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AN ITERATIVE METHOD FOR VARIATIONAL INEQUALITIES WITH APPLICATION TO TRAFFIC EQUILIBRIUM PROBLEMS

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Abstract Variational inequalities have extensively been studied to formulate equilibrium problems which arise in many fields including economics and operations research. Also, various numerical methods such as projection method and diagonalization method have recently been developed for the solution of variational inequalities. This paper considers modified variational inequalities which allow the constraint set to be non-convex and therefore contain classical variational inequalities as a special case. First, a solution method is presented for unconstrained problems and conditions for global convergence are established. Then, for inequality constrained variational inequalities, a solution method is proposed by modifying the multiplier methods for constrained optimization, and its convergence property is examined. When this method is applied to a dual formulation of the asymmetric traffic equilibrium problem, in which variables are travel costs and all constraints are inequalities, path flows can be obtained as the optimal Lagrange multipliers of the variational inequality problem. Finally, some numerical examples containing traffic equilibrium problems of medium size are solved to exhibit the effectiveness of the proposed methods.

1. Introduction

Many equilibrium problems in operations research and economics can be formulated as variational inequalities and numerical methods for solving them have also been extensively studied [3,4,10]. Most of the existing methods such as projection methods [3,4] and diagonalization methods [5,7] find a solution of variational inequalities by successively solving symmetric variational inequalities, for which equivalent minimization problems exist. Thus, these methods are doubly iterative, because they require at each major iteration solving a minimization subproblem by another suitable iterative method. Also, the amount of computation may considerably increase as problems become large. Recently, a singly iterative method called the relaxed projection method [8] was proposed for asymmetric variational inequalities. Unlike the above mentioned methods, this method generates a sequence of points converging to a solution by a very simple iterative scheme, but the generated sequence is not necessarily contained in the constraint set. Also, Hammond and Magnanti [12] consider systems of nonlinear equations from the viewpoint of variational inequalities, and present a singly iterative solution method which is a modification of the steepest descent method in unconstrained optimization. But this method can only solve unconstrained problems and requires the mapping involved to satisfy more restrictive conditions than those for existence and uniqueness of a solution.

In this paper, we first present a formulation of modified variational inequalities which may contain unlike the classical variational inequalities non-convex constraint sets. Such modified variational inequalities are also considered in Fukushima [9] and may be reduced to the usual variational inequalities, when constraint sets are convex.

Next, we consider unconstrained variational inequalities and present a method of steepest descent type. This method differs from the one proposed by Hammond and Magnanti [12] only in the line search criterion and this difference yields weaker convergence conditions than those of [12].

For the modified variational inequalities with inequality constraints, we then propose a solution method which is a modification of the multiplier methods for constrained optimization. This method updates solutions and Lagrange multipliers alternatively and can be implemented very easily. We give some updating schema for Lagrange multipliers and establish convergence conditions for the method with a particular multiplier updating scheme.

As an application, the last method is used to solve the dual formulation of the traffic equilibrium problem in which variables are arc costs and costs for origin-destination (O/D) pairs and Lagrange multipliers correspond to path flows [10]. Since this problem may contain an enormous number of constraints, we also incorporate a technique based on a shortest path algorithm which generates only needed constraints at each iteration.

Finally, we report numerical results for some examples including a traffic equilibrium problems of medium size. These results indicate that the proposed methods are practical and effective in solving asymmetric variational inequality problems.

Throughout the paper, $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ denote the inner product and the ordinary Euclidean norm in R^n, respectively. Also, transposition of a matrix is denoted by T.

2. Modified Variational Inequalities

The classical variational inequality problem is to find a point $\boldsymbol{x} \in \boldsymbol{S}$ such that

$$\langle \boldsymbol{F}(\boldsymbol{x}) , \boldsymbol{x}' - \boldsymbol{x} \rangle \geq 0 \quad \text{for all } \boldsymbol{x}' \in \boldsymbol{S}, \tag{2.1}$$

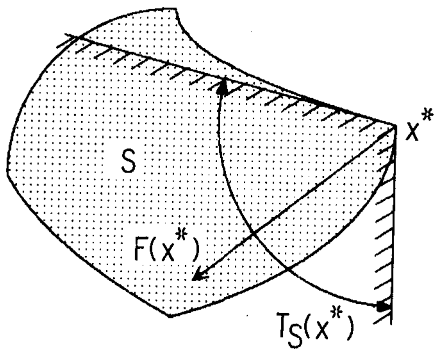


Figure 1.

where $\boldsymbol{S}$ is a nonempty closed set in R^n and $\boldsymbol{F}$ is a mapping from R^n into R^n, and the work on variational inequalities has primarily been concerned with problems of the form (2.1). However, this formulation is adequate only when the constraint is convex. Here we consider, as in [9] a modification of (2.1) which allows the constraint set $\boldsymbol{S}$ to be non-convex. Specifically, the problem is to find a point $\boldsymbol{x} \in \boldsymbol{S}$ such that

$$\langle \boldsymbol{F}(\boldsymbol{x}) , \boldsymbol{y} \rangle \geq 0 \quad \text{for all } \boldsymbol{y} \in \boldsymbol{T_S}(\boldsymbol{x}), \tag{2.2}$$

where $\boldsymbol{T_S}(\boldsymbol{x})$ is the tangent cone [16] of $\boldsymbol{S}$ at $\boldsymbol{x}$, i.e.,

$$\boldsymbol{T_S}(\boldsymbol{x}) = \{ \boldsymbol{y} \mid \forall \boldsymbol{x}^k \to \boldsymbol{x},\ \boldsymbol{x}^k \in \boldsymbol{S},\ t_k \downarrow 0,\ \exists \boldsymbol{y}^k \to \boldsymbol{y} \text{ with } \boldsymbol{x}^k + t_k \boldsymbol{y}^k \in \boldsymbol{S} \}. \tag{2.3}$$

Of course, when the set $\boldsymbol{S}$ is convex, problem (2.1) and (2.2) are equivalent.

From now on, we assume that the mapping $\boldsymbol{F}$ is continuously differentiable and the set $\boldsymbol{S}$ consists of points $\boldsymbol{x}$ satisfying

$$g_i(\boldsymbol{x}) \leq 0, \quad i = 1,2,\dots,m, \tag{2.4}$$

where g_i: $R^n \to R$ are twice continuously differentiable.

Let us consider the next problem:

Find $\boldsymbol{x} \in R^n$, $u_i \in R$, $i = 1,2,\dots,m$, such that

$$\boldsymbol{F}(\boldsymbol{x}) + \sum_{i=1}^{m} u_i \nabla g_i(\boldsymbol{x}) = 0, \tag{2.5}$$

$$g_i(\boldsymbol{x}) \leq 0, \quad u_i \geq 0, \quad u_i g_i(\boldsymbol{x}) = 0, \quad i = 1,2,\dots,m.$$

Under a suitable constraint qualification such as linear independence of active constraint gradients, the problem (2.5) is easily shown to be equivalent to the modified variational inequality problem (2.2).

We define the mapping $H(x,u)$ from R^{n+m} into R^n by:

$$H(x,u) = F(x) + \sum_{i=1}^{m} \max(\ 0\ ,\ u_i + \sigma g_i(x)\ )\nabla g_i(x), \tag{2.6}$$

where σ is a positive penalty parameter. It may easily be shown that any solution (x^*,u^*) of (2.5) also solves the equation

$$H(x,u) = 0.$$

We shall prove that under some assumptions the augmented mapping $H(x,u)$ is locally monotone with respect to x near the solution. To establish this result, we need the next lemma.

Lemma 1. (Finsler) Let B and A are n by n and n by m matrices, respectively. Then

$$y^T B y > 0 \quad \text{for all} \quad y \in \{\ y \mid A^T y = 0\ \},\ y \neq 0,$$

if and only if, there exists $r_0 \geqq 0$ such that

$$y^T(\ B + rAA^T\ )y > 0 \quad \text{for all} \quad y \in R^n,\ y \neq 0,\ r \geqq r_0. \quad \square$$

Theorem 1. Let $x^* \in R^n$ and $u_i^* \in R$, $i=1,2,\ldots,m$, be the solution of (2.5). Assume that

$$u_i^* > 0 \quad \text{for all} \quad i \in I(x^*)$$

and

$$<\ y\ ,\ \{\ \nabla F(x^*) + \sum_{i=1}^{m} u_i^* \nabla^2 g_i(x^*)\ \}^T\ y\ >\ > 0, \tag{2.7}$$

for all $y \in C(x^*)$, $y \neq 0$,

where

$$C(x) = \{\ y \in R^n \mid <\ \nabla g_i(x)\ ,\ y\ > = 0,\ \ i \in I(x)\ \},$$
$$I(x) = \{\ i \mid g_i(x) = 0,\ i = 1,2,\ldots,m\ \}.$$

Then for all σ large enough, the matrix $\nabla_x H(x^*,u^*)$, the Jacobian of H with respect to x, is positive definite.

Proof: Since

$$\nabla_x H(x^*,u^*) = \nabla F(x^*) + \sum_{i=1}^{m} u_i^* \nabla^2 g_i(x^*) + \sigma \sum_{i \in I(x^*)} \nabla g_i(x^*) \nabla g_i(x^*)^T,$$

Lemma 1 guarantees that for all sufficiently large σ, $\nabla_x H(x^*, u^*)$ is positive definite. □

This theorem states that $H(x, u^*)$ is a locally monotone mapping of x in a neighborhood of x^*. Note that x^* is a solution of

$$H(x, u^*) = 0,$$

we may therefore obtain x^* by applying such an algorithm to this equation that accounts for the monotonicity of the mapping $H(x, u^*)$. This fact motivates the solution method for the modified variational inequalities to be developed in Section 4.

3. Method for Unconstrained Problems

In this section, we present a solution method for unconstrained variational inequality problems, which are actually systems of nonlinear equations

$$F(x) = 0, \tag{3.1}$$

where $F: R^n \to R^n$. Note that if the Jacobian of F is symmetric for each x, then there exists a function $f: R^n \to R$ such that $F(x) = \nabla f(x)$, and hence (3.1) is considered the first order necessary condition for optimality of the unconstrained problem

$$\text{minimize} \quad f(x) \quad \text{over} \quad x \in R^n. \tag{3.2}$$

Furthermore, if F is monotone, then (3.1) and (3.2) are equivalent because (3.2) becomes a convex minimization problem.

From an optimization viewpoint, Hammond and Magnanti [12] propose a descent-like method for solving problem (3.1) that involves an asymmetric mapping. The following is an outline of this method in its simplest form. For a given point x, a search direction d is chosen as $-F(x)$, and line search is then performed to determine the step length α satisfying the condition

$$\langle F(x + \alpha d), d \rangle = 0. \tag{3.3}$$

The next iterate is set $x + \alpha d$. (A similar method is considered in [13, p.164].)

Note that, in the symmetric case, the search direction $d = -F(x)$ is just the steepest descent direction of the objective function f of (3.2) and the criterion (3.3) corresponds to the exact line search rule

$$\langle \nabla f(x + \alpha d), d \rangle = 0$$

in unconstrained minimization. Hammond and Magnanti [12] also consider a generalization of the above method in which the search direction is chosen as $\boldsymbol{d} = -\boldsymbol{G}\cdot\boldsymbol{F}(\boldsymbol{x})$, where $\boldsymbol{G}$ is a nonsingular scaling matrix, and the line search criterion is modified to be

$$< \boldsymbol{G}\cdot\boldsymbol{F}(\boldsymbol{x} + \alpha\boldsymbol{d}) \ , \ \boldsymbol{d} > = 0. \tag{3.4}$$

They show that a convergence condition for this method is the positive definiteness of both $\boldsymbol{G}\cdot\nabla\boldsymbol{F}(\boldsymbol{x})$ and $(\boldsymbol{G}\cdot\nabla\boldsymbol{F}(\boldsymbol{x}))^2$. In what follows, we consider a line search criterion which slightly differs from (3.4), and show that such a modification yields a weaker convergence condition.

Now consider the following algorithm.

Algorithm 1

Step 0. Choose an n by n nonsingular matrix $\boldsymbol{G}$ and an initial point $\boldsymbol{x}^0$. Set $k := 0$.

Step 1. Calculate a search direction $\boldsymbol{d}^k = -\boldsymbol{G}\cdot\boldsymbol{F}(\boldsymbol{x}^k)$. If $\|\boldsymbol{d}^k\|$ is small enough, then terminate.

Step 2. Determine a step length α, such that

$$< \nabla\boldsymbol{F}(\boldsymbol{x}^k + \alpha\boldsymbol{d}^k)^{\mathrm{T}}\cdot\boldsymbol{G}^{\mathrm{T}}\cdot\boldsymbol{G}\cdot\boldsymbol{F}(\boldsymbol{x}^k + \alpha\boldsymbol{d}^k) \ , \ \boldsymbol{d}^k > = 0, \tag{3.5}$$

and set $\boldsymbol{x}^{k+1} := \boldsymbol{x}^k + \alpha\boldsymbol{d}^k$. Set $k := k + 1$, and return to Step 1. □

Note that finding α such that (3.5) is satisfied amounts to solving the one-dimensional problem

$$\text{minimize} \quad \|\boldsymbol{G}\cdot\boldsymbol{F}(\boldsymbol{x}^k + \alpha\boldsymbol{d}^k)\|^2.$$

Convergence properties of this algorithm are seriously affected by the eigenvalue structure of the matrices $\nabla\boldsymbol{F}(\boldsymbol{x})\cdot\boldsymbol{G}^{\mathrm{T}}$. In fact, if we can choose $\nabla\boldsymbol{F}(\boldsymbol{x})\cdot\boldsymbol{G}^{\mathrm{T}}$ in such a way that it is positive definite and has a small condition number, then convergence of the above algorithm may be greatly enhanced. A practical choice of the matrix $\boldsymbol{G}$ would be, for example,

$$\boldsymbol{G} = (\nabla\boldsymbol{F}(\boldsymbol{x}^0)^{-1})^{\mathrm{T}},$$

or more simply

$$\boldsymbol{G} = \boldsymbol{D}^{-1},$$

where $\boldsymbol{D}$ is the diagonal matrix which consists of the diagonal part of the matrix $\nabla\boldsymbol{F}(\boldsymbol{x}^0)$. The latter choice seems better from a practical viewpoint and is adopted in the numerical examples of Section 6.

We establish a global convergence theorem for the above method.

Theorem 2. Suppose that $\boldsymbol{G}$ is chosen such that $\boldsymbol{G}\cdot\boldsymbol{F}(\boldsymbol{x})$ is strongly monotone or uniformly monotone, that is, there exists $\beta > 0$ such that

$$< G\cdot F(x^1) - G\cdot F(x^2) \ , \ x^1 - x^2 > \geq \beta \ \| x^1 - x^2 \|^2 \quad \text{for all} \ x^1 \ , \ x^2 \in R^n. \tag{3.6}$$

Then, $\{x^k\}$ generated by Algorithm 1 globally converges to the unique solution of (3.1).

Proof: Define $h:R^n \to R$ by

$$h(x) = \frac{1}{2} \ \| G\cdot F(x) \|^2.$$

Since

$$\nabla h(x) = \nabla F(x)\cdot G^T\cdot G\cdot F(x) \tag{3.7}$$

and (3.6) implies that $\nabla F(x)\cdot G^T$ and G are both nonsingular, the system of equation (3.1) is equivalent to the problem

$$\text{minimize} \ \ h(x) \ . \tag{3.8}$$

It is well known [14, p.142] that (3.6) implies that, for any x^k,

$$< y \ , \ \nabla F(x^k)\cdot G^T y > \geq \beta \| y \|^2 \quad \text{for all} \ \ y \in R^n. \tag{3.9}$$

Since $d^k = -G\cdot F(x^k)$, it then follows from (3.7) and (3.9) that

$$\begin{aligned} < d^k \ , \ \nabla h(x^k) > &= - < G\cdot F(x^k) \ , \ \nabla F(x^k)\cdot G^T\cdot G\cdot F(x^k) > \\ &\leq - \beta \| G\cdot F(x^k) \|^2 < 0. \end{aligned} \tag{3.10}$$

Consequently, we conclude that the search direction d^k is a direction of sufficient decrease of the objective function $h(x)$ of (3.8). Moreover, by (3.7), we see that (3.5) is a restatement of the condition

$$< d^k \ , \ \nabla h(x^k + \alpha d^k) > = 0, \tag{3.11}$$

that is, the line search in Step 2 is the exact minimization of the function $h(x^k + \alpha d^k)$ with respect to α.

We define the level set T by

$$T = \{ \ x \in R^n \ | \ \| G\cdot F(x) \|^2 \leq \| G\cdot F(x^0) \|^2 \ \},$$

then strong monotonicity condition (3.6) implies that T is compact set. In view of Theorem 14.3.2 of [14], global convergence of the algorithm follows form (3.10) and (3.11). □

4. Method for Inequality Constrained Problems

In this section, we propose a solution method for the modified variational inequality problem (2.2) in which the set S is specified by the system of inequalities (2.4). For this problem, we also consider a modification of methods for optimization problems. Specifically, we focus upon the multiplier methods in nonlinear programming and extend them to the modified variational inequalities.

We define the augmented mapping $\boldsymbol{H}(\boldsymbol{x},\boldsymbol{u})$ by (2.6). The variables $\boldsymbol{u}$ are Lagrange multipliers associated with the inequality constraints (2.4). If $(\boldsymbol{x}^*,\boldsymbol{u}^*)$ satisfies (2.5), then $(\boldsymbol{x}^*,\boldsymbol{u}^*)$ is a solution of

$$\boldsymbol{H}(\boldsymbol{x},\boldsymbol{u}) = 0.$$

Moreover, if the assumptions of Theorem 1 are satisfied, then $\boldsymbol{H}(\boldsymbol{x},\boldsymbol{u}^*)$ is a monotone mapping of $\boldsymbol{x}$ in a neighborhood of $\boldsymbol{x}^*$. Therefore, if the value of $\boldsymbol{u}^*$ is *a priori* known, we may obtain $\boldsymbol{x}^*$ as a solution of the unconstrained problem

$$\boldsymbol{H}(\boldsymbol{x},\boldsymbol{u}^*) = 0,$$

by using the method described in the previous section. In practice, however, the exact value of $\boldsymbol{u}^*$ cannot be known *a priori*. To obtain an estimate $\boldsymbol{x}$ of the solution $\boldsymbol{x}^*$, we may therefore apply a single iteration of the method of Section 3 to the mapping $\boldsymbol{H}(\boldsymbol{x},\boldsymbol{u})$ with $\boldsymbol{u}$ being fixed. For the value of $\boldsymbol{x}$ thus obtained, we then calculate a new estimate of $\boldsymbol{u}^*$ by updating the value of $\boldsymbol{u}$. We repeat these two steps alternatively. This method is explicitly stated as follows:

Algorithm 2

Step 0. Choose $\boldsymbol{x}^0 \in R^n$ and $\boldsymbol{u}^0 \in R^m$, and set $k := 0$.

Step 1. Calculate the search direction $\boldsymbol{d}^k = -\boldsymbol{H}(\boldsymbol{x}^k,\boldsymbol{u}^k)$. If $\|\boldsymbol{d}^k\|$ and $|\max(-u_i^k, \sigma g_i(\boldsymbol{x}^k))|$, $i=1,2,\ldots,m$, are all small enough, then terminate.

Step 2. Obtain the step length α_k, and set $\boldsymbol{x}^{k+1} := \boldsymbol{x}^k + \alpha_k\boldsymbol{d}^k$.

Step 3. Update $\boldsymbol{u}^k$ to get $\boldsymbol{u}^{k+1}$, set $k := k + 1$, and return to Step 1. □

In this algorithm, several choices are possible in the way of selecting the step length α_k and updating the Lagrange multiplier estimate $\boldsymbol{u}^k$. Concerning the step length, we may either simply use a fixed step length or perform the line search using the same criterion as in Algorithm 1.

As far as the update of $\boldsymbol{u}^k$ is concerned, we may choose one of the following schemes:

(1) First order iteration: This is a generalization of the first order iteration scheme used in the multiplier method for constrained optimization [2]. In the presented context, this scheme is explicitly stated as

$$u_i^{k+1} := u_i^k + \max(\sigma g_i(\boldsymbol{x}^{k+1}), -u_i^k), \tag{4.1}$$

which, in the case of optimization, corresponds to the steepest descent method for the dual problem.

(2) First order multiplier estimate: This estimate satisfies the condition that if $\| x^k - x^* \|$ is of order ε for sufficiently small ε then $\| u^k - u^* \|$ is also of order ε [11]. Let $\nabla\hat{g}$ consist of the gradients of the constraints which are supposed to be active at x^*. Assume that $\nabla\hat{g}$ is of full rank. Then, a typical example of the first order estimate of u^* may be obtained by

$$u^k = -\left(\nabla\hat{g}(x^k)^{\mathrm{T}}\cdot\nabla\hat{g}(x^k) \right)^{-1}\cdot\nabla\hat{g}(x^k)^{\mathrm{T}}\cdot F(x^k). \tag{4.2}$$

(3) Second order multiplier estimate: This estimate satisfies the condition that if $\| x^k - x^* \|$ is of order ε for sufficiently small ε then $\| u^k - u^* \|$ is of order ε^2 [11]. Let $\nabla\hat{g}$ be the same as above. Also suppose that the matrix $L(x^k,\tilde{u}) := \nabla F(x^k) + \sum \tilde{u}_i\nabla^2 g_i(x^k)$ is nonsingular, where $\tilde{u}$ is a first order estimate as given above. Then the second order estimate may be obtained by

$$u^k = \left(\nabla\hat{g}(x^k)^{\mathrm{T}}\cdot L(x^k,\tilde{u})^{-1}\cdot\nabla\hat{g}(x^k) \right)^{-1} \cdot\left(\hat{g}(x^k) - \nabla\hat{g}(x^k)\cdot L(x^k,\tilde{u})^{-1}\cdot F(x^k) \right). \tag{4.3}$$

(Similar schemes are also considered in [18].)

Among these three schemes, the second order estimate (4.3) is the most expensive to compute, and the first order iteration (4.1) is the least. From a practical viewpoint, (4.1) is most favorable and is therefore used in the numerical experiments of Section 6.

Now, we establish a convergence theorem for Algorithm 2.

Theorem 3. Suppose that (x^*,u^*) is a solution of the variational inequality problem (2.2) where the constraint set S is specified by the inequalities (2.4). Assume also that the conditions of Theorem 1 are satisfied. Then the sequence (x^k,u^k) generated by Algorithm 2 converges to the solution (x^*,u^*), provided that an initial point (x^0,u^0) is chosen sufficiently close to (x^*,u^*), the step length is selected as $\alpha_k \equiv \alpha$, where $\alpha > 0$ is small enough, and the second order multiplier estimate is used to update u^k.

Proof: Since

$$x^{k+1} = x^k + \alpha H(x^k,u^k)$$

and

$$H(x^*,u^*) = 0,$$

we have

$$\begin{aligned}\| x^{k+1} - x^* \| \le\ & \| (I - \alpha\nabla_x H(x^*,u^k)^{\mathrm{T}})(x^k - x^*) \| \\ & + \alpha\| \nabla_u H(x^*,u^*)^{\mathrm{T}}(u^k - u^*) \| \\ & + O(\| x^k - x^* \|^2) + O(\| u^k - u^* \|^2).\end{aligned}$$

From the basic property of the second order multiplier estimate, the last inequality yields

$$\| x^{k+1} - x^* \| \le \| I - \alpha\nabla_x H(x^*,u^k)^T \| \cdot \| x^k - x^* \| + O(\| x^k - x^* \|^2). \tag{4.4}$$

Since $\nabla_x H(x^*,u^*)$ is positive definite, $\nabla_x H(x^*,u^k)$ is also positive definite when u^k is sufficiently close to u^*. Therefore, there exists $\rho > 0$ such that

$$\| I - \alpha\nabla_x H(x^*,u^k)^T \| \le \rho < 1,$$

for each α small enough. Consequently, it follows from (4.4) that $\| x^k - x^* \|$ converges to 0. □

This theorem guarantees local convergence of Algorithm 2 with specific updating schema of x^k and u^k. We have been unable to prove convergence for the other schema mentioned above. But the numerical experiments to be presented in Section 6 suggest that Algorithm 2 remains to be convergent for some classes of problems, even if the simpler scheme (4.1) is used to update u^k and the line search criterion similar to (3.5) is adopted to determine step lengths α_k.

5. Application to Traffic Equilibrium Problems

The traffic equilibrium problem on a road network is to find a traffic pattern characterized by the following property: For every origin-destination (O/D) pair, the travel costs of the paths with positive flow are equal to each other and no greater than those of the paths without flow. Here the travel cost on a road may be given as a composition of various things such as travel time, operating cost, and risks incurred [1].

In the transportation literature, various formulations of the traffic equilibrium problems have been presented by assuming that (i) the travel cost of each arc in the network is dependent on the traffic flow pattern and (ii) the travel demand between each O/D pair may either be fixed regardless of the travel cost (fixed demand) or depend on the travel costs (elastic demand). The assumption (i) reveals the congestion effect which may lead to the monotonicity property of the cost mappings. Similarly, in the case of elastic demand, the traffic demand between an O/D pair would usually decrease as the travel cost between that pair increases [1]. (The reader may recall demand curves in economics.) In general, the negative of the demand mappings may thus be considered monotone with respect to travel costs.

In the classical traffic equilibrium model, it is usually assumed that the travel cost on each arc is dependent only on the traffic volume on that arc and, in the elastic demand case, the traffic demand between each O/D pair is dependent only on the travel cost between that pair. Under these assumptions, the traffic equilibrium problem can be formulated as an equivalent nonlinear programming problem [1]. However, when interactions between different arcs and O/D pairs are present, the travel cost of each arc and the traffic demand of each O/D pair should be treated as functions of the entire flow pattern and travel costs of all O/D pairs, respectively, and hence the traffic equilibrium problem may not be reduced to an equivalent optimization problem unless the Jacobian matrices of those functions satisfy the symmetry conditions.

To deal with the traffic equilibrium problems containing asymmetric cost and demand functions, the variational inequality formulations have been proposed and studied extensively (see [3,7,17] for the fixed demand case and [4,5] for the elastic demand case). In those models, the variables are flows on arcs and demands for O/D pairs. Recently, from the viewpoint of duality, an alternative variational inequality formulation was proposed [10], in which costs for travel on each arc and between each O/D pair were involved as variables. In the latter formulation, the constraint set is specified by inequality constrains only, and hence, we may obtain the equilibrium solution by applying the method described in the previous section.

We summarize the dual formulation of the traffic equilibrium problem with elastic demand. Let us introduce the following:

A : the set of q directed arcs,
T : the set of s O/D pairs,
c_a : the travel cost on arc a,
$\boldsymbol{c}$: the vector of all arc costs, i.e., $\boldsymbol{c}=(c_1,c_2,..,c_q)$,
f_a : the flow on arc a,
$\boldsymbol{f}$: the vector of all arc flows, i.e., $\boldsymbol{f}=(f_1,f_2,..,f_q)$,
v_{ik} : the travel cost for O/D pair (i,k),
$\boldsymbol{v}$: the vector of travel costs for all O/D pairs, i.e., $\boldsymbol{v}=(v_1,v_2,..,v_s)$,
d_{ik} : the travel demand for O/D pair (i,k),
$\boldsymbol{d}$: the vector of travel demands for all O/D pairs, i.e., $\boldsymbol{d}=(d_1,d_2,..,d_s)$,
P_{ik} : the set of paths which join O/D pair (i,k),
δ_{ap} : the (a,p) element of the arc-path incidence matrix, i.e.,

$$\delta_{ap} = \begin{cases} 1 & \text{if arc } a \text{ is contained in path } p, \\ 0 & \text{otherwise.} \end{cases}$$

We suppose that every arc flow f_a depends on travel costs of all arcs and every travel demand d_{ik} is determined by travel costs of all O/D pairs, i.e., there exist mappings $\hat{\boldsymbol{f}}:\mathrm{R}^q \to \mathrm{R}^q$ and $\hat{\boldsymbol{d}}:\mathrm{R}^s \to \mathrm{R}^s$ such that

$$\boldsymbol{f} = \hat{\boldsymbol{f}}(\boldsymbol{c}) = (\hat{f}_1(\boldsymbol{c}),\ldots\ldots,\hat{f}_q(\boldsymbol{c}))$$

and

$$\boldsymbol{d} = \hat{\boldsymbol{d}}(\boldsymbol{v}) = (\hat{d}_1(\boldsymbol{v}),\ldots\ldots,\hat{d}_s(\boldsymbol{v})),$$

respectively. Then, the traffic equilibrium conditions may be formulated as the variational inequality problem of finding a vector $(\boldsymbol{c},\boldsymbol{v}) \in \boldsymbol{S}$ such that

$$< \hat{\boldsymbol{f}}(\boldsymbol{c}) \ , \ \boldsymbol{c}' - \boldsymbol{c} > \ \geq \ < \hat{\boldsymbol{d}}(\boldsymbol{v}) \ , \ \boldsymbol{v}' - \boldsymbol{v} > \quad \text{for all} \quad (\boldsymbol{c}',\boldsymbol{v}') \in \boldsymbol{S}, \qquad (5.1)$$

where $\boldsymbol{S}$ is the set of $(\boldsymbol{c},\boldsymbol{v})$ satisfying

$$v_{ik} \leq \sum_{a \in A} c_a \delta_{ap}, \quad p \in P_{ik}, \quad (i,k) \in T.$$

It is noted that problem (5.1) contains as many constraints as paths between all O/D pairs in the network. Thus the number of constraints in problem (5.1) may become prohibitively large as the size of the problem increases. This fact causes a serious difficulty in applying Algorithm 2 to large problems directly. In the following, let us propose a modification of Algorithm 2, which does not require explicit enumeration of the constraints. At the equilibrium solution, each active constraint corresponds to a shortest path for some O/D pair. Hence, at each iteration, we may generate needed constraints by using a shortest path algorithm such as Dijkstra and Warshall-Floyd methods [6], and add them to the list, say M, of constraints currently considered to be active. Also the multiplier estimate $\boldsymbol{u}^k$ associated with those constraints are updated using the first order iteration scheme described in Section 4, i.e., for $p \in P_{ik}$, we put

$$u_p^{k+1} = \max\{ \ 0 \ , \ u_p^k + \sigma(\ v_{ik} - \sum_{a \in A} c_a \delta_{ap} \ ) \ \}, \qquad (5.2)$$

where σ is a penalty constant used in the definition (2.6) of the augmented mapping $\boldsymbol{H}$. To be more specific, if path p is generated by the shortest path calculation and was not included in the list M at the previous iteration, then we append such p to the list M and update u_p^{k+1} by (5.2) with $u_p^k = 0$. On the other hand, if the Lagrange multiplier vanishes, then such a path p will be eliminated from the list M. In this approach, we only have to keep the data associated with the paths which belong to the current list M. This implies that the memory actually used will be greatly saved even for large problems.

It is worth pointing out that since Lagrange multipliers in problem (5.1) represent path flows, we are able to predict not only the flow on each arc and the demand between each O/D pair but also the distribution of the demand into shortest paths at equilibrium.

6. Computational Results

In this section, we present computational results for several problems. Computer programs were coded in PASCAL and the run was executed on a personal computer called NEC PC-9801. Firstly, we solve the next unconstrained problem using the method described in Section 3.

Problem 1. Find x^* such that $F(x^*) = 0$,
where

$$F(x) = \begin{pmatrix} 0.04x_1^3 + 3.0x_1 - 3.0x_2 + 5.0x_3 - 6.0x_4 - 4.0x_5 + 50.0 \\ 0.08x_2^3 + 3.0x_1 + 15.0x_2 - 10.0x_3 - 10.0x_4 + 5.0x_5 + 25.0 \\ 0.04x_3^3 - 1.0x_1 + 12.0x_2 + 3.0x_3 - 2.0x_4 + 1.0x_5 + 70.0 \\ 0.08x_4^3 - 6.0x_1 + 10.0x_2 + 2.0x_3 + 6.0x_4 + 1.0x_5 + 75.0 \\ 0.12x_5^3 + 4.0x_1 - 5.0x_2 - 1.0x_3 + 3.0x_4 + 15.0x_5 + 95.0 \end{pmatrix}.$$

The solution of this problem is

$$x^* = (-5.0 , -5.0 , -5.0 , -5.0 , -5.0) .$$

The computational results contain the following three cases:

Case 1 : Algorithm 1 in which G is set to be the unit matrix,

Case 2 : Algorithm 1 in which G is set to be the inverse of the diagonal part of $\nabla F(x^0)$, and

Case 3 : The method of [12] in which G is set to be the unit matrix.

Note that $\nabla F(x^*)$ is positive definite and $\nabla F(x^*)^2$ is not so. Therefore, convergence of the method proposed in [12] is not guaranteed as long as the unit matrix is used as a scaling matrix, while convergence of the method of Section 3 is guaranteed. In all cases, starting point x^0 is chosen to be (0,0,0,0,0). The results are summarized in Table 1, where ACC is the relative error calculated by

$$\| x^k - x^* \| \ / \ \| x^* \|.$$

Now, we proceed to solve inequality constrained variational inequality problems by Algorithm 2. Recall that the following two rules may be used to select the step length α_k:

(a): Line search by solving $\min \| H(x^k+\alpha d^k, u^k) \|$, and

Table 1. Results for Problem 1

	Algorithm 1		Algorithm of [12]
	Case 1	Case 2	
ITE. k	ACC (%)	ACC (%)	ACC (%)
0	100.00000	100.00000	100.00000
1	39.50811	60.53221	35.88979
2	21.05737	58.87709	25.88979
3	19.05737	55.29373	28.67679
4	17.76561	49.98032	35.04149
5	15.82546	43.39982	38.78901
6	13.12302	36.75468	41.80199
7	10.02321	30.02752	41.03937
8	7.41510	23.10537	40.85535
9	5.01001	17.71936	42.14634
10	2.70386	13.70874	42.41044
15	0.98543	6.63961	51.13631
20	0.17887	2.51458	85.88710
25	0.05021	1.02865	155.36122
30	0.01280	0.40731	163.15505
35	0.00297	0.18666	132.91551
40	0.00079	0.09120	75.48271
50	0.00004	0.01766	62.33397

Case 1: $G \equiv I$

x^{50} = (-4.999998, -5.000005, -5.000004, -5.000001, -5.000000)

Case 2: $G \equiv$ [diagonal part of $\nabla F(x^0)]^{-1}$

x^{50} = (-5.000172, -4.999561, -5.001710, -5.001307, -4.999689)

(b): fixed step length, i.e., $\alpha_k \equiv \alpha$ for some $\alpha > 0$.
We shall primarily employ rule (a) in solving the following examples, and the both rules will be compared for Problem 4. As to the update of u^k, we gave three schema in Section 4. In the present numerical experiments, we prefer to use the first order iteration scheme (4.1) because of its simplicity of implementation.

Problem 2. Find x^* such that $\langle F(x^*), y \rangle \geq 0$ for all $y \in T_S(x^*)$, where

$$F(x) = \begin{pmatrix} 3x_1 + x_2 + 12 \\ x_1 + 5x_2 + 2 \end{pmatrix},$$

$$S = \{ x \mid - x_1 - x_2^2 \leq 0 \}.$$

Table 2. Results for Problem 2

	Algorithm 2 [G ≡ I, σ = 0.500, step size rule (a)]					
	$\boldsymbol{x}^0$ = (-10, 5), $\boldsymbol{u}^0$ = 0			$\boldsymbol{x}^0$ = (-10, -10), $\boldsymbol{u}^0$		
ITE. k	x_1	x_2	u	x_1	x_2	u
0	-10.000000	5.000000	0.000000	-10.000000	-10.000000	0.000000
1	-5.257541	-1.201677	1.906757	-4.506526	1.379338	1.301977
2	-2.751413	-1.472958	2.594548	-4.131664	1.760908	1.817409
3	-2.528436	-1.513578	2.713307	-4.024330	1.932186	1.962902
4	-2.544297	-1.623098	2.668233	-3.999225	1.983669	1.995044
5	-2.568648	-1.593982	2.682167	-3.995187	1.995765	2.001098
6	-2.569483	-1.605866	2.677506	-3.995500	1.998633	2.001582
7	-2.572109	-1.602930	2.678868	-3.999878	2.000860	1.999801
8	-2.572217	-1.604121	2.678375	-4.000189	1.999997	1.999902
9	-2.572492	-1.603808	2.678521	-4.000124	2.000020	1.999924
10	-2.572502	-1.603934	2.678470	-3.999992	1.999969	1.999982
11	-2.572531	-1.603902	2.678485	-3.999988	1.999987	2.000002
12	-2.572532	-1.603915	2.678480	-3.999988	1.999996	2.000004
solutions	-2.572536	-1.603913	2.678481	-4.000000	2.000000	2.000000

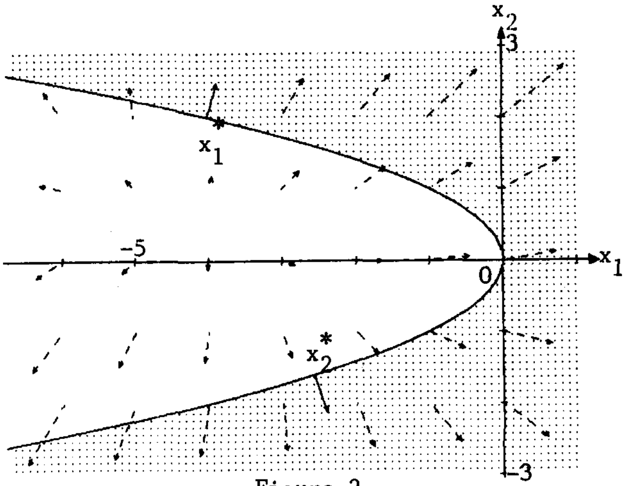


Figure 2.

Note that this problem contains a non-convex constraint set and therefore can not be treated by earlier methods. Because of the lack of convexity, the solution of this problem is not necessarily unique. In fact, it has two solutions as shown in Figure 2, i.e.,

$$\boldsymbol{x}_1^* = (\ -4.000000\ ,\ \ 2.000000\ )\ ,\quad u_1^* = 2.000000\ ,$$

and

$$\boldsymbol{x}_2^* = (\ -2.572536\ ,\ -1.603913\ )\ ,\quad u_2^* = 2.678481\ .$$

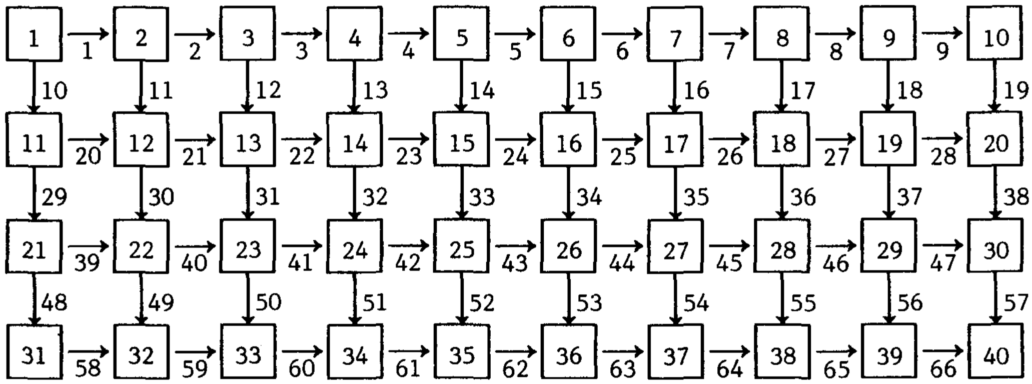
Thus, we may get either of the solutions depending on the choice of an initial estimate $(\boldsymbol{x}^0, u^0)$. The numerical results for two different starting points are shown in Table 2, where

Case 1 : $\boldsymbol{x}^0 = (\ -10,\ \ 5\ ),\ u^0 = 0$, and

Case 2 : $\boldsymbol{x}^0 = (\ -10,\ -10\ ),\ u^0 = 0$.

For each case, the scaling matrix $\boldsymbol{G}$ is chosen to be the unit matrix, the parameter σ is set at 0.500, and the step length α_k is determined by rule (a).

Problem 3. The traffic equilibrium problem illustrated in Figure 3 is solved. In this problem, arc costs are determined by a linear monotone mapping of arc flows. The monotonicity property of the cost mapping reveals the congestion effect on each road. (Note that the cost mapping often contains higher order terms in more realistic models.) Since $\nabla \boldsymbol{c}(\boldsymbol{f})$ is constant, we calculate the inverse matrix of $\nabla \boldsymbol{c}(\boldsymbol{f})$ to obtain the arc flow function, and then set a scaling matrix $\boldsymbol{G}$ to be the inverse of the diagonal part of $\nabla \boldsymbol{F}(\boldsymbol{x}^0)$, where $\boldsymbol{x} = (\boldsymbol{c}, \boldsymbol{v})$ and $\boldsymbol{F}(\boldsymbol{x}) = (\hat{\boldsymbol{f}}(\boldsymbol{c}), -\hat{\boldsymbol{d}}(\boldsymbol{v}))$. Since this example contains a convex constraint set, earlier methods for solving the ordinary variational inequalities may also be used to obtain equilibrium solutions. Among the existing methods, the relaxed projection method [8,10] seems to be appropriate as a measure of evaluating Algorithm 2, since they are both singly iterative methods. Table 3



Arc Costs

$c_1(\boldsymbol{f}) = 5f_1 + 2f_2 + 500$	$c_2(\boldsymbol{f}) = 4f_2 + f_1 + 200$	$c_3(\boldsymbol{f}) = 3f_3 + f_4 + 350$
$c_4(\boldsymbol{f}) = 6f_4 + 3f_5 + 400$	$c_5(\boldsymbol{f}) = 6f_5 + 4f_6 + 600$	$c_6(\boldsymbol{f}) = 7f_6 + 3f_7 + 500$
$c_7(\boldsymbol{f}) = 8f_7 + 2f_8 + 400$	$c_8(\boldsymbol{f}) = 5f_8 + 2f_9 + 650$	$c_9(\boldsymbol{f}) = 6f_9 + 2f_{10} + 700$
$c_{10}(\boldsymbol{f}) = 4f_{10} + f_{12} + 800$	$c_{11}(\boldsymbol{f}) = 7f_{11} + 4f_{12} + 650$	$c_{12}(\boldsymbol{f}) = 8f_{12} + 2f_{13} + 700$
$c_{13}(\boldsymbol{f}) = 7f_{13} + 3f_{18} + 600$	$c_{14}(\boldsymbol{f}) = 8f_{14} + 3f_{15} + 500$	$c_{15}(\boldsymbol{f}) = 9f_{15} + 2f_{14} + 200$
$c_{16}(\boldsymbol{f}) = 8f_{16} + 5f_{12} + 300$	$c_{17}(\boldsymbol{f}) = 7f_{17} + 2f_{15} + 450$	$c_{18}(\boldsymbol{f}) = 5f_{18} + f_{16} + 300$
$c_{19}(\boldsymbol{f}) = 8f_{19} + 3f_{17} + 600$	$c_{20}(\boldsymbol{f}) = 6f_{20} + f_{21} + 300$	$c_{21}(\boldsymbol{f}) = 4f_{21} + f_{22} + 400$
$c_{22}(\boldsymbol{f}) = 6f_{22} + f_{23} + 500$	$c_{23}(\boldsymbol{f}) = 9f_{23} + 2f_{24} + 350$	$c_{24}(\boldsymbol{f}) = 8f_{24} + f_{25} + 400$
$c_{25}(\boldsymbol{f}) = 9f_{25} + 3f_{26} + 450$	$c_{26}(\boldsymbol{f}) = 7f_{26} + 4f_{27} + 300$	$c_{27}(\boldsymbol{f}) = 8f_{27} + 3f_{28} + 500$
$c_{28}(\boldsymbol{f}) = 7f_{28} + 650$	$c_{29}(\boldsymbol{f}) = 3f_{29} + f_{30} + 450$	$c_{30}(\boldsymbol{f}) = 7f_{30} + 2f_{31} + 600$
$c_{31}(\boldsymbol{f}) = 4f_{31} + f_{32} + 750$	$c_{32}(\boldsymbol{f}) = 8f_{32} + 3f_{33} + 650$	$c_{33}(\boldsymbol{f}) = 9f_{33} + 2f_{34} + 750$
$c_{34}(\boldsymbol{f}) = 8f_{34} + 2f_{35} + 650$	$c_{35}(\boldsymbol{f}) = 6f_{35} + f_{36} + 500$	$c_{36}(\boldsymbol{f}) = 8f_{36} + 2f_{37} + 450$
$c_{37}(\boldsymbol{f}) = 7f_{37} + 3f_{38} + 700$	$c_{38}(\boldsymbol{f}) = 8f_{38} + 4f_{39} + 800$	$c_{39}(\boldsymbol{f}) = 7f_{39} + 3f_{40} + 550$
$c_{40}(\boldsymbol{f}) = 8f_{40} + 3f_{41} + 600$	$c_{41}(\boldsymbol{f}) = 8f_{41} + 4f_{42} + 750$	$c_{42}(\boldsymbol{f}) = 5f_{42} + f_{43} + 350$
$c_{43}(\boldsymbol{f}) = 7f_{43} + 2f_{45} + 400$	$c_{44}(\boldsymbol{f}) = 8f_{44} + 3f_{46} + 500$	$c_{45}(\boldsymbol{f}) = 9f_{45} + 6f_{47} + 350$
$c_{46}(\boldsymbol{f}) = 6f_{46} + 2f_{48} + 400$	$c_{47}(\boldsymbol{f}) = 3f_{47} + f_{49} + 800$	$c_{48}(\boldsymbol{f}) = 7f_{48} + 2f_{50} + 700$
$c_{49}(\boldsymbol{f}) = 6f_{49} + f_{50} + 600$	$c_{50}(\boldsymbol{f}) = 8f_{50} + 2f_{51} + 500$	$c_{51}(\boldsymbol{f}) = 7f_{51} + 3f_{52} + 600$
$c_{52}(\boldsymbol{f}) = 6f_{52} + 2f_{53} + 700$	$c_{53}(\boldsymbol{f}) = 9f_{53} + 3f_{54} + 650$	$c_{54}(\boldsymbol{f}) = 7f_{54} + 4f_{55} + 700$

Figure 3. Problem 3

$c_{55}(f) = 8f_{55} + 3f_{56} + 250$	$c_{56}(f) = 6f_{56} + f_{57} + 350$	$c_{57}(f) = 5f_{57} + 2f_{58} + 200$
$c_{58}(f) = 7f_{58} + f_{59} + 250$	$c_{59}(f) = 6f_{59} + 3f_{60} + 400$	$c_{60}(f) = 4f_{60} + f_{61} + 300$
$c_{61}(f) = 7f_{61} + 2f_{62} + 400$	$c_{62}(f) = 5f_{62} + f_{63} + 200$	$c_{63}(f) = 4f_{63} + 2f_{64} + 100$
$c_{64}(f) = 5f_{64} + 100$	$c_{65}(f) = 4f_{65} + 200$	$c_{66}(f) = 7f_{66} + 300$

Demands

O/D pair 1 [1,20], $d_1(v) = -0.010v_1 + 60.0$

O/D pair 2 [1,25], $d_2(v) = -0.009v_2 + 70.0$

O/D pair 3 [16,30], $d_3(v) = -0.010v_3 + 50.0$

O/D pair 4 [24,40], $d_4(v) = -0.008v_4 + 50.0$

O/D pair 5 [4,27], $d_5(v) = -0.010v_5 + 50.0$

O/D pair 6 [22,40], $d_6(v) = -0.008v_6 + 50.0$

Figure 3. (Continued) Problem 3

contains the computational results for (i) Algorithm 2 with a modification described in Section 5, where the parameter σ is set at 0.010 and the step length α_k is determined by rule (a), and (ii) the relaxed projection method, where a controlling parameter is adjusted in the same manner as in [10].

In Table 3, comparison is made in terms of the number of iterations to obtain approximate solutions with various degrees of accuracy (ACC) evaluated by

$$\text{ACC} = \max(\| c^k - c^* \| / \| c^* \| , \| v^k - v^* \| / \| v^* \|).$$

Since the two methods require almost the same CPU time per iteration, the number of iterations seems to be a good measure of estimating the computational efficiency. Table 3 shows that Algorithm 2 exhibits faster convergence.

Moreover, to see how convergence of Algorithm 2 is affected by the choice of the parameter σ, the same problem is solved using various parameter values. The results, which are summarized in Figure 4, suggest that the choice of the parameter σ considerably influences convergence properties of Algorithm 2. However, since suitable parameter value actually depends on the problem to be solved, we have to resort to an ad hoc method to determine a desirable value of σ.

Table 3. Results for Problem 3

ACC	Algorithm 2*	Algorithm of [10]
10%	6	15
5%	11	27
2%	14	37
1%	20	41
0.5%	26	46

*$\sigma = 0.001$ and α_k is determined by rule (a).

Problem 4. A larger traffic equilibrium problem which has 100 nodes, 360 arcs and 10 O/D pairs is also solved by Algorithm 2 with a modification described in Section 5. We first apply the algorithm using step length rule (a), and summarize the computational results for various values of parameter σ in Figure 5. Then, we also solved the same problem by adopting the fixed step length rule (b). Figure 6 compares the results for various values of the fixed step length α with that of using rule (a). (In all calculations, the parameter σ was set at 0.010.) In Figure 6, the graph corresponding to each run is marked every 20 iterations. The results shown in Figure 6 indicate that step length rule (b) usually requires more iterations than rule (a) to attain the same level of accuracy. But because rule (b) spends less CPU time per iteration, approximate solutions with high accuracy may be obtained quickly if we choose suitable step length α. Also it is noted that the algorithm may fail to converge if the fixed step length is too large. In fact, Figure 6 shows that convergence is not obtained for $\alpha = 0.018$.

7. Conclusion

The proposed method for unconstrained variational inequality problems is constructed by modifying the line search criterion of the algorithm due to Hammond and Magnanti [12]. We have shown that the present method is convergent under weaker conditions than those for the method of [12]. The numerical result given in Section 6 also indicates this.

For the inequality constrained problems, a new solution method is also presented, which is a modification of the multiplier methods in nonlinear programming. This method may deal with problems containing non-convex constraints and can be shown to converge under appropriate conditions. Moreover, this method can be applied to the traffic equilibrium problems by using a shortest path calculation. The numerical results of Section 6 are quite satisfactory and would encourage further study of solution methods for variational inequalities form optimization viewpoints.

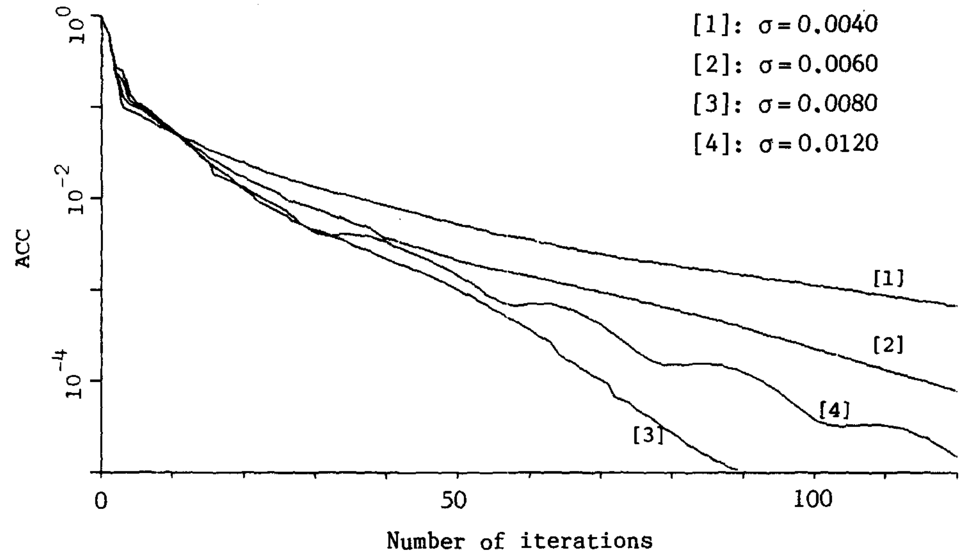


Figure 4. Results for Problem 3

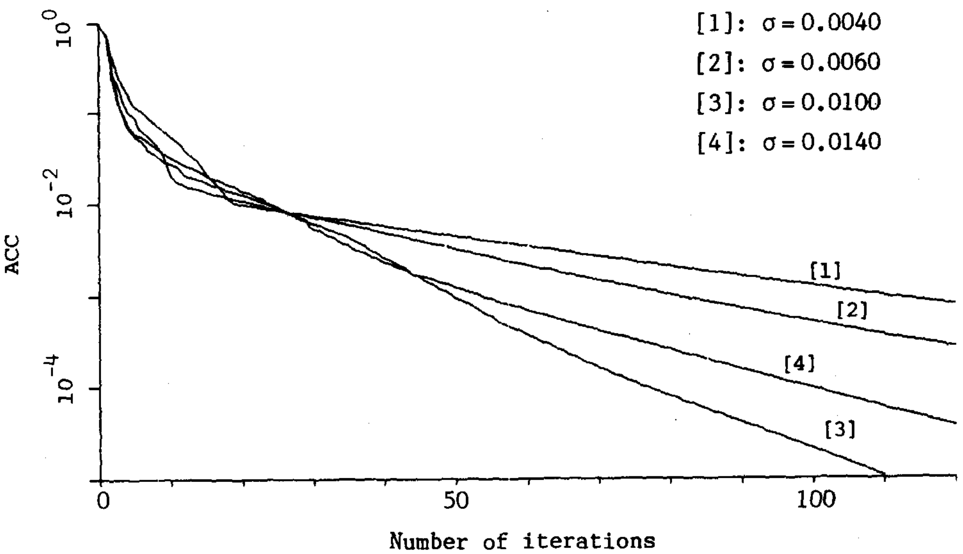


Figure 5. Results for Problem 4

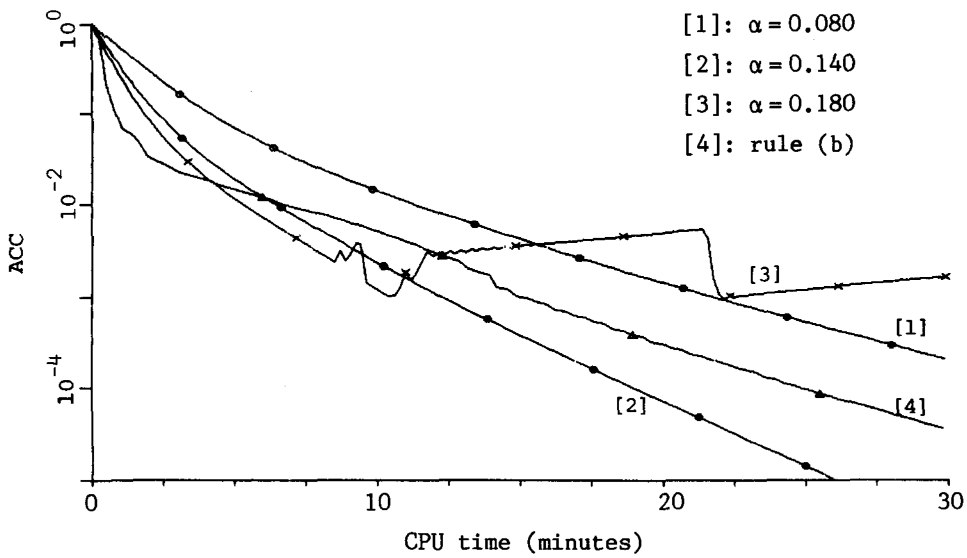


Figure 6. Results for Problem 4

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