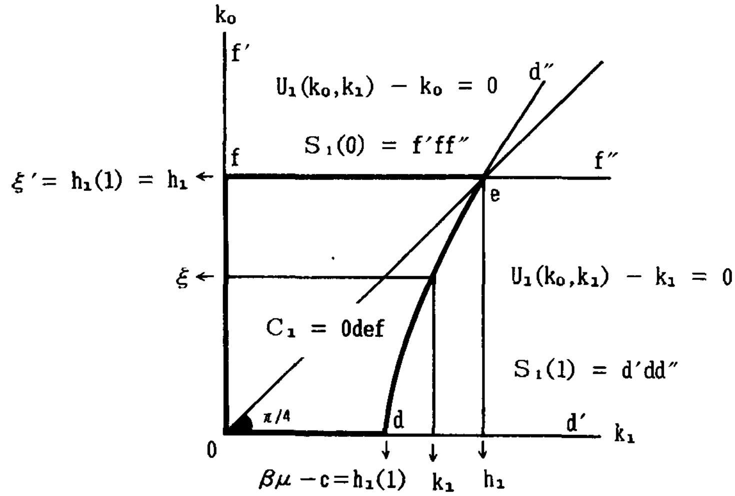

Fig. 2 Continuation Region C_1 (N = 1)

Now, suppose $h_1 = h_1(1)$. Then since the curved line $\overline{fe}$ is reduced to a straight line perpendicular to k_1-axis, it goes without saying that the optimal stopping rule has not DRV-property. Next, suppose $h_1 > h_1(1)$. Then it is easily realized from Figure 2 that the optimal stopping rule has DRV−property. In fact, for any k_1 with $h_1(1) < k_1 < h_1$, the two critical numbers ξ and ξ' characterizing the property are given by, respectively, the solution of $U_1(\xi,k_1) = k_1$ and the solution of $U_1(\xi',k_1) = \xi'$; i.e., $\xi' = h_1$. Here $a < \xi < \xi' < b$ is proved by reductive absurdity as follows. If $b \leq \xi'$, then $\xi' = \beta(p_0\mu+(1-p_0)\xi') - c$, yielding $\xi' = (\beta p_0\mu-c)/(1-\beta(1-p_0)) = \mu - ((1-\beta)\mu+c)/(1-\beta(1-p_0)) \leq \mu < b$, if $\xi' \leq \xi$, then $\xi' \leq U_1(\xi,k_1) = k_1 < h_1 = \xi'$, and if $\xi \leq a$, then $k_1 = \beta(p_0\mu+(1-p_0)\mu) - c = \beta\mu - c = h_1(1)$. From the above, it follows that it is only when $h_1 > h_1(1)$ that the optimal stopping rule of time 1 has DRV-property.

If $\beta\mu - c \leq a$, then since $U_1(\beta\mu - c,k_1) = \beta\mu - c$, we get $h_1 = \beta\mu - c = h_1(1)$ from the uniqueness of the solution to $U_1(h,k_1) = h$. On the contrary, if $\beta\mu - c > a$, then $h_1 = U_1(h_1,k_1) \geq U_1(h_1(1),k_1) = U_1(\beta\mu - c,k_1) > U_1(a,k_1) = \beta\mu - c = h_1(1)$. Consequently, it is concluded that the necessary and sufficient condition for the optimal stopping rule of time 1 to have DRV−property is $\beta\mu - c > a$. It is proved in Section 6.4. that this holds for any $t \geq 1$ and any $N \geq 1$.

5. Conclusions and Considerations

Define the following sets:

(5.1) $A = \{K | 0 \leq k_j \leq a,\ j = 0,1,\cdots,N\}$,

(5.2) $B = \{K | 0 \leq k_j \leq b,\ j = 0,1,\cdots,N\}$,

(5.3) $M = \{K | 0 \leq k_j \leq \beta\mu - c,\ j = 0,1,\cdots,N\}$,

(5.4) $H^* = \{K | 0 \leq k_j \leq h^*,\ j = 0,1,\cdots,N\}$,

(5.5) $H_t = \{K | 0 \leq k_j \leq h_t,\ j = 0,1,\cdots,N\}$,

(5.6) $H_t' = \{K | 0 \leq k_j \leq h_t(j),\ j = 0,1,\cdots,N\}$,

in which $h_t(j)$ and h_t are the solutions of, respectively, $Q_t((0,\cdots,0,k_j,0,\cdots,0),j) = 0$ and $Q_t((h,h,\cdots,h),j) = 0$ where h_t is independent of j. Both solutions are positive and unique (Lemma 3(c)). The conclusions obtained in this paper are as follows:

a. Structure of continuation region

i. Suppose $\beta\mu - c \leq a$. Then $C_t = M\ (\subset A)$, a *perfect cube*, for all $t \geqq 1$ (Theorem 6(d)). Accordingly, in the case, not only has not the optimal stopping rule the DRV-property, but when the search process starts without any offer, stopping with accepting the first offer is optimal because, once an offer w_0 is made, the current state $(0, 0,\cdots,0)$ changes into $(w_0,0,\cdots,0) \notin M = C_t$ due to $a < w_0$.

ii. Suppose $\beta\mu - c > a$. Then

(1). For all $t \geqq 1$, the continuation region C_t is given by a *hollowed cube* enclosed by $N+1$ coordinate planes and $N+1$ hollowed planes as shown in Figure 3 ($N = 2$)

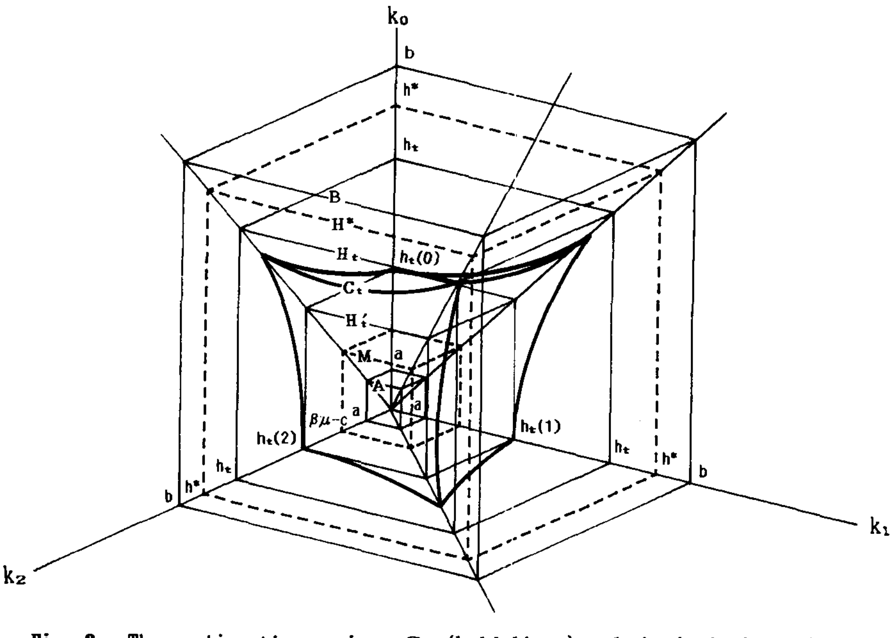

Fig. 3 The continuation region C_t (bold lines) and the inclusion relation among its related regions ($N = 2$); for all t, if $\beta\mu - c > a$, then $B \supset H^* \supset H_t \supset C_t \supset H_t' \supset M \supsetneqq A$; otherwise, $C_t = M \subset A$

where each hollowed plane is a set of K satisfying $Q_t(K,j) = 0$ and $k_j = k$, $j = 0, 1,\cdots,N$; the outer space of the plane, $S_t(j)$, is convex (Theorem 1). In the figure, $h_t(j)$ is a point on k_j-axis at which state $K = (0,\cdots,0,k_j,0,\cdots,0)$ is transferred from the continuation region to the stop region when k_j travels from its origin in a positive direction, and the (h_t,h_t,h_t) is an intersection point of the surface of the continuation region C_t and the straight line emerging from the origin at angle of $\pi/4$ with each coordinate axis.

(2). $B \supset H^* \supset H_t \supset C_t \supset H_t' \supset M \supsetneqq A$ (Theorem 3, Lemma 10(d), Figure 3) and $C_{t+1} \supset C_t$ for $t \geqq 1$ (Theorem 4(a)) where, in general, $X \supsetneqq Y$ means that Y is a proper subset of X. If $(1-\beta)^2 + c^2 \neq 0$, then $B \supsetneqq H^*$ (Lemma 0(c)); otherwise, $B = H^*$ (lemma 0(d)).

(3). If $N = 1$, then $h_t = h_t(0)$ and $h_t > h_t(1)$ (Theorem 5(a), Figure 2). If $N \geqq 2$, then $h_t > h_t(j) > h_t(N)$ for $j = 0,1,\cdots,N-1$ (Theorem 5(b)).

iii. If $p_j >(=) p_i$, then $h_1(j) <(=) h_1(i)$ (Theorem 5(c)). It will be quite difficult to examine whether or not the relationship holds for all $t \geqq 1$.

b. Necessary and sufficient coindition for an optimal stopping rule to have DRV-property

The above conclusion means that the necessary and sufficient condition for the optimal stopping rule to have DRV-property for all $t \geqq 1$ is $\beta\mu - c > a$. Here it should be noted that the conclusion claims that the DRV-property may appears even in the simplest case of $\beta = 1$ and $c = 0$. Figure 4 illustrates how the property appears in case of $N = 2$ where ξ' and ξ are values of k_0 at which the straight line LL' intersects with the surface of the continuation region.

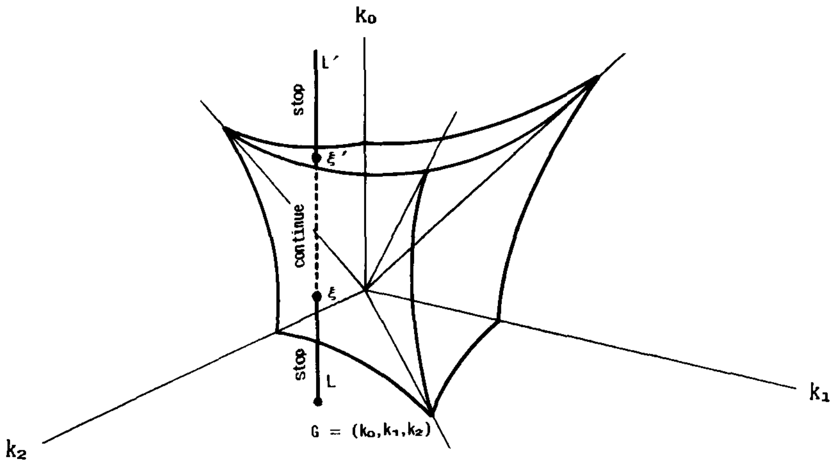

Fig. 4 DRV-property caused by the curvature of continuation region C_t ($N = 2$)

In almost all models of an optimal stopping problem presented so far, it has been demonstrated that the optimal stopping rules are characterized in such a fashion that, if continuing is optimal for a given present offer w_0, then so also is continuing for any present offer $w < w_0$. This implies that there exists the superior ξ of present offers w_0 for which continuing is optimal. The ξ is commonly called a *reservation value*, and when the optimal stopping rule is characterized by such a reservation value, it is usually said to have a *reservation value property*. Applying the concept to our model will follows that, for any given G, presently available past offers, there exists a G-dependent critical number $\xi(G)$ such that, if $w_0 < \xi(G)$, then continue; otherwise, stop. The conclusion we stated above, however, claims that the optimal stopping rule of the optimal stopping problem with uncertain recall has not always the reservation value property.

c. DRV-property gradually disappears as a planning horizon tends to infinity

When $\beta\mu - c > a$, the continuation region C_t, a *hollowed cube*, increases and converges to H^*, a *perfect cube*, as $t \to \infty$ (Theorems 4(a),6(c)), implying that DRV-property gradually disappears as a planning horizon becomes larger and totally vanishes in its limit.

d. Reduction to with-no-recall case in the limit of a planning horizon

The above conclusion implies that when $\beta\mu - c > a$, the optimal stopping rule is reduced in the limit of the planning horizon to that, if $K \in H^*$, stop; otherwise, continue. This is the same to saying that, if $k \leqq h^*$, stop; otherwise, continue. Now assume that the process has continued up to the present time, following the optimal stopping strategy. The assumption means that the continuation decision was also made at the previous time; accordingly, it must be that $k'' \leqq h^*$. Then we can show $k \leqq(>) h^* \Leftrightarrow w_0 \leqq(>) h^*$. It is clear that $k \leqq h^*$ yields $w_0 \leqq h^*$. Suppose $h^* < k$. Since $g = \max\{k_1,k_2,\cdots,k_N\} \leqq \max\{k''_0,k''_1,\cdots,k''_{N-1}\} \leqq \max\{k''_0,k''_1,\cdots,k''_{N-1},k''_N\} = k''$, it follows that $g \leqq h^*$ by the assumption. Accordingly, we have $k = \max\{w_0,g\} \leqq \max\{w_0,h^*\}$, from which we get $h^* < w_0$ because $w_0 \leqq h^*$ leads to the contradiction of $k \leqq h^*$. The equivalent relation above implies that the optimal stopping rule is reduced in the limit of a planning horizon to a mere comparison of w_0 and h^*; in other words, it suffices to memorize only the current offer w_0 with neglecting all past ones. This eventually means that, in the limit of an planning horizon, the optimal stopping rule of the optimal stopping problem with uncertain recall becomes substantially identical with that of an optimal stopping problem with no recall.

e. Expected discounted gain, search amount, and value realization

In with-no-recall case (with-recall-case), let $v_t(k_0|0)$ $(v_t(k|1))$ represent the expected discounted gain starting from time t with the present offer k_0 (with the best offer k). Then $v_t(k|1) \geqq v_t(K) \geqq v_t(k_0|0)$ for all $t \geqq 0$ (Lemma 10(a)). If $K \in H^*$, then $v_t(k|1)$, $v_t(K)$, and $v_t(k_0|0)$ converge to h^* as $t \to \infty$ (Theorem 6(a), Lemma 9(a1,b1)).

In with-uncertain-recall case (with-recall-case), when the process starts from time t with offers K (with the best offer k), let the *search amount*, the expected number of searches, be represented by $\rho_t(K)$ $(\rho_t(k|1))$, and the *value realization*, the expectation of the offer w accepted, be denoted by $\upsilon_t(K)$ $(\upsilon_t(k|1))$. Then $\rho_t(K)$, $\rho_t(k|1)$, $\upsilon_t(K)$, and $\upsilon_t(k|1)$ are all increasing in t for all K and all k (Theorems 8(b), 9(b)), and $\rho_t(K) \leqq \rho_t(k|1)$ and $\upsilon_t(K) \leqq \upsilon_t(k|1)$ for all $t \geqq 0$ and all K (Theorems 8(a),9(a)).

Suppose $(1-\beta)^2 + c^2 \neq 0$. Then $\rho_t(K)$, $\rho_t(k|1)$, $\nu_t(K)$, and $\nu_t(k|1)$ converge as $t \to \infty$ where the limits of $\rho_t(k|1)$ and $\nu_t(k|1)$ are, respectively, $(1-F(h^*))^{-1}$ and $h^* + c(1-F(h^*))^{-1}$ (Theorem 8(c),9(c)). In addition, if $\beta\mu - c > a$, let K be an inner point of H^*; otherwise, let $K \in H^*$. Then, as $t \to \infty$, $\rho_t(K)$ and $\nu_t(K)$ converge to, respectively, $(1-F(h^*))^{-1}$ and $h^* + c(1-F(h^*))^{-1}$.

f. Numerical Examples

i. Let $N = 1$, $\beta = 0.99$, $p_0 = 0.02$, $c = 0.01$, and $F(w) = w$ on $0 \leqq w \leqq 1$, a uniform distribution with $a = 0$ and $b = 1$, where $x + T(x) = (1+x^2)/2$ on $0 \leqq w \leqq 1$, and suppose the search process has only one period to go; i.e., it starts from time 1. In the case, since the inequality $\beta\mu - c > a$ holds, the optimal stopping rule has DRV-property for all $t \geqq 1$. Then the continuation region C_1 becomes as in Figure 5 where $h_1(1) = 0.485$ and $h_1(0) = h_1 = 0.78$. Now consider the following four different states: $K_1 = (0.23,0.61)$, $K_2=(0.65,0.61)$, $K_3=(0.82,0.61)$, and $K_4=(0.72,0.65)$.

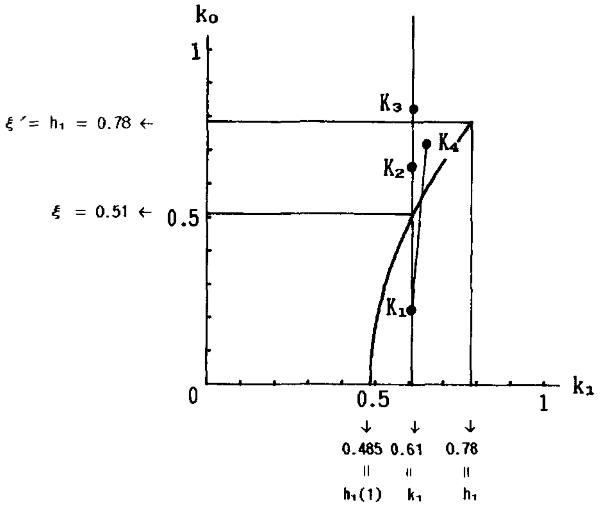

Fig. 5 Continuation region C_1 ($N = 1$) and DRV-property where $\beta = 0.99$, $p_0 = 0.02$, $c = 0.01$, and uniform distribution $F(w)$ with $a = 0$ and $b = 1$ (K_1 = stop, K_2 = continue, K_3 = stop, K_4 = continue)

Then we have

$v_1(K_1) = \max\{0.61, 0.51066179\}$, so stop with accepting offer 0.61 ($= k_1$)
$v_1(K_2) = \max\{0.65, 0.68995475\}$, so continue,
$v_1(K_3) = \max\{0.82, 0.81118124\}$, so stop with accepting offer 0.82 ($= k_0$),
$v_1(K_4) = \max\{0.72, 0.73647584\}$, so continue

where no computational error is involved.

First, compare $v_1(K_1)$, $v_1(K_2)$, and $v_1(K_3)$ with $K_1 \leqq K_2 \leqq K_3$, suggesting the existence of DRV-property. Two critical numbers characterizing the property, $\xi = 0.51$

and $\xi' = 0.78$, are given by the solutions of $U_0(k_0,0.61) = \max\{k_0,0.61\}$ (Figure 6). Then the optimal stopping rule is that if $0.51 \leq k_0 \leq 0.78$, then continue; otherwise, stop.

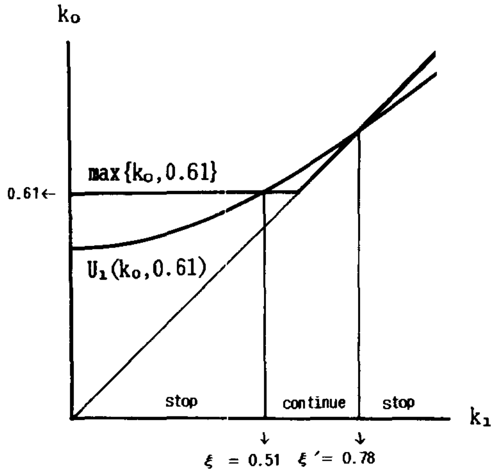

Fig. 6 Expected discounted gain from continuing, $U_0(k_0,0.61)$, and gain from stopping, $\max\{k_0,0.61\}$

Next, let us compare $v_1(K_1)$ and $v_1(K_4)$ with $K_1 < K_4$. In the case, we are liable to think that, since stopping is optimal when in state K_1, so also will be stopping when in state K_4 that is strictly greater than K_1. The result of the above numerical examples, however, indicates that this is not always true.

ii. Figures 7(a,b,c) are continuation regions C_1 for, respectively,

$$(\beta,p_0,c) = \begin{cases} (\beta,0,0) \text{ with } \beta = 0.75, 0.80, \cdots, 1.00, \\ (1,p_0,0) \text{ with } p_0 = 0.00, 0.20, \cdots, 1.00, \\ (1,0,c) \text{ with } c = 0.00, 0.05, \cdots, 0.25, \end{cases}$$

which illustrate the relationship between the continuation region and the parameters β, p_0, and c.

iii. Figure 7(d) shows how the continuation region increases as a planning horizon becomes greater and converges to H^* in its limit.

6. Analysis

6.1. Monotonicity of continuation region

LEMMA 1. For all t,

(a) $V_t(G) \geq \mu$ and $V_t(G) \geq k_j$ for all G and $j = 1,2,\ldots,N$,

(b) $V_t(G) \leq b$ for all G, and $U_t(K) \leq b$ for all K,

(c) $V_t(G)$ is increasing and convex in G, and $U_t(K)$ is increasing and convex in K,

(d) $Q_t(K,j)$ is increasing in k_i for $i \neq j$ and convex in K.

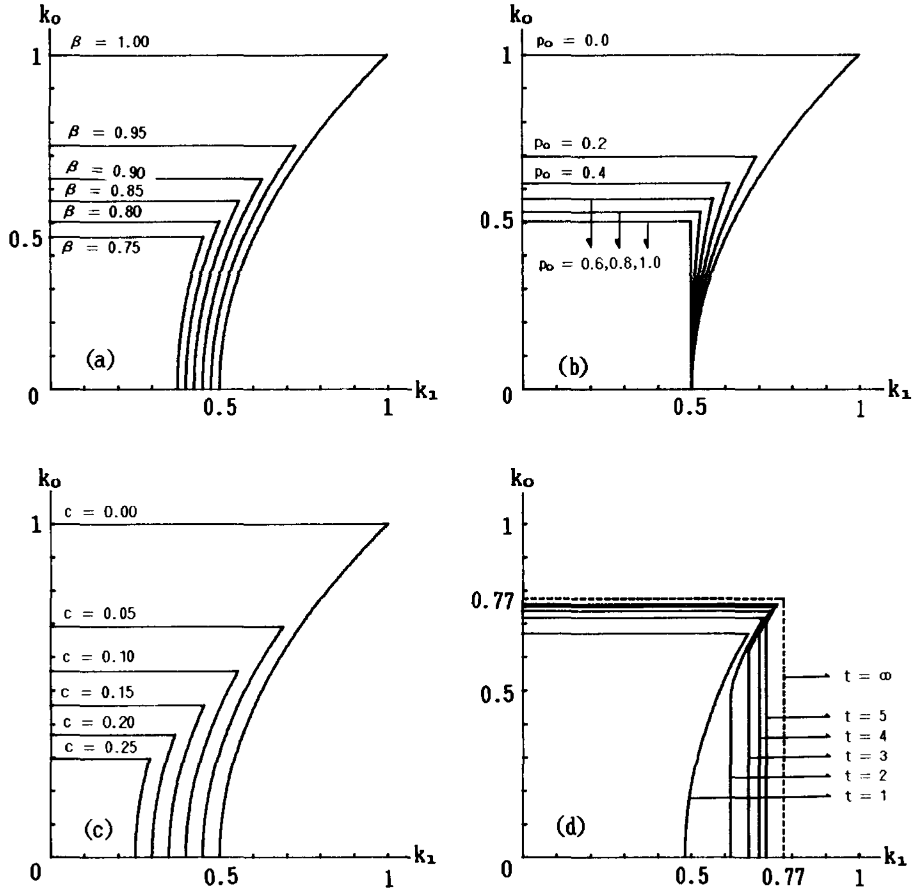

Fig. 7 Numerical examples illustrating the relationship between the structure of continuation region and parameters β, p_0, and c ($N = 1$).

(a) $\beta = 0.75\sim1.00$, $p_0 = 0$, and $c = 0$,
(b) $\beta = 1$, $p_0 = 0.0\sim1.0$, and $c = 0$,
(c) $\beta = 1$, $p_0 = 0$, and $c = 0.00\sim0.25$,
(d) $\beta = 0.98$, $p_0 = 0.15, c = 0.01$, and $t = 1,2,3,4,5,\infty$

Proof: (a). Immediate from $v_t(K) \geq k_j$ for all $t \geq 0$, all K, and $j = 0,1,\ldots,N$. (b). Since $v_0(k_0,G) = k \leq b$, we have $V_0(G) \leq b$. Suppose $V_{t-1}(G) \leq b$ for all G. Then since $U_t(K) \leq \beta b - c \leq b$ for all K, we have $v_t(K) \leq \max\{b,b\} = b$ for all K. Hence, $V_t(G) \leq b$ for all G. (c). g, the function of G, is increasing and convex in G, and $V_0(G)$ $(= g + T(g))$ is increasing and convex in g. Hence, $V_0(G)$ is also increasing and convex in G. Suppose $V_{t-1}(G)$ is increasing and convex in G. Then $U_t(K)$ also becomes increasing and convex in K. In addition, since k, the function of K, is also increasing and convex in K, it follows that $v_t(K)$ is increasing and convex in K. Therefore, $V_t(G)$ is also increasing and convex in G. (d). Obvious from (c). □

THEOREM 1. $S_t(j)$ is a convex set for all $t \geq 1$ and all j.

Proof: Immediate from Lemma 1(d). □

LEMMA 2. For all t and all j,
(a) $\beta V_{t-1}(G') - k_j$ is decreasing in k_j,
(b) $Q_t(K,j)$ is strictly decreasing in k_j and tends to $-\infty$ as $k_j \to \infty$,
(c) $Q_t(K,j) = 0$ with unknown k_j has a positive unique solution.

Proof: (a,b). These statements are always true for $j = N$ because G' is independent of k_N. Below assume $0 \leq j \leq N-1$. Let $\tau(k_j) = \beta V_0(G') - k_j = \beta(g' + T(g')) - k_j$. If $k'_{j+1} = 0$, then since G' is independent of k_j, $\tau(k_j)$ is decreasing in k_j. Suppose $k'_{j+1} = k_j$, and let $\bar{g}' = \max\{k'_1,\ldots,k'_j,k'_{j+2},\ldots,k'_N\}$. Then since $g' = \max\{k'_{j+1},\bar{g}'\} = \max\{k'_j,\bar{g}'\}$, $\tau(k_j) = \beta(\bar{g}' + T(\bar{g}')) - k_j$ on $0 \leqq k_j \leqq \bar{g}'$ and $\tau(k_j) = \beta T(k_j) - (1-\beta)k_j$ on $\bar{g}' \leqq k_j$. Thus, $\tau(k_j)$ is decreasing in k_j both on $0 \leqq k_j \leqq \bar{g}'$ and on $\bar{g}' \leqq k_j$. In addition, since $\tau(k_j)$ is continuous on $k_j \geqq 0$, it follows that $\tau(k_j)$ is decreasing in $k_j \geqq 0$. Therefore, (a) holds for $t = 1$. Suppose (a) holds for a given $t \geqq 1$. Now note that (3.17) is expressed as

$$(6.1)\qquad Q_t(K,j) = \sum_{G' \in \Delta(K)} P(G')(\beta V_{t-1}(G') - k_j) - c.$$

Here consider the term $P(G')(\beta V_{t-1}(G') - k_j)$ with $P(G') = p_0 p_1 \cdots p_{N-1} > 0$. Then since $G' = (0,0,\cdots,0)$, it follows that, for all j, the term is strictly decreasing in k_j and tends to $-\infty$ as $k_j \to \infty$. In addition, the other terms are all decreasing in k_j from the induction hypothesis. Thus it follows that $Q_t(K,j)$ is strictly decreasing in k_j and tends to $-\infty$ as $k_j \to \infty$. Now we have

$$(6.2)\qquad \beta v_t(k_0,G) - k''_j = \max\{\beta\max\{k_0,g\} - k''_j,\ \rho\},\qquad \rho = \beta U_t(K) - k''_j$$

where $\beta\max\{k_0,g\} - k''_j$ is decreasing in k''_j. If $k_{j+1} = 0$, then since $U_t(K)$ is independent of k''_j, ρ is decreasing in k''_j. If $k_{j+1} = k''_j$, then $\rho = Q_t(K,j+1)$, decreasing in k''_j. Accordingly, (6.2) is decreasing in k''_j; therefore, $\beta V_t(G) - k''_j$ is also decreasing in k''_j. Thus the induction completes. (c). For $k_j = 0$, since $Q_t(K,j) \geqq \beta\mu - c > 0$ from Lemma 1(a) and the assumption (2.1), the assertion becomes true from (b). □

For any subset L of $\{0,1,\cdots,N\}$, possibly an empty set, if $j \in L$, let $u_j = h$; otherwise, $u_j = 0$. Then for any given $K = (k_0,k_1,\cdots,k_N)$, define

$$(6.3)\qquad K(L,i,h) = (k_0,k_1,\cdots,k_{i-1},u_i,u_{i+1},\cdots,u_N),\quad 0 \leqq i \leqq N+1$$

in which $K(L,0,h) = (u_0,u_1,\cdots,u_N)$ and $K(L,N+1,h) = (k_0,k_1,\cdots,k_N) = K$, and let $G(L,i,h) = (k_1,k_2,\cdots,k_{i-1},u_i,u_{i+1},\cdots,u_N)$ for $1 \leqq j \leqq N+1$. Similarly, also define $K'(L,i,h) = (k'_0,k'_1,\cdots,k'_{i-1},u_i,u_{i+1},\cdots,u_N)$ and $G'(L,i,h) = (k'_1,k'_2,\cdots,k'_{i-1},u_i,u_{i+1},\cdots,u_N)$. Then for $0 \leqq i \leqq N+1$,

$$(6.4)\qquad \beta v_t(K(L,i,h)) - h = \max\{\beta\max\{k_0,k_1,\cdots,k_{i-1},\rho\} - h,\ \beta Q_t(K(L,i,h)) - (1-\beta)h\}$$

where

$$(6.5)\qquad Q_t(K(L,i,h)) = U_{t-1}(K(L,i,h)) - h = \sum_{G' \in \Delta(K(L,i,h))} P(G')(\beta V_{t-1}(G') - h) - c$$

and $\rho = 0$ ($\rho = h$) if $K(L,i,h) = (k_0,k_1,\cdots,k_{i-1},0,\cdots,0)$ (otherwise).

LEMMA 3. For all t, all L, and all i,

(a) $\beta V_{t-1}(G'(L,i,h)) - h$ is decreasing in h ($i \neq 0$),

(b) $Q_t(K(L,i,h))$ is strictly decreasing in h and tends to $-\infty$ as $h \to \infty$,

(c) $Q_t(K(L,i,h)) = 0$ with unknown h has a positive unique solution.

Proof: (a,b). $\beta V_0(G'(L,i,h)) - h = \beta \int \max\{w,k_1',k_2',\cdots,k_{i-1}',\rho\} dF(w) - h$ with $\rho = 0$ ($\rho = h$) if $G'(L,i,h) = (k_1',k_2',\cdots,k_{i-1}',0,\cdots,0)$ (otherwise). This is decreasing in h. Thus, (a) holds for $t = 1$. Suppose it is true for a given $t \geq 1$. Then (b) can be proved in quite the similar way as in Lemma 2(b). Accordingly, (6.4) is decreasing in h; therefore, $\beta V_t(G(L,i,h)) - h$ is also decreasing in h. Thus, the induction completes. (c). Almost the same as the proof of Lemma 2(c). □

Let $h_t(L)$ denote the solution to the equation $Q_t(K(L,0,h)) = 0$ with unknown h where $h_t(L)$ is positive and unique from Lemma 3(c), and let $h_t(j) = h_t(\{j\})$ for $j = 0,1,\cdots,N$ and $h_t = h_t(\{0,1,\cdots,N\})$. Since $Q_t(K(\{N\},0,h)) = U_t(0,0,\cdots,0,h) - h$, we have

$$
\begin{aligned}
(6.6)\qquad h_t(N) &= U_t(0,0,\cdots,0,h_t(N)) \\
&= U_t(0,0,\cdots,0) = \beta V_{t-1} - c \geq \beta\mu - c > 0.
\end{aligned}
$$

THEOREM 2. For all t,

(a) $h_t(j) \leqq h_t$ for all j,

(b) For a given K,

1. If $h_t < k_j$ for at least one $j \in \{0,1,\cdots,N\}$, then $K \in S_t$,
2. If $k_j \leqq h_t(j)$ for all j, then $K \in C_t$.

Proof: (a). If $h_t < h_t(j)$ for a certain j, then we get the contradiction of

$$
\begin{aligned}
0 &= Q_t(K(\{0,1,\cdots,N\},0,h_t)) = Q_t((h_t,h_t,\cdots,h_t),j) \\
&> Q_t((h_t,\cdots,h_t,h_t(j),h_t,\cdots,h_t),j) \geqq Q_t((0,\cdots 0,h_t(j),0,\cdots,0),j) \\
&= Q_t(K(\{j\},0,h_t(j))) = 0.
\end{aligned}
$$

(b1). Let $k_j = k$. Then since $h_t < k_j$ and $k_i \leq k_j$ for $i = 0,1,\cdots,N$, we have

$$
\begin{aligned}
Q_t(K,j) &= Q_t((k_0,\cdots,k_j,\cdots,k_N),j) \leq Q_t((k_j,\cdots,k_j,\cdots,k_j),j) \\
&= Q_t(K(\{0,1,\cdots,N\},0,k_j)) < Q_t(K(\{0,1,\cdots,N\},0,h_t)) = 0,
\end{aligned}
$$

implying $K \in S_t(j)$; therefore, $K \in S_t$. (b2). For all j, we have

$$
\begin{aligned}
Q_t(K,j) &= Q_t((k_0,\cdots,k_j,\cdots,k_N),j) \geq Q_t(0,\cdots,0,k_j,0,\cdots,0),j) \\
&= Q_t(K(\{j\},0,k_j)) \geq Q_t(K(\{j\},0,h_t(j))) = 0,
\end{aligned}
$$

implying $K \notin S_t(j)$; therefore, $K \in C_t$. □

LEMMA 4. For all t,

(a) If $K \in H^*$, then $v_t(K) \leq h^*$; otherwise, $v_t(K) = k$.

(b) $h_t(L) \leqq h^*$ for all L.

Proof: (a). Clear for $t = 0$ because $v_0(K) = k$ for all K. Assume the assertion holds for $t-1$, so $v_{t-1}(K') \leqq \max\{h^*,k'\}$ ...(*) for all K'. First, suppose $K' \in H^*$, so $k'_j \leqq k' \leqq h^*$ for $1 \leqq j \leqq N$. If $w > h^*$, then $(w,G') \notin H^*$ and $\max\{w,G'\} = w$, and if $h^* \geqq w$, then $(w,G') \in H^*$. Accordingly,

$$V_{t-1}(G') = \int (v_{t-1}(w,G')I(w>h^*) + v_{t-1}(w,G')I(h^* \geqq w))\,dF(w)$$
$$\leqq \int (wI(w>h^*) + h^*I(h^* \geqq w))\,dF(w) = h^* + T(h^*),$$

from which $U_t(K) \leqq \beta(h^* + T(h^*)) - c = H(h^*) + h^* = h^*$. Therefore, we have $v_t(K) \leqq \max\{k,h^*\} = h^*$. Next, suppose $K' \notin H^*$, implying $h^* < k' \leqq k$. Then, noticing $g' \leqq k$ and (*), we have

$$V_{t-1}(G') = \int v_{t-1}(w,G')\,dF(w) \leqq \int \max\{h^*, \max\{w,g'\}\}\,dF(w)$$
$$= \int \max\{w, \max\{h^*,g'\}\}\,dF(w) \leqq \int \max\{w, \max\{h^*,k\}\}\,dF(w)$$
$$= \int \max\{w,k\}\,dF(w) = k + T(k),$$

from which $U_t(K) \leqq \beta(k + T(k)) - c = H(k) + k \leqq k$ due to $H(k) \leqq 0$ for $k > h^*$. Consequently, it follows that $v_t(K) = k$. (b). Suppose $h \leqq h^*$. Then since $K(L,0,h) \in H^*$, it follows that $U_t(K(L,0,h)) \leqq h^*$ (see the proof of (a)); therefore, $Q_t(K(L,0,h)) \leqq h^* - h$, yielding $Q_t(K(L,0,h^*)) \leqq 0$. Accordingly, $h_t(L) \leqq h^*$ from Lemma 3(b,c). □

THEOREM 3. For all $t \geqq 1$,
(a) $B \supset H^* \supset H_t \supset C_t \supset H'_t$,
(b) If $(1-\beta)^2 + c^2 \neq 0$, then $B \supsetneqq H^*$. In addition, if $\beta\mu - c > a$, then $a < h_t < b$; hence, $B \supsetneqq H_t \supsetneqq A$,
(c) If $(1-\beta)^2 + c^2 = 0$, then $H^* = B$.

Proof: (a). $B \supset H^* \supset H_t$ is clear from Lemma 0(c,d) and Lemma 4(b). Consider any $K \in C_t$, for which $Q_t(K,j) \geqq 0$ for $j = 0,1,\cdots,N$. Then, if $K \notin H_t$, then $h_t < k_i$ for $k_i = k$, leading to the contradiction of

$$0 \leqq Q_t(K,i) = Q_t((k_0,\cdots,k_i,\cdots,k_N),i) \leqq Q_t((k_i,k_i,\cdots,k_i),i)$$
$$= Q_t(\{0,1,\cdots,N\},0,k_i)) < Q_t(K(\{0,1,\cdots,N\},0,h_t)) = 0.$$

Therefore, $K \in H_t$ must hold, implying $H_t \supset C_t$. $C_t \supset H'_t$ is obvious from Theorem 2(b2). (b). $B \supsetneqq H^*$ is clear from Lemma 0(c). If $\beta\mu - c > a$, then $h_t \geqq h_t(N) \geqq \beta\mu - c > a$ from Theorem 2(a) and (6.6), and $h_t \leqq h^* < b$ from Lemma 4(b) and Lemma 0(c). (c). Clear from Lemma 0(d). □

LEMMA 5. For all K, all L, and all i,
(a) $v_t(K)$ is increasing in t; therefore, so also are $Q_t(K,j)$ and $Q_t(K(L,i,h))$,
(b) $h_t(L)$ is increasing in t.

Proof: (a). Easily proved by induction starting with $v_1(K) \geqq k = v_0(K)$ for any K. (b). Clear from Lemma 3(b,c) and (a). □

THEOREM 4. We have

(a) S_t is decreasing in t; therefore, C_t is increasing in t,

(b) H_t and H'_t are increasing in t.

Proof: (a). Since $Q_t(K,j) < 0$ leads to $Q_{t-1}(K,j) < 0$ from Lemma 5(a), $S_{t+1}(j) \subset S_t(j)$ for all j, which implies $S_{t+1} \subset S_t$; therefore, $C_{t+1} \supset C_t$. (b). Immediate from Lemma 5(b). □

6.2. Structure of continuation region

LEMMA 6. For any continuous function $g(w)$, if $h > g(w)$ for $a < w < a + \varepsilon < b$ with an infinitesimal $\varepsilon > 0$, then $\int \max\{g(w),h\} dF(w) > \int g(w) dF(w)$.

Proof: Clear from $\max\{g(w),h\} > g(w)$ and $f(w) > 0$ for such w where $f(w)$ is the probability density function of $F(w)$. □

Let $K_a = (k_0,\cdots,k_{j-1},a,k_{j+1},\cdots,k_N)$ and $K_0 = (k_0,\cdots,k_{j-1},0,k_{j+1},\cdots,k_N)$ for $0 \leqq j \leqq N$, and let $G_a = (k_1,\cdots,k_{j-1},a,k_{j+1},\cdots,k_N)$ and $G_0 = (k_1,\cdots,k_{j-1},0,k_{j+1},\cdots,k_N)$ for $1 \leqq j \leqq N$. Let the maximum elements in G_a and G_0 be denoted by g_a and g_0, respectively. Then

LEMMA 7. $V_t(G_a) = V_t(G_0)$ and $U_t(K_a) = U_t(K_0)$ for all t and j.

Proof: Let $s = \max\{k_1,\cdots,k_{i-1},k_{i+1},\cdots,k_N\}$. Then, on $a < w < b$, we have $\max\{w,g_a\} = \max\{w,s,a\} = \max\{w,s,0\} = \max\{w,g_0\}$. Hence, $V_0(G_a) = \int \max\{w,g_a\}I(a<w<b) dF(w) = \int \max\{w,g_0\}I(a<w<b) dF(w) = V_0(G_0)$. Suppose $V_{t-1}(G_a) = V_{t-1}(G_0)$ for all pairs (G_a,G_0). Then, since $U_t(K_a) = U_t(K_0)$, we have $V_t(G_a) = \int v_t(k_0,G_a) dF(k_0) = \int \max\{\max\{k_0,g_a\}, U_{t-1}(K_a)\}I(a<k_0<b) dF(k_0) = \int \max\{\max\{k_0,g_0\}, U_{t-1}(K_0)\}I(a<k_0<b) dF(k_0) = \int v_t(k_0,G_0) dF(k_0) = V_t(G_0)$. □

In general, let $(x_1,x_2,\cdots,x_n)_j = (x_1,x_2,\cdots,x_n)$ with $x_j = h$ for $n \geqq j \geqq 1$.

LEMMA 8. Suppose $\beta\mu - c > a$, and let $h > h_t(N)$ $(\geqq a)$ for $t \geqq 1$. Then

$$V_t(0,\cdots,0,h\ 0,\cdots,0)_j > V_t, \quad 1 \leqq j \leqq N, \quad t \geqq 1.$$

Proof. We have $h_t(N) = U_t(0,0,\cdots,0,h_t(N)) = U_t(a,0,\cdots,0,h_t(N)) = U_t(a,0,\cdots,0)$ from (6.6), and Lemma 7, and $U_{t-1}(w_0,0,\cdots,0)$ is continuous and increasing in w_0 (Lemma 1(c)). Therefore, it follows that there exists an infinitesimal $\varepsilon > 0$ for which $h_t(N) \leqq U_t(w_0,0,\cdots,0) < h$ for $a < w_0 < a + \varepsilon < h$ with $a + \varepsilon < b$. This leads to $\max\{w_0, U_t(w_0,0,\cdots,0)\} < h$ for such w_0. Hence, from Lemma 6, we have for $1 \leqq j \leqq N$

$$\begin{aligned} V_t(0,\cdots,0,h,0,\cdots,0)_j &= \int \max\{\max\{w_0,h\}, U_t(w_0,0,\cdots,0,h,0,\cdots,0)_{j+1}\} dF(w_0) \\ &= \int \max\{\max\{w_0, U_t(w_0,0,\cdots,0,h,0,\cdots,0)_{j+1}\}, h\} dF(w_0) \\ &\geqq \int \max\{\max\{w_0, U_t(w_0,0,\cdots,0)\}, h\} dF(w_0) \\ &> \int \max\{w_0, U_t(w_0,0,\cdots,0)\} dF(w_0) \\ &= V_t(0,\cdots,0) = V_t. \quad \square \end{aligned}$$

THEOREM 5. Suppose $\beta\mu - c > a$. Then

(a) If $N = 1$, then $h_t = h_t(0)$ and $h_t > h_t(1)$ for all $t \geqq 1$,

(b) If $N \geqq 2$, then $h_t > h_t(L) > h_t(N)$ for all $t \geqq 1$ and for all L such as $L \neq \phi$(empty set), $L \neq \{N\}$, $L \neq \{0,1,\cdots,N-1\}$, and $L \neq \{0,1,\cdots,N\}$; hence, $h_t > h_t(j) > h_t(N)$ for $j = 0,1,\cdots,N-1$ and all $t \geqq 1$,

(c) If $p_j >(=) p_i$ for $i, j \in \{0,1,\cdots,N-1\}$, then $h_1(j) <(=) h_1(i)$.

Proof: (a). $Q_t(h,h) = 0$ and $Q_t(h,0) = 0$ are expressed as, respectively, $U_t(h,h) - h = 0$ and $U_t(h,0) - h = 0$. The two equations are identical because $U_t(k_0, k_1)$ is independent of k_1. Hence, $h_t = h_t(0)$. Consider any $h > h_t(1)$. Then since $h > h_{t-1}(1)$ from Lemma 5(b), we have $V_{t-1}(h) > V_{t-1}$ from Lemma 8. Therefore, for $h > h_t(1)$, we have $Q_t(h,h) = \beta(p_0V_{t-1} + (1-p_0)V_{t-1}(h)) - c - h > \beta V_{t-1} - c - h = Q_t(0,h)$, implying $h_t > h_t(1)$ from Lemma 3(b,c) (Figure 8(a)). (b). First, note the next three expressions:

$$U_t(K(\{0,1,\cdots,N\},0,h)) = \beta \sum_{G'\in\Delta(K(\{0,1,\cdots,N\},0,h))} P(G')V_{t-1}(G') - c \cdots(1^*),$$

$$U_t(K(L,0,h)) = \beta \sum_{G'\in\Delta(K(L,0,h))} P(G')V_{t-1}(G') - c \cdots(2^*),$$

$$U_t(K(\{N\},0,h)) = \beta \sum_{G'\in\Delta(K(\{N\},0,h))} P(G')V_{t-1}(G') - c \cdots(3^*).$$

Since $K(\{0,1,\cdots,N\},0,h) \geqq K(L,0,h)$ for any L, any term $P(G')V_{t-1}(G')$ in (1^*) is greater than or equal to the corresponding term in (2^*) from Lemma 1(c). Hence, $(1^*) \geqq (2^*)$. Since all $V_{t-1}(G')$ in (3^*) equal V_{t-1}, any term in (2^*) is greater than or equal to the corresponding term in (3^*). Therefore, $(2^*) \geqq (3^*)$. Next, let us show $(1^*) > (2^*) > (3^*)$ on $h > h_t(N)$. For the L defined in the lemma, let $D = L\cap\{0,1,\cdots, N-1\}$ and $D^* = L^c\cap\{0,1,\cdots,N-1\}$ where L^c is the complementary set of L. Here $D \neq \phi$ and $D^* \neq \phi$; its reason is as follows. If $D = \phi$ ($D^* = \phi$), then $L = \phi$ or $\{N\}$ ($L^c = \phi$ or $\{N\}$, i.e., $L = \{0,1,\cdots,N\}$ or $\{0,1,\cdots,N-1\}$), contradicting the assumption in the lemma. Below, assume $h > h_t(N)$; therefore, $h > h_{t-1}(N)$ from Lemma 5(b). First, consider the following corresponding terms in (1^*) and (2^*):

$$(\prod_{j\in D} p_j)\times(\prod_{j\in D^*}(1-p_j))V_{t-1}(G') \qquad (1^{**}),$$

$$(\prod_{j\in D} p_j)\times(\prod_{j\in D^*}(1-p_j))V_{t-1} \qquad (2^{**})$$

where, for any $j \in D^*$ $(\neq \phi)$, $V_{t-1}(G') \geqq V_{t-1}(0,\cdots,0,h,0,\cdots,0)_{j+1} > V_{t-1}$ from Lemma 8. Accordingly, it follows that $(1^{**}) > (2^{**})$; hence,

$$Q_t(K(\{0,1,\cdots,N\},0,h)) > Q_t(K(L,0,h)) \qquad (^{**}).$$

Similarly consider the following corresponding terms in (2^*) and (3^*):

$$(\prod_{j\in D^*} p_j)\times(\prod_{j\in D}(1-p_j))V_{t-1}(G') \qquad (2^{***}),$$

$$(\prod_{j\in D^*} p_j)\times(\prod_{j\in D}(1-p_j))V_{t-1} \qquad (3^{***}).$$

where, for any $j \in D\ (\neq \phi)$, $V_{t-1}(G') \geqq V_{t-1}(0,\cdots,0,h,0,\cdots,0)_{j+1} > V_{t-1}$ from Lemma 8. Therefore, we get $(2^{***}) \geqq (3^{***})$; hence,,

$$Q_t(K(L,0,h)) > Q_t(K(\{N\},0,h)) \cdots (^{***}).$$

Accordingly, it follows from $(^{**})$, $(^{***})$, and Lemma 3(b,c) that $h_t > h_t(L) > h_t(N)$ (Figure 8(b)). (c). For $0 \leqq j \leqq N-1$,

$$\begin{aligned} Q_1(K(\{j\},0,h)) &= \beta((1-p_j)V_0(0,\cdots,0,h,0,\cdots,0)_{j+1} + p_jV_0) - c - h \\ &= \beta(1-p_j)(V_0(0,\cdots,0,h,0,\cdots,0)_{j+1} - V_0) + \beta V_0 - c - h \\ &= \beta(1-p_j)(h + T(h) - \mu) + \beta\mu - c - h. \end{aligned}$$

Suppose $h > h_1(N)$. Then, since $h > a$ from (6.6), we have $h + T(h) - \mu > 0$ from Lemma 0(b). Therefore, it follows that $Q_0(K\{i\},0,h) >(=) Q_1(K(\{j\},0,h))$ for $p_j >(=) p_i$ with $i, j \in \{0,1,\cdots,N-1\}$ on $h > h_1(N)$. In addition, $Q_1(K(\{j\},0,h_1(N))) > Q_1(K(\{j\},0,h_1(j))) = 0$ because of $h_1(j) > h_1(N)$ from (b). Consequently, the assertion becomes true (Figure 8(c)). □

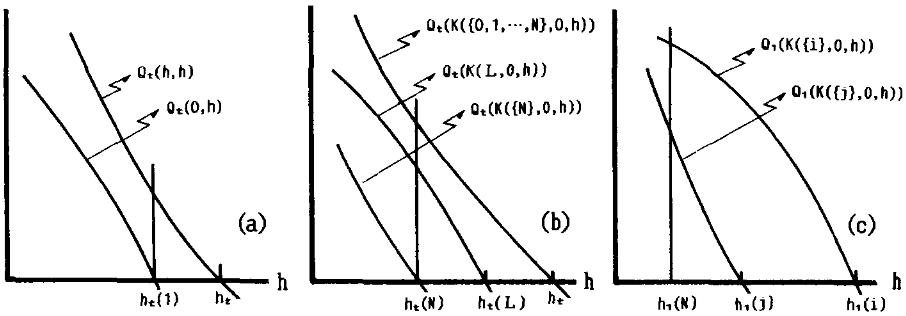

Fig. 8. (a) $h_t(1) < h_t$, (b) $h_t(N) < h_t(L) < h_1$, (c) $h_1(j) < h_1(i)$ if $p_j > p_i$

6.3. Continuation region near the limit of a planning horizon

First, let us summarize some well-known properties of the standard, discrete-time stopping problem, usually classified into two cases as already stated in Section 1: *with-no-recall case* and *with-recall case.*

Since with-no-recall case is the special case with $N = 0$ of with-uncertain-recall case, $v_t(k_0|0) = v_t(k_0)$ (see the definitions in e of Section 5). Let $V_{t-1}(-|0) = \int v_{t-1}(w|0)dF(w)$. Then we have

(6.7) $\quad v_t(k_0|0) = \max\{k_0, \beta V_{t-1}(-|0) - c\}, \quad t \geqq 1,$

where $v_0(k_0|0) = k_0$ and $V_0(-|0) = \mu$.

In with-recall case, the maximum expected discounted gain attained only depends on the best offer k so far. Hence, let us denote it by $v_t(k|1)$, and define $V_{t-1}(k|1) =$

$\int v_{t-1}(\max\{w,k\}|1)dF(w)$. Then we have

(6.8) $v_t(k|1) = \max\{k, \beta V_{t-1}(k|1) - c\}$, $t \geq 1$,

where $v_0(k|1) = k$ and $V_0(k|1) = k + T(k)$.

LEMMA 9. We have

(a) In with-no-recall case,

1. $v_t(k_0|0)$ and $V_{t-1}(-|0)$ increase in t and k_0 and converge to, respectively, $\max\{k_0,h^*\}$ and $(h^* + c)/\beta$ as $t \to \infty$,
2. The reservation value, $\beta V_{t-1}(-|0) - c$, converges to h^* as $t \to \infty$.

(b) In with-recall case,

1. $v_t(k|1)$ and $V_{t-1}(k|1)$ increase in t and k and converge to, respectively, $\max\{k,h^*\}$ and $\max\{k + T(k), h^* + T(h^*)\}$ as $t \to \infty$,
2. The reservation value are given by h^* for all $t \geq 1$.

Proof: Refer to [2], [13]. □

In with-no-recall case (with recall case), the optimal stopping rule is given as follows. If $k_0 > \beta V_{t-1}(-|0) - c$ ($k > h^*$), then stop with accepting the present offer k_0 (the best offer k); otherwise, continue.

LEMMA 10. For all t,

(a) $v_t(k|1) \geq v_t(K) \geq v_t(k_0|0)$,

(b) $V_t(g|1) \geq V_t(G) \geq V_t(-|0)$,

(c) $h_t(L) \geq \beta V_{t-1}(-|0) - c \geq \beta\mu - c$ (> 0) for all L,

(d) $H_t' \supset M$.

Proof: (a,b). $v_0(k|1) = k = v_0(K) \geq k_0 = v_0(k_0|0)$, from which $V_0(G) \geq V_0(-|0)$. Since $v_0(\max\{k_0,g\}|1) = \max\{k_0,g\} = v_0(k_0,G)$, we have $V_0(g|1) = V_0(G)$. Thus, (a) and (b) are true for $t = 0$. Suppose that (b) is true for $t-1$. Then since $V_{t-1}(G') \geq V_{t-1}(-|0)$ for all G', we have $U_t(K) \geq \beta V_{t-1}(-|0) - c$. In addition, since $k \geq k_0$, we have $v_t(K) \geq \max\{k_0,\beta V_{t-1}(-|0) - c\} = v_t(k_0|0)$, from which $V_t(G) \geq V_t(-|0)$. Since $V_{t-1}(G') \leq V_{t-1}(g'|1) \leq V_{t-1}(k|1)$ due to $g' \leq k$, we have $U_t(K) \leq \beta V_{t-1}(k|1) - c$, from which $v_t(K) \leq \max\{k,\beta V_{t-1}(k|1) - c\} = v_t(k|1)$. This is written as $v_t(k_0,G) \leq v_t(\max\{k_0,g\}|1)$; therefore, $V_t(G) \leq V_t(g|1)$. Thus the induction completes. (c). Since $Q_t(K(L,0,h)) \geq \beta V_{t-1}(-|0) - c - h$ from (b), we get $h_t(L) \geq \beta V_{t-1}(-|0) - c$ from Lemma 3(b,c) (Figure 9). Furthermore, since $V_t(-|0)$ is increasing in t, $\beta V_{t-1}(-|0) - c \geq \beta V_0(-|0) - c = \beta\mu - c$. (d). Clear from $h_t(j) \geq \beta\mu - c$ for all j from (c). □

THEOREM 6. We have

(a) If $K \in H^*$, then $v_t(K)$ converge to h^* as $t \to \infty$,

(b) $h_t(L)$ converge to h^* as $t \to \infty$ for all L,

(c) H_t, C_t, and H_t' converge to H^* as $t \to \infty$,

(d) If $\beta\mu - c \leq a$, then $H^* = H_t = C_t = H_t' = M \subset A$ for all $t \geq 1$.

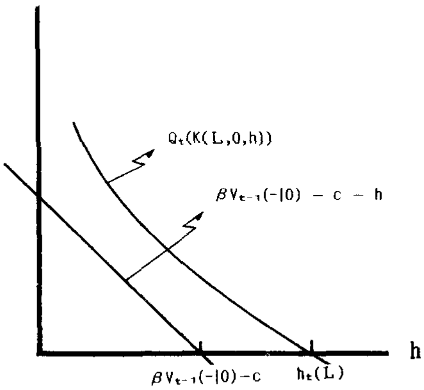

Fig. 9 $\beta v_{t-1}(-|0)-c < h_t(L)$

Proof: (a). Suppose $K \in H^*$, implying $k \leqq h^*$; hence, $k_0 \leqq h^*$. Then from Lemma 9(a1,b1), $v_t(k_0|0)$ and $v_t(k|1)$ tend to h^* as $t \to \infty$; therefore, so also does $v_t(K)$ from Lemma 10(a). (b). Since $\beta V_{t-1}(-|0) - c \to h^*$ as $t \to \infty$ from Lemma 9(a1), it follows from Lemma 4(b) and Lemma 10(c) that $h_t(L) \to h^*$ as $t \to \infty$. (c). From (b), h_t and $h_t(j)$ for $j = 0,1,\cdots,N$ converge to h^*, implying that both H_t and H_t' converge to H^*. Consequently, C_t also converge to H^* from Theorem 3(a). (d). Since the assumption in the lemma means $(1-\beta)^2 + c^2 \neq 0$, we have $h^* = \beta\mu - c \leqq a$ from Lemma 0(c); therefore, $H^* = M \subset A$. From this, Lemma 10(d), and Theorem 3(a), the assertion becomes true for all t. □

6.4. DRV-property

LEMMA 11. If $(1-\beta)^2 + c^2 = 0$, then $V_{t-1}(G') < b$ for all t and all G' with $g' < b$.

Proof: For any G' with $g' < b$, we have $V_0(G') = g' + T(g') < b + T(b) = b$ from Lemma 0(b). Suppose the assertion is true for a certain t. Then we shall show that, for G with $g < b$, the equality $b = V_t(G) = \int v_t(w_0,G)I(a<w_0<b)dF(w_0)$ leads to a contradiction. Since $b \geqq v_t(K)$ for all K (see the proof of Lemma 1(b)), the above equality yields $b = v_t(w_0,G)$ $(= v_t(K)) = \max\{k, \Sigma P(G')V_{t-1}(G')\}$ on $a < w_0 < b$, from which $\Sigma P(G')V_{t-1}(G') = b$ because $k = \max\{w_0,g\} < b$ on $a < w_0 < b$. Consequently, it must follow from Lemma 1(b) that $V_{t-1}(G') = b$ for all G' where $g' < b$ because $g' \leqq k = \max\{w_0,g\} < b$ on $a < w_0 < b$. This contradicts the induction hypothesis. Therefore, it must be that $V_t(G) < b$ for G with $g < b$. Thus, the induction completes. □

Let $K[w_0] = (w_0,h_t(L)+\varepsilon,\cdots,h_t(L)+\varepsilon)$ with $L = \{1,2,\cdots,N\}$ where ε is an positive infinitesimal number such as $h_t(L)+\varepsilon < h_t$; the existence of such ε is obvious from Theorem 5(a,b). Then

LEMMA 12. If $\beta\mu - c > a$. Then, for all $t \geq 1$, we get $Q_t(K[h_t(L)+\varepsilon],j) > 0$ for $0 \leqq j \leqq N$, $Q_t(K[a],N) < 0$, and $Q_t(K[b],0) < 0$.

Proof: For $0 \leqq j \leqq N$, we have $Q_t(K[h_t(L)+\varepsilon],j) = Q_t(K(\{0,1,\cdots,N\},0,h_t(L)+\varepsilon)) > Q_t(K(\{0,1,\cdots,N\},0,h_t)) = 0$. From Lemma 7, $Q_t(K[a],N) = Q_t(K[0],N) = Q_t(0, h_t(L)+\varepsilon,\cdots,h_t(L)+\varepsilon) = Q_t(K(L,0,h_t(L)+\varepsilon)) < Q_t(K(L,0,h_t(L)) = 0$ (Lemma 3(b)). If $(1-\beta)^2 + c^2 = 0$, then $Q_t(K[b],0) < b-b = 0$ from Lemma 1(b) and $V_{t-1}(0,0,\cdots,0) < b$ (Lemma 11). If $(1-\beta)^2 + c^2 \neq 0$, then $Q_t(K[b],0) \leqq \beta b-c-b < 0$. □

THEOREM 7. The necessary and sufficient condition for the optimal stopping rule to have DRV-property for all $t \geqq 1$ is $\beta\mu - c > a$.

Proof: When $\beta\mu - c \leqq a$, it is clear that the optimal stopping rule has not DRV-property because the continuation region C_t is then reduced to a perfect cube M for all t from Theorem 6(d). Below assume $\beta\mu - c > a$. Suppose the search process starts when in state $K[w_0]$. Here, note that $Q_t(K[w_0],N)$ and $Q_t(K[w_0],0)$ are continuous functions of w_0, respectively, increasing in w_0 and decreasing in w_0 (Lemma 1(d) and Lemma 2(b)). Then, it is easily seen from these and Lemma 12 that there exist finite numbers ξ and ξ' such as $\xi = \sup\{w_0 | Q_t(K[w_0],N) < 0\}$ and $\xi' = \inf\{w_0 | Q_t(K[w_0],0) < 0\}$ where $a < \xi < h_t(L)+\varepsilon < \xi' < b$ (Figure 10). If $w_0 < \xi$, then $Q_{t-1}(K[w_0],N) < 0$, implying $K[w_0] \in S_t(N) \subset S_t$. If $\xi' < w_0$, then $Q_{t-1}(K[w_0],0) < 0$, meaning $K[w_0] \in S_t(0) \subset S_t$. If $\xi \leqq w_0 \leqq \xi'$, then $Q_{t-1}(K[w_0],0) \geqq 0$ and $Q_{t-1}(K[w_0],j) = Q_{t-1}(K[w_0],N) \geqq 0$ for $1 \leqq j \leqq N$; therefore, $K(w_0) \notin S_t$ for $0 \leqq j \leqq N$, leading to $K[w_0] \in C_t$. Consequently, it follows that if either $w_0 < \xi$ or $\xi' < w_0$, then stop; otherwise, continue. Thus the proof completes. □

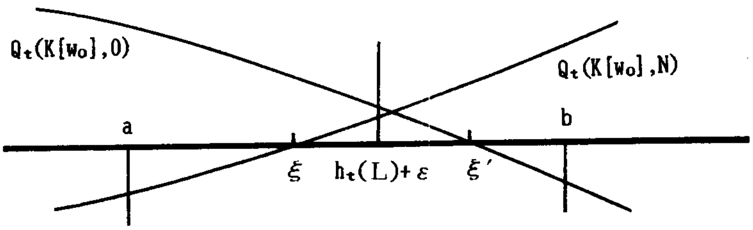

Fig. 10 $a < \xi < h_t(L)+\varepsilon < \xi' < b$

6.5. Search amount and value realization

The approach employed here is a generalization of that in [6]. First, the following are clear from the definitions in e of Section 5: $\rho_0(K) = 0$ for all K, $\rho_t(K) = 0$ for all $K \in S_t$ and all $t \geq 1$, $\rho_0(k|1) = 0$ for all k, and $\rho_t(k|1) = 0$ for all $k > h^*$ and all $t \geq 1$. For all $t \geq 1$, if $K \in C_t$ ($\subset H^*$), then

$$(6.9) \qquad \rho_t(K) = 1 + \sum_{G' \in \Delta(K)} P(G') \int \rho_{t-1}(w,G') I((w,G') \in H^*)\, dF(w).$$

Similarly for all $t \geqq 1$, if $k \leqq h^*$, then

$$(6.10) \qquad \rho_t(k|1) = 1 + \rho_{t-1}(k|1)F(h^*).$$

THEOREM 8. We have

(a) $\rho_t(K) \leqq \rho_t(k|1)$ for all $t \geqq 0$ and all K,

(b) $\rho_t(K)$ and $\rho_t(k|1)$ are increasing in t for all K,

(c) Suppose $(1-\beta)^2 + c^2 \neq 0$. Then

1. If $K \in H^*$, then $\rho_t(K)$ and $\rho_t(k|1)$ converge as $t \to \infty$ where the limit of $\rho_t(k|1)$ is $(1-F(h^*))^{-1}$,
2. If $\beta\mu - c > a$, let K be an inner point of H^*; otherwise, let $K \in H^*$. Then $\rho_t(K)$ converges to $(1-F(h^*))^{-1}$ as $t \to \infty$.

Proof: (a). Clear for $t = 0$. Suppose the assertion is true for $t-1$. If $K \in S_t$, then $\rho_t(K) = 0 \leqq \rho_t(k|1)$. Assume $K \in C_t$ ($\subset H^*$), so $k \leqq h^*$. Now since $\rho_t(k|1)$ is constant on $k \leqq h^*$, we shall denote it by $\rho_t(-|1)$. Then we have

$$\begin{aligned}\rho_t(K) &\leqq 1 + \sum_{G'\in\Delta(K)} P(G')\int \rho_{t-1}(\max\{w,g'\}|1)I((w,G')\in H^*)\,dF(w)\\ &= 1 + \rho_{t-1}(-|1)\sum_{G'\in\Delta(K)} P(G')\int I((w,G')\in H^*)\,dF(w)\\ &= 1 + \rho_{t-1}(k|1)\sum_{G'\in\Delta(K)} P(G')\int I(w\leqq h^*)\,dF(w)\\ &= 1 + \rho_{t-1}(k|1)F(h^*) = \rho_t(k|1).\end{aligned}$$

(b). $\rho_1(K) \geqq 0 = \rho_0(K)$ for all K. Suppose $\rho_{t-1}(K) \geqq \rho_{t-2}(K)$ for all K. If $K \in S_{t-1}$, then $\rho_t(K) \geqq 0 = \rho_{t-1}(K)$. If $K \in C_{t-1}$, then since $K \in C_t$ from Theorem 4(a),

$$\rho_t(K) \geqq 1 + \sum_{G'\in\Delta(K)} P(G')\int \rho_{t-2}(w,G')I((w,G')\in H^*)\,dF(w) = \rho_{t-1}(K).$$

Almost similarly proved also for $\rho_t(k|1)$. (c). First, note $F(h^*) < 1$ because of $h^* < b$ from Lemma 0(c). (c1). Suppose $K \in H^*$. Then, from (6.10) and (a), immediately we get $\rho_t(K) \leqq \rho_t(k|1) = (1-F(h^*)^t)/(1-F(h^*)) \leqq (1-F(h^*))^{-1} \cdots (*)$ for all t; therefore, $\rho_t(K)$ and $\rho_t(k|1)$ are bounded above. Thus it follows that $\rho_t(K)$ and $\rho_t(k|1)$ converge in t. It is clear from (*) that $\rho_t(k|1)$ converges to $(1-F(h^*))^{-1}$. (c2). Let the limit of $\rho_t(K)$ be denoted by $\rho(K)$. If $\beta\mu - c > a$, then for any inner point K of H^*, there exists such an integer $T > 0$ that $K \in C_t$ for all $t \geqq T$; otherwise, for any $K \in H^*$, we have $K \in C_t$ for all t because of $C_t = H^*$ for all t from Theorem 6(d). Accordingly, for any K defined in the lemma, since (6.9) holds for all t that are sufficiently large, we have

$$\rho(K) = 1 + \sum_{G'\in\Delta(K)} P(G')\int \rho(w,G')I((w,G')\in H^*)\,dF(w).$$

Now suppose the equation has two different solutions, $\rho(K)$ and $\tau(K)$, and let $\delta = \sup_{K\in K^*}|\rho(K)-\tau(K)| > 0$ where K^* is the set of K defined in the lemma. Then, from the above equation, immediately we get $\delta \leqq \delta F(h^*)$, leading to the contradiction of $1 \leqq F(h^*) < 1$. Hence, the solution must be unique. It is easy to see that $(1-F(h^*))^{-1}$ satisfies the above equation. Accordingly, $\rho_t(K)$ must converge to $(1-F(h^*))^{-1}$. □

From the definitions, we have, for all $t \geqq 0$ and all K,

(6.11) $$v_t(K) = \upsilon_t(K) - c\rho_t(K),$$

(6.12) $$v_t(k|1) = \upsilon_t(k|1) - c\rho_t(k|1).$$

THEOREM 9. We have

(a) $\upsilon_t(K) \leqq \upsilon_t(k|1)$ for all $t \geqq 0$ and all K,

(b) $\upsilon_t(K)$ and $\upsilon_t(k|1)$ are increasing in t for all K,

(c) Suppose $(1-\beta)^2 + c^2 \neq 0$. Then

1. If $K \in H^*$, then $\upsilon_t(K)$ and $\upsilon_t(k|1)$ converge as $t \to \infty$, where the limit of $\upsilon_t(k|1)$ is $h^* + c(1-F(h^*))^{-1}$.
2. If $\beta\mu - c > a$, let K be an inner point of H^*; otherwise, let $K \in H^*$. Then $\upsilon_t(K)$ converges to $h^* + c(1-F(h^*))^{-1}$ as $t \to \infty$.

Proof: (a). $\upsilon_t(K) - \upsilon_t(k|1) = v_t(K) - v_t(k|1) + c(\rho_t(K) - \rho_t(k|1)) \leqq 0$ from Lemma 10(a) and Theorem 8(a). (b). Obvious from Theorem 8(b), Lemma 5(a), Lemma 9(b1), and (6.11). (c). Clear from Theorem 6(a), Lemma 9(b1), Theorem 8(c), (6.11), and (6.12). □

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