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ORIENTABILITY OF PSEUDOMANIFOLD AND GENERALIZATIONS OF SPERNER'S LEMMA

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Abstract We propose a combinatorial framework for fixed point algorithms and constructive proofs of combinatorial lemmas in topology. The framework consists of two sets of pseudomanifolds and an operator relating them. They have lattice structures which are dual to each other. We show that the set of "joins" of pseudomanifolds related by the operator is a homogeneous and orientable pseudomanifold under several conditions. By exploiting this framework we generalize Sperner's lemma on convex polytope. Namely, let C be a convex polytope with m facets $F_1, \ldots, F_m$, S be a finite triangulation of C and $\bar{S} = \{\sigma \mid \sigma$ is a face of some simplex of S$\}$. Given a nondenerate vertex v of C and a labelling function ℓ from the set of vertices of S to $\{1, \ldots, m\}$, the set of indices of facets, there is an odd number of simplices σ of $\bar{S}$ such that $\ell(\sigma) \cup \{i \mid 1 \leqq i \leqq m, \sigma \subset F_i\}$ strictly includes $\{i \mid 1 \leqq i \leqq m, v \in F_i\}$ We also prove the generalization of Sperner's lemma by Fan and van der Laan-Talman-Van der Heyden's lemma as corollaries to the result.

1. Introduction

Since a new class of fixed point algorithms was proposed by van der Laan-Talman [13], a framework unifying the existing algorithms has been studied by several researchers, Kojima-Yamamoto [11], van der Laan-Talman [14], Freund [3], Yamamoto [20]. The framework is useful not only to interpret the existing algorithms but also to develop new algorithms (see Kojima-Yamamoto [12]), to prove the convergence of the algorithms (see van der Laan-Talman [16]) and also to give constructive proofs for fixed point theorems (see Freund [16]). One of the frameworks is a subdivision of a space with an artificial dimension. It is a weakness, however, of this framework that it heavily depends upon the geometrical structure of the subdivision. The framework in Kojima-Yamamoto [11] is based on the "product" of two subdivided manifolds. This framework is simple and useful for the interpretation and development of fixed point algorithms with vector labelling but it still depends on the facial structure of polytopes. It should be stated that they have generalized it in [10] to overcome this weakness. The framework in Freund [3] consists of a pseudomanifold and a label set. It is combinatorial and free from geometrical structure. Hence it is suitable to fixed point algorithms with integer labelling and to the proof of combinatorial lemmas in topology.

In this paper after reviewing several basic properties of orientation of pseudomanifolds we propose a combinatorial structure which consists of two sets of pseudomanifolds and an operator relating them. We will call it primal-dual pseudomanifold and abbreviate it by pdpm. We show in Sections 4 and 5 that the set of "joins" of the pseudomanifolds is also a pseudomanifold and under some conditions it is homogeneous and orientable. In Section 6 some examples of pdpm are constructed from triangulations of a convex polytope and its polar. In Section 7 we generalize Sperner's lemma on a general convex polytope by using pdpm. It is shown that the generalization of Sperner's lemma on the cross product of simplices in van der Laan-Talman-Van der Heyden [15] and Freund [5] is readily obtained from the theorem. We also show the generalized Sperner's lemma by Fan [2] and Sperner's lemma as corollaries to the theorem. Finally we sketch another scheme to prove the theorem and corollaries by aggregating facets of the polytope.

2. Preliminaries

Let σ be an abstract simplex of n vertices, i.e., $\sigma = \{v_1,\dots,v_n\}$. The simplex σ is usually called an (n-1)-dimensional simplex, but here we will call it an n-cardinal simplex (abbreviated by #n-simplex) to avoid the confusion of the dimension and the number of vertices. Any subset of σ is called a face of σ and an #(n-1)-face is especially called a facet. A set K of #n-simplices is called an #n-pseudomanifold (abbreviated by #n-pm) if for any facet τ of any simplex of K there are at most two simplices of K having τ. We, however, mean a set of one #1-simplex by a #1-pm. The boundary of K, denote by δK, is the set of #(n-1)-simplices each of which is a facet of exactly one simplex of K. An

#n-pm K is said to be finite if it consists of a finite number of simplices. It is said to be locally finite if for each vertex of K the number of simplices of K having v is finite.

Two simplices σ and η of K are said to be neighboring if they share a facet, or equivalently $\eta = \sigma \Delta \{v,u\}$ for some $v \in \sigma$ and $u \notin \sigma$, where Δ means symmetric difference. An #n-pm K is said to be homogeneous if for any pair σ and $\eta \in K$ there is a sequence σ_0, $\sigma_1,\ldots,\sigma_t$ of simplices of K such that $\sigma_o = \sigma$, $\sigma_t = \eta$, σ_{i-1} and σ_i are neighboring simplices for $i = 1,\ldots,t$.

The set of all #k-faces of all simplices of K is called the #k-skelton of K and denoted by $K^{\#k}$. We also denote $\bigcup_{k=0}^{n} K^{\#k}$ by $\bar{K}$.

Let C be a convex polytope of R^n. A set L of geometrical n-dimensional simplices is said to be a triangulation of C if the union of all simplies of L is C and for any two simplices of L their intersection is their common face. For a triangulation L of C let $L' = \{\{ v \mid v \text{ is a vertex of } \sigma \} \mid \sigma \in L \}$. Then L' is a pm. In the sequel we will make no distinction between L and L'.

3. Orientation

For an #n-simplex $\sigma = \{v_1,\ldots,v_n\}$ let $Or(\sigma,.)$ be a function from the set of orderings of the n vertices of σ to the set $\{-1,+1\}$ such that

$$(3.1) \qquad Or(\sigma,(v_1,\ldots,v_n)) = \mathrm{sgn}\,(\pi)\, Or(\sigma,(v_{\pi(1)},\ldots,v_{\pi(n)}))$$

for any permutation π of $\{1,\ldots,n\}$, where $\mathrm{sgn}(\pi)$ is the sign of π. Let σ and $\eta = \sigma \Delta \{v_k, u\}$ be neighboring simplices, where $v_k \in \sigma$, $u \notin \sigma$. $Or(\sigma,.)$ and $Or(\eta,.)$ are said to be coherent iff

$$(3.2) \qquad Or(\eta,(v_1,\ldots,v_{k-1},u,v_{k+1},\ldots,v_n)) = -Or(\sigma,(v_1,\ldots,v_n)).$$

An #n-pm K is orientable if there exists an orientation function $Or(.,.)$ such that it is coherent for any neighboring simplices. Let $\tau = \sigma \setminus \{v_k\}$ for some $v_k \in \sigma$. $Or(\tau,.)$ is called the induced orientation of τ from $Or(\sigma,.)$ if

$$(3.3) \qquad Or(\tau,(v_1,\ldots,v_{k-1},v_{k+1},\ldots,v_n)) = (-1)^k Or(\sigma,(v_1,\ldots,v_n)).$$

In the followings we give several lemmas about the orientation. Some of them are obtained directly from the above definitions. The readers who are familiar with the orientation theory could skip the following lemmas.

Lemmma 3.1. Let σ and η be neighboring #n-simplices and $\tau = \sigma \cap \eta$. Then $Or(\sigma,.)$ and $Or(\eta,.)$ are coherent if and only if the induced orientations $Or^{\sigma}(\tau,.)$ and $Or^{\eta}(\tau,.)$ of τ from $Or(\sigma,.)$ and $Or(\eta,.)$, respectively, have opposite signs.

Proof. Let $\sigma = \{v_1,\ldots,v_n\}$ and $\eta = \sigma \Delta \{v_k,u\}$. Suppose $Or(\sigma,.)$ and $Or(\eta,.)$ are coherent. Then by (3.3)

$$\begin{aligned}
&Or^{\eta}(\tau,(v_1,\ldots,v_{k-1},v_{k+1},\ldots,v_n))\\
&= (-1)^k Or(\eta,(v_1,\ldots,v_{k-1},u,v_{k+1},\ldots,v_n))\\
&= (-1)^{k+1} Or(\sigma,(v_1\ ,\ldots,v_n))\\
&= -Or^{\sigma}(\tau,(v_1,\ldots,v_{k-1},v_{k+1},\ldots,v_n)).
\end{aligned}$$

Next suppose $Or^{\sigma}(\tau,.) = -Or^{\eta}(\tau,.)$. Then

$$\begin{aligned}
&Or(\sigma,(v_1,\ldots,v_n))\\
&= (-1)^k Or^{\sigma}(\tau,(v_1,\ldots,v_{k-1},v_{k+1},\ldots,v_n))\\
&= (-1)^{k+1} Or^{\eta}(\tau,(v_1,\ldots,v_{k-1},v_{k+1},\ldots,v_n))\\
&= -Or(\eta,(v_1,\ldots,v_{k-1},u,v_{k+1},\ldots,v_n)).
\end{aligned}$$

Lemma 3.2. Let K be a homogeneous pm and orientable with respect to two distinct orientations Or^1 and Or^2. Then $Or^1 = -Or^2$.

Proof. Since $Or^1 \neq Or^2$, there is a simplex $\sigma \in K$ with $Or^1(\sigma,.) = -Or^2(\sigma,.)$. Let $\eta = \sigma \Delta \{v_k,u\} \in K$ for some $v_k \in \sigma$, $u \notin \sigma$. Then since both pairs $(Or^1(\sigma,.), Or^1(\eta,.))$ and $(Or^2(\sigma,.), Or^2(\eta,.))$ are coherent,

$$\begin{aligned}
&Or^1(\eta,(v_1,\ldots,v_{k-1},u,v_{k+1},\ldots,v_n\ ))\\
&= -Or^1(\sigma,(v_1,\ldots,v_n)) = Or^2(\sigma,(v_1,\ldots,v_n))\\
&= -Or^2(\eta,(v_1,\ldots,v_{k-1},u,v_{k+1},\ldots,v_n)).
\end{aligned}$$

Let η be an arbitrary simplex of K. Since K is homogeneous, there is a sequence of neighboring simplices from σ to η. By applying the above argument along this sequence we have the desired result.

Lemma 3.3. (compare with Lemma 26 in Freund [3])
Let K be a locally finite and orientable #n-pm. If δK is an #(n-1)-pm, then it is orientable with respect to the induced orientation.

Proof. Let $\alpha = \{v_1,\ldots,v_{n-1}\}$ and $\beta = \alpha \Delta \{v_1,v_n\}$ be neighboring simplices of δK. Let ℓ be a function defined on the set of vertices of K into

$\{1,2,\dots,n\}$ such that

$$\ell(v_1) = \ell(v_n) = 1$$
$$\ell(v_i) = i \quad \text{for} \quad i = 2,\dots,n-1$$
$$\ell(v) = 1 \quad \text{for any} \quad v \notin \alpha \cup \beta.$$

Then we will see that only α and β are the simplices of δK with $\ell(\alpha) = \ell(\beta) = \{1,\dots,n-1\}$ where $\ell(\alpha) = \{\ \ell(v) \mid v \in \alpha\ \}$. Suppose $\ell(\gamma) = \{1,\dots,n-1\}$ for some $\gamma \in \delta K$. Then by the definition of ℓ $\{v_2,\dots,v_{n-1}\} \subset \gamma$. Since δK is an #(n-1)-pm, there are at most two #(n-1)-simplices of δK having $\{v_2,\dots,v_{n-1}\}$. Therefore $\gamma = \alpha$ or β. Furthermore by the local finiteness of K, the number of simplices having $\{v_2,\dots,v_{n-1}\}$ is finite. Therefore there is a finite sequence of simplices $\sigma_o, \sigma_1,\dots, \sigma_t$ of K such that $\alpha \subset \sigma_o$, $\beta \subset \sigma_t$, σ_{i-1} and σ_i are neighboring and $\ell(\sigma_{i-1} \cap \sigma_i) = \{1,\dots,n-1\}$ for all $i = 1,\dots,t$. By applying (3.2) and (3.3) to these simplices (see for example Lemke [17], Gould-Tolle [8] and Eaves [1]), we see that the induced orientations $Or(\alpha,.)$ and $Or(\beta,.)$ are coherent.

Lemma 3.4. Let K and L be locally finite and orientable #n-pm's such that $K \cap L = \emptyset$. Suppose δK and δL are #(n-1)-pm's, $K \cup L$ is an #n-pm, and $\delta K \cap \delta L$ is a homogeneous pm. Then $K \cup L$ is orientable.

Proof. First note that $K^{\#(n-1)} \cap L^{\#(n-1)} = \delta K \cap \delta L$ since $K \cup L$ is a pm and $K \cap L = \emptyset$. Let $M = \delta K \cap \delta L$, Or^K and Or^L be the orientations of K and L, respectively. Let $Or^{M<K}$ and $Or^{M<L}$ be the induced orientations of M from Or^K and Or^L. By Lemma 3.2 M is orientable with respect to both $Or^{M<K}$ and $Or^{M<L}$. Then by Lemma 3.2 $Or^{M<K} = Or^{M<L}$ or $-Or^{M<L}$. If $Or^{M<K} = Or^{M<L}$, then reverse the orientation of one of K and L. Then we have $Or^{M<K} = -Or^{M<L}$. By Lemma 3.1 we see that $K \cup L$ is orientable.

4. Primal-Dual Pseudomanifolds

Let K and L are #p- and #d-pm's which share no vertices. Let P_p be a finite partition of K into #p-pm's. Let P_{p-1} be a finite partition of $\bigcup\{\ \delta X \mid X \in P_p\ \}$ into #(p-1)-pm's such that $\delta X_1 \cap \delta X_2$ is also partitined by P_{p-1} for any $X_1, X_2 \in P_p$. In general let P_k be a finite partition of $\bigcup\{\ \delta X \mid X \in P_{k+1}\ \}$ into #k-pm's such that $\delta X_1 \cap \delta X_2$ is also partitioned by P_k for any $X_1, X_2 \in P_{k+1}$. For consistency of notation we define $P_0 = \{\emptyset\}$. Let $D_d, D_{d-1}, \ldots, D_0$ be defined for L in the same way. Let $P = \bigcup_{k=0}^{p} P_k$ and $D = \bigcup_{k=0}^{d} D_k$. When there are a subset $A \subset P \cup D \setminus \{\emptyset\}$, an operator denoted by $*$ on A and a positive integer n satisfying the following conditions, (P,D,A,*,n) is called an #n-primal-dual pseudomanifolds (will be abbreviated by #n-pdpm):

(4.1) If $X \in P_k \cap A$ ($Y \in D_k \cap A$) and $0 < k \leq n$, then $X^* \in D_{n-k}$ ($Y^* \in P_{n-k}$).

(4.2) If $X \in A$ and $X^* \neq \emptyset$, then $X^{**} = X$.

(4.3) If $X, Y \in A$ and $X \subset \delta Y$, then $Y^* \subset \delta X^*$.

(4.4) For each $X \in P_{n-1}$ ($Y \in D_{n-1}$) there is at most one member $U \in P_n$ ($V \in D_n$) such that $X \subset \delta U$ ($Y \subset \delta V$).

For $X \in P \cap A$ let

$$X \cdot X^* = \{\ \sigma \cup \eta \mid \sigma \in X,\ \eta \in X^*\ \},$$

which coincides with $X^{**} \cdot X^*$ when $X^* \neq \emptyset$.

Lemma 4.1.

(4.5) $X \cdot X^*$ is an #n-pm.

(4.6) $\delta(X \cdot X^*) = (\delta X \cdot X^*) \cup (X \cdot \delta X^*)$.

(4.7) $(X \cdot X^*) \cap (Y \cdot Y^*) = \emptyset$ if $X \neq Y$.

Proof. Let $X \in P_k$ and $X^* \in D_{n-k}$. Since $X \cdot X^* = X$ if $X^* = \emptyset$, (4.5) and (4.6) are clear in this case. We suppose $X^* \neq \emptyset$. Let $\tau \cup \eta$ be an #(n-1)-simplex of $\overline{X \cdot X^*}$. Then we have exactly one of the following two cases: case 1: $|\tau| = k$ and $|\gamma| = n - k - 1$, case 2 : $|\tau| = k - 1$ and $|\gamma| = n - k$. It is sufficient to condsider case 1. Suppose $\tau \cup \gamma \subset \sigma \cup \eta \in X \cdot X^*$. Then $\sigma = \tau$. Since X^* is a pm, there are at most two $\eta \in X^*$ having γ. Therefore we have (4.5). The assertion (4.6) also follows from the above argument. The assertion (4.7) is also clear since $X \cap Y = \emptyset$ when $X \neq Y$ and any simplex of $(X \cdot X^*) \cap (Y \cdot Y^*)$ whould have n vertices.

Lemma 4.2. Suppose that $X \in P_k \cap A$, $Y \in P_h \cap A$, $X \neq Y$, $h \leq k$, and $\overline{X \cdot X^*}$ and $\overline{Y \cdot Y^*}$ share an #(n-1)-simplex. Then $k = h + 1$ and $Y \subset \delta X$.

Proof. Let $\tau \cup \gamma$ be an #(n-1)-simplex in $\overline{X \cdot X^*} \cap \overline{Y \cdot Y^*}$. Then there exist $\sigma_1 \cup \eta_1 \in X \cdot X^*$ and $\sigma_2 \cup \eta_2 \in Y \cdot Y^*$ having $\tau \cup \gamma$. Therefore $\tau \subset \sigma_1 \cap \sigma_2$ and $\gamma \subset \eta_1 \cap \eta_2$. Suppose that $k = h$. Then $|\tau| \leqq |\sigma_1| - 1 = k - 1$ and $|\gamma| \leqq |\eta_1| - 1 = n - k - 1$. This implies that $|\tau \cup \gamma| \leqq n - 2$, which is a contradiction. Suppose that $k \geqq h + 2$. Then $|\tau| \leqq |\sigma_2| = h$ and $|\gamma| \leqq |\eta_1| \leqq n - k$. Hence $|\tau \cup \gamma| \leqq h + n - k \leqq n - 2$, again a contradiction. Therefore we have $k = h + 1$. If $Y \not\subset \delta X$, then $|\sigma_1 \cap \sigma_2| \leqq |\sigma_2| - 1 = h - 1$ and $|\eta_1 \cap \eta_2| \leqq |\eta_1| - 1 = n - k - 1$. This is contrary to $|\tau \cup \gamma| = n - 1$.

Now let

$$
\begin{aligned}
M = & \bigcup \{ X \cdot X^* \mid X \in P \cup A,\ X^* \neq \emptyset \} \\
& \cup \bigcup \{ X \mid X \in P \cap A,\ X^* = \emptyset \} \\
& \cup \bigcup \{ Y \mid Y \in D \cap A,\ Y^* = \emptyset \}
\end{aligned}
$$

Then we have the following theorem.

Theorem 4.3. M is an #n-pm.

Proof. Let $\tau \cup \gamma$ be an arbitrary #(n-1)-simplex of $X \cdot X^*$ for some $X \in P \cap A$. Suppose $\tau \cup \gamma \subset \sigma \cup \eta \in M$. We can assume without loss of generality $X \in P_k$, $X^* \in D_h$, $|\tau| = k - 1$ and $|\gamma| = h$, where $h = n - k$.

We first consider the case where $k > 1$, which implies $\tau \neq \emptyset$. When $\tau \notin \delta X$, $\tau \cup \gamma \notin \delta(X \cdot X^*)$ by (4.6) of Lemma 4.1. Then we have two #n-simplices of $X \cdot X^*$ having $\tau \cup \gamma$ by (4.5). We also see that no other #n-simplices of M have $\tau \cup \gamma$ by (4.7). When $\tau \in \delta X$, there is a unique $\sigma \in X$ with $\tau \subset \sigma$, and consequently $\tau \cup \gamma \subset \sigma \cup \gamma \in X \cdot X^* \subset M$. Since $\tau \in \delta X$, there is a unique $Y \in P_{k-1}$ such that $\tau \in Y \subset \delta X$ by the construction of P_{k-1}. If $Y \in A$ there is $Y^* \in D_{h+1}$ and $X^* \subset \delta Y^*$. Therefore there is a unique $\eta \in Y^*$ with $\gamma \subset \eta$. Thus we obtain $\tau \cup \eta \in Y \cdot Y^* \subset M$ having $\tau \cup \gamma$. Finally suppose that $\tau \cup \gamma \subset \alpha \cup \beta \in M$. Then either $|\alpha| = |\tau| + 1$ and $|\beta| = |\gamma|$ or $|\alpha| = |\tau|$ and $|\beta| = |\gamma| + 1$ holds. In the first case $\beta = \gamma \in X^*$ and $\alpha \cup \beta \in X^{**} \cdot X^* = X \cdot X^*$. Therefore $\alpha \cup \beta$ must be $\sigma \cup \gamma$ above. In the second case we have $\alpha \cup \gamma = \tau \cup \eta$. Thus we have seen that $\tau \cup \gamma$ is contained in at most two simplices of M when $k > 1$.

Next, we consider the case where $k = 1$. Note that $h = n - 1$. By the definition of #1-pm, X consists of a single #1-simplex, say $\{v\}$. Therefore we obtain $\{v\} \cup \gamma \in X \cdot X^* \subset M$ having $\tau \cup \gamma = \gamma$. If $D_n \neq \emptyset$, at most one member $Y \in D_n$ such that $X^* \subset \delta Y$ belongs to A from condition (4.4). Therefore we have at most one simplex $\eta \in Y$ such that $\tau \cup \gamma = \gamma \subset \eta = \emptyset \cup \eta \in M$. Finally suppose that $\tau \cup \gamma \subset \alpha \cup \beta \in M$. Then either $|\alpha| = 1$ and $|\beta| = |\gamma|$ or $|\alpha| = 0$ and $|\beta| = |\gamma| + 1$ holds, so that we see that $\alpha \cup \beta$ is either $\{v\} \cup \gamma$ or $\emptyset \cup \eta$.

Corollary 4.4. δM is the set of #(n-1)-simplices $\tau \cup \gamma$ satisfying one of the following conditions. Here $\tau \cup \gamma \in (X \cdot X^*)^{\#(n-1)}$, $X \in P_k$ and $X^* \in D_h$.

(4.8) $k > 1$, $\tau \in \delta X$ and if $\tau \in Y \subset \delta X$, then $Y \notin A$.

(4.9) $h > 1$, $\gamma \in \delta X^*$ and if $\gamma \in Y \subset \delta X^*$, then $Y \notin A$.

(4.10) $k = 1$, $\tau = \emptyset$ and if $X^* \subset \delta Y$, then $Y \notin A$.

(4.11) $h = 1$, $\gamma = \emptyset$ and if $X \subset \delta Y$, then $Y \notin A$.

Proof. The asssertion is readily obtained from the proof of Theorem 4.3.

5. Orientability of M

In this section we will show that M is a homogeneous and orientable pm under following conditions.

(5.1) Each $X \in A$ is homogeneous and orientable.

(5.2) For any pair X and $Y \in P \cap A \cup \{\emptyset\}$ there is a sequence $X_0, X_1, \ldots, X_t$ of $P \cap A \cup \{\emptyset\}$ such that $X_0 = X$, $X_t = Y$ and either $X_{i-1} \subset \delta X_i$ or $X_i \subset \delta X_{i-1}$ for $i = 1, \ldots, t$.

(5.3) δX is a pm for any $X \in A$.

(5.4) (Shellability) There is an ordering $X_1 \cdot Y_1, \ldots, X_s \cdot Y_s$ of #n-pm's constructing M such that for $i = 2, \ldots, s$

$$\delta(\textstyle\bigcup_{j \leq i-1} X_i \cdot Y_j) \cap \delta(X_i \cdot Y_i)$$

is a homogeneous #(n-1)-pm and

$$(\delta(\textstyle\bigcup_{j \leq i-1} X_j \cdot Y_j) \cap \delta(X_i \cdot Y_i))^{\#(n-2)}$$
$$= \delta(\textstyle\bigcup_{j \leq i-1} X_j \cdot Y_j)^{\#(n-2)} \cap \delta(X_i \cdot Y_i)^{\#(n-2)}.$$

Lemma 5.1. $X \cdot X^*$ is homogeneous.

Proof. We prove only the case where $X^* \neq \{\emptyset\}$. Let $\sigma \cup \eta$ and $\sigma' \cup \eta'$ be arbitrary simplices of $X \bullet X^*$. Since X and X^* are homogeneous from condition (5.1), we can find sequences of neighboring simplices $\sigma = \sigma_0, \sigma_1, \ldots, \sigma_k = \sigma'$ and $\eta = \eta_0, \eta_1, \ldots, \eta_h = \eta'$ such that $\sigma_i \in X$, $\eta_j \in X^*$. Then the sequence $\sigma \cup \eta = \sigma_0 \cup \eta_0, \sigma_1 \cup \eta_0, \ldots, \sigma_k \cup \eta_0 = \sigma' \cup \eta_0, \sigma' \cup \eta_1, \ldots, \sigma' \cup \eta_h = \sigma' \cup \eta'$ is the desired sequence.

Lemma 5.2. $X \bullet X^*$ is orientable.

Proof. Let $\sigma \cup \eta \in X \bullet X^*$, $\sigma = \{v_1, \ldots, v_k\}$, $\eta = \{u_1, \ldots, u_h\}$, where $h = n - k$. We define the orientation $Or(\sigma \cup \eta, .)$ by

$$Or(\sigma \cup \eta, (v_1, \ldots, v_k, u_1, \ldots, u_h)) = Or(\sigma, (v_1, \ldots, v_k)) \times Or(\eta, (u_1, \ldots, u_h)).$$

Note that $Or(\sigma \cup \eta, .)$ is uniquely defined by the above formula. Suppose $\sigma' \cup \eta'$ is a neighboring simplex of $\sigma \cup \eta$. Then either $\sigma' \cup \eta' = (\sigma \Delta \{v_i, v\}) \cup \eta$ for some $v_i \in \sigma$, $v \notin \sigma$ or $\sigma' \cup \eta' = \sigma \cup (\eta \Delta \{u_i, u\})$ for some $u_i \cup \eta$, $u \notin \eta$. Since X is orientable by (5.1), we have for the first case

$$\begin{aligned} &Or(\sigma' \cup \eta', (v_1, \ldots, v_{i-1}, v, v_{i+1}, \ldots, v_k, u_1, \ldots, u_h)) \\ &= Or(\sigma', (v_1, \ldots, v_{i-1}, v, v_{i+1}, \ldots, v_k)) \times Or(\eta', (u_1, \ldots, u_h)) \\ &= - Or(\sigma, (v_1, \ldots, v_k)) \times Or(\eta, (u_1, \ldots, u_h)) \\ &= - Or(\sigma \cup \eta, (v_1, \ldots, v_k, u_1, \ldots, u_h)). \end{aligned}$$

We have the same result also for the second case.

Lemma 5.3. M is homogeneous.

Proof. Let $\sigma \cup \eta$ and $\sigma' \cup \eta'$ be arbitrary simplices of M. We first consider the case where $\sigma \neq \emptyset$ and $\sigma' \neq \emptyset$. Then there are X and $Y \in P \cap A$ such that $\sigma \cup \eta \in X \bullet X^*$ and $\sigma' \cup \eta' \in Y \bullet Y^*$. Since we have seen that there is a sequence of neighboring simplices between $\sigma \cup \eta$ and $\sigma' \cup \eta'$ by Lemma 5.1 when $X = Y$, we suppose $X \neq Y$. By condition (5.2) we obtain a sequence $X = X_0, X_1, \ldots, X_t = Y$ of members of $P \cap A$ such that $X_{i-1} \subset \delta X_i$ or $X_i \subset \delta X_{i-1}$ for $i = 1, \ldots, t$. As the inductive hypothesis we assume that there is a sequence of neighboring simplices from $\sigma \cup \eta$ to any $\tau \cup \eta \in X_{t-1} \bullet X^*_{t-1}$.

Now consider the case where $Y = X_t \subset \delta X_{t-1}$. Then there is a simplex τ of X_{t-1} with $\sigma' \subset \tau$. On the other hand let γ be an arbitrary simplex of X^*_{t-1} (let $\gamma = \emptyset$ when $X^*_{t-1} = \{\emptyset\}$). Then there is a simplex

$\alpha \in X_t^*$ having γ. By Lemma 5.1 there is a sequence of neighboring simplices from $\sigma' \cup \eta'$ to $\sigma' \cup \alpha$. Furthermore $\sigma' \cup \alpha$ and $\tau \cup \gamma$ are also neighboring. Thus we have a sequence of neighboring simplices from $\sigma \cup \eta$ to $\sigma' \cup \eta'$.

When $X_{t-1} \subset \delta X_t = \delta Y$, exchange the roles of X_t and X_{t-1}, then we have the proof.

Next we consider the case where $\eta = \emptyset$, $\sigma' = \emptyset$, and hence $\sigma \cup \eta = \sigma \in X \in P \cap A$ and $\sigma' \cup \eta' = \eta' \in Y \in D \cap A$. By condition (5.2) we obtain $U \in P \cap A$ and $V \in D \cap A$ such that $U \subset \delta X$ and $V \subset \delta Y$. Note that there is a sequence of neighboring simplices between any simplices in $U \cdot U^*$ and $V^* \cdot V$. Furthermore there are such sequences between σ and any simplices in $U \cdot U^*$ and between η' and any simplex of $V^* \cdot V$. Thus there is a sequence of neighboring simplices.

Lemma 5.4. $\delta(X \cdot X^*)$ is an #(n-1)-pm.

Proof. When $X^* = \{\emptyset\}$, the lemma is an immediate consequence of (5.3). Then we suppose $X^* \neq \{\emptyset\}$. Let $\tau \cup \gamma \subset \sigma \cup \eta \in \delta(X \cdot X^*)$ and $|\tau \cup \gamma| = n - 2$. Then by Lemma 4.1 either $\sigma \in \delta X$ and $\eta \in X^*$ or $\sigma \in X$ and $\eta \in \delta X^*$. We suppose the first case without loss of generality. Let $X \in P_k$, $X^* \in Q_h$, where $h = n - k$. Then we have two possibilities: case 1: $|\tau| = k - 1$, $|\gamma| = h - 1$, $\tau = \sigma$; case 2: $|\tau| = k - 2$, $|\gamma| = h$, $\gamma = \eta$. Now consider case 1. When $\gamma \notin \delta X^*$, there are exactly two simplices η_1 and η_2 of X^* having τ. Therefore we find two simplices $\tau \cup \eta_1$ and $\tau \cup \eta_2 \in \delta(X \cdot X^*)$ having $\tau \cup \gamma$. When $\gamma \in \delta X^*$, there is a unique simplex η of X^* having τ. Therefore $\tau \cup \gamma \subset \tau \cup \eta \in \delta(X \cdot X^*)$. On the other hand there is a unique simplex $\tau' \in X$ with $\tau \subset \tau'$, and consequently $\tau' \cup \gamma \in \delta(X \cdot X^*)$ has $\tau \cup \gamma$. We have seen that $\tau \cup \gamma$ is contained in at least two simplices of $\delta(X \cdot X^*)$. Suppose $\tau \cup \gamma \subset \alpha \cup \beta \in \delta(X \cdot X^*)$. Then either $|\alpha| = |\tau| + 1$ or $|\beta| = |\gamma| + 1$. In each case $\alpha \cup \beta$ must coincide with one of the simplices of $\delta(X \cdot X^*)$ in the above argument. Therefore $\tau \cup \gamma$ is contained in exactly two simplices of $\delta(X \cdot X^*)$ in case 1.

Next consider case 2. Since δX is a pm from (5.3), there are at most two simplices σ_1 and σ_2 of δX having τ. Hence there are at most two simplices of $\delta(X \cdot X^*)$ having $\tau \cup \eta$.

Now let $X_1 \cdot Y_1, X_2 \cdot Y_2, \ldots, X_s \cdot Y_s$ be the shelling order in condition (5.4). Let

$K_1 = X_1 \cdot Y_1,$

$K_i = K_{i-1} \cup (X_i \cdot Y_i),$

$L_i = \delta K_i \cap \delta(X_{i+1} \cdot Y_{i+1}).$

Then we have seen that

(5.5) K_1 is an orientable pm,

(5.6) δK_1 is a pm,

(5.7) $X_i \cdot Y_i$ is an orientable pm for any i,

(5.8) $\delta(X_i \cdot Y_i)$ is a pm for any i.

Furthermore by condition (5.4)

(5.9) L_i is a homogeneous pm for any i.

Thus we assume as the inductive hypothesis that

(5.5)' K_i is an orientable pm,

(5.6)' δK_i is a pm.

By (5.5)',(5.6)',(5.7),(5.8),(5.9), Lemma 3.4 and Theorem 4.3 we see that K_{i+1} is orientable if each $X_i \cdot Y_i$ is locally finite. To complete the induction we have to show that δK_{i+1} is also a pm.

Lemma 5.5. Let K and L be #n-pm's with $K \cap L = \emptyset$. Suppose δK and δL are pm's, $\delta K \cap \delta L$ is an #(n-1)-pm and $\delta K^{\#(n-2)} \cap \delta L^{\#(n-2)} = (\delta K \cap \delta L)^{\#(n-2)}$. Then $\delta(K \cup L)$ is an #(n-1)-pm.

Proof. Let $\tau \in \delta(K \cup L)^{\#(n-2)}$. Since $\delta(K \cup L)^{\#(n-2)} \subset \delta K^{\#(n-2)} \cup \delta L^{\#(n-2)}$, we have two cases: case 1: $\tau \in \delta K^{\#(n-2)}$ and $\tau \notin \delta L^{\#(n-2)}$; case 2: $\tau \in \delta K^{\#(n-2)} \cap \delta L^{\#(n-2)}$. Since δK is a pm, we find at most two simplices of $\delta(K \cup L)$ having τ in case 1. For case 2 suppose that there are three distinct #(n-1)-simplices σ_1, σ_2 and η of $\delta(K \cup L)$ having τ. If an #(n-1)-simplex σ is in $\delta(K \cup L)$, it lies in exactly one simplex of $K \cup L$. Therefore $\sigma \notin \delta K \cap \delta L$, whose simplices lie in exactly two simplices of $K \cup L$. Therefore we see that $\sigma_1, \sigma_2, \eta \notin \delta K \cap \delta L$. We assume without loss of generality that $\sigma_1, \sigma_2 \in \delta K \setminus \delta L$, $\eta \in \delta L \setminus \delta K$. Since δK is a pm and it has already two simplices σ_1 and σ_2 containing τ, $\delta K \cap \delta L$ has no simplex containing τ. Hence $\tau \notin (\delta K \cap \delta L)^{\#(n-2)} = \delta K^{\#(n-2)} \cap \delta L^{\#(n-2)}$. This is a contradiction. Therefore we have seen that there are at most two simplices of $\delta(K \cup L)$ having τ also in case 2.

When $X_i \neq \{\emptyset\}$ and $Y_i \neq \{\emptyset\}$, $X_i \cdot Y_i$ is locally finite if and only if both X_i and Y_i are finite . Thus we have the following theorem.

Theorem 5.6. Let $X_1 \cdot Y_1, X_2 \cdot Y_2, \ldots, X_s \cdot Y_s$ be the shelling order in (5.4). Suppose that for each $i = 1,\ldots,s$ either

(i) X_i and Y_i are finite or

(ii) one of them is $\{\emptyset\}$ and the other is locally finite.

Then $M = \bigcup_{i=1}^{s}(X_i \cdot Y_i)$ is a homogeneous and orientable #n-pm.

6. Polytopes and Polars

In this section we investigate pdpm's derived from convex polytopes and their polars. Through this section we denote an n-dimensional convex polytope of R^n by C and its polar by C^o. It is well-known that there is a one-to-one and inclusion-reversing mapping ψ from the set of faces C to that of C^o such that $\psi(C) = \emptyset$, $\psi(\emptyset) = C^o$ and $\dim F + \dim \psi(F) = n - 1$ for any face F of C (see Grünbaum [9]). The set of all faces of C forms a lattice, called the face lattice of C and denoted by $F(C)$, with the partial order of inclusion relation. In terms of lattice theory the face lattice $F(C^o)$ of C^o is said to be dual to $F(C)$ and ψ be dual isomorphism.

Now let S and T be finite triangulations of C and C^o, respectively. Let

$$X_F = \{ \sigma \mid \sigma \in \bar{S},\ \sigma \subset F,\ \dim \sigma = \dim F \},$$

$$Y_G = \{ \eta \mid \eta \in \bar{T},\ \eta \subset G,\ \dim \eta = \dim G \}$$

for each face F of C and for each face G of C^o. Then X_F and Y_G are #(dim F + 1)-pm and #(dim G + 1)-pm, respectively. For $k = 0,1,\ldots,n+1$ let

$$P_k = \{ X_F \mid F \text{ is a (k-1)-dimensional face of } C \},$$

$$D_k = \{ Y_G \mid G \text{ is a (k-1)-dimonsional face of } C^o \}$$

and let

$$P = \bigcup_{k=0}^{n+1} P_k, \quad D = \bigcup_{k=0}^{n+1} D_k.$$

When we introduce the partial order $\prec$ on P such that $X_F \prec X_{F'}$ iff $X_F \subset \delta X_{F'}$, P forms a lattice. It is clear that this is isomorphic to the face lattice $F(C)$. The set D with the same partial order is also a lattice isomorphic to $F(C^o)$. Let the operator $*$ be defined by $(X_F)^* = Y_{\psi(F)}$ and $(Y_G)^* = X_{\psi^{-1}(G)}$. Then both $(P, D, P \cup D \setminus \{\emptyset\}, *, n+1)$ and $(P, D, P \cup D \setminus (\{\emptyset\} \cup D_{n+1}), *, n+1)$ are #(n+1)-pdpm's. Let us denote the #(n+1)-pm's derived from these pdpm's by M_1 and M_2, respectively. The pm M_2 has a close relation to the H-complex in Freund [3]. By Corollary 4.4 we see that $\delta M_1 = \emptyset$ and $\delta M_2 = \delta T$. We also see that conditions (5.1) to (5.3) are satisfied.

We will show that condition (5.4) is also satisfied if we choose the decreasing order of dimension of X_i's as the shelling order. Following the definitions in the previous section let

$$K_1 = X_1 \cdot Y_1 = X_C \cdot \{\emptyset\} = X_C,$$
$$K_i = K_{i-1} \cup (X_i \cdot Y_i),$$
$$L_i = \delta K_i \cap \delta(X_{i+1} \cdot Y_{i+1}).$$

By Lemma 5.4 $\delta(X_{i+1} \cdot Y_{i+1})$ is an #n-pm. Then L_i, a subset of $\delta(X_{i+1} \cdot Y_{i+1})$, is also an #n-pm. Therefore it suffices to show that L_i is homogeneous and $\delta K_i^{\#(n-1)} \cap \delta(X_{i+1} \cdot Y_{i+1})^{\#(n-1)} = L_i^{\#(n-1)}$ in order to see that M_1 and M_2 are orientable.

Lemma 6.1. L_i is homogeneous for $i = 1, \ldots, s-1$.

Proof. Let F and G be faces of C and suppose that $\delta(X_F \cdot (X_F)^*) \cap \delta(X_G \cdot (X_G)^*) \neq \emptyset$. Then by Lemma 4.2 either F is a facet of G or G is a facet of F. Now let us denote X_{i+1} by X_F. Since the shelling order is the decreasing order of dimension, we have by Lemma 4.1 (4.6)

$$\begin{aligned} L_i &= \delta K_i \cap \delta(X_F \cdot (X_F)^*) \\ &= \bigcup\{\delta(X_G \cdot (X_G)^*) \cap \delta(X_F \cdot (X_F)^*) \mid F \text{ is a facet of } G\} \\ &= \bigcup\{(X_F \cdot (X_G)^*) \mid F \text{ is a facet of } G\} \\ &= X_F \cdot \bigcup\{(X_G)^* \mid F \text{ is a facet of } G\}. \end{aligned}$$

Note that $\bigcup\{(X_G)^* \mid F \text{ is a facet of } G\} = \bigcup\{Y_H \mid H \text{ is a facet of } \psi(F)\}$ and hence it is homogeneous. Therefore in the same way as in the proof of Lemma 5.1 we obtain that L_i is homogeneous.

Lemma 6.2. $\delta K_i^{\#(n-1)} \cap \delta(X_{i+1} \cdot Y_{i+1})^{\#(n-1)} = L_i^{\#(n-1)}$ for $i = 1, \ldots, s-1$.

Proof. From the definition of L_i it is clear that $L_i^{\#(n-1)} \subset \delta K_i^{\#(n-1)} \cap \delta(X_{i+1} \cdot Y_{i+1})^{\#(n-1)}$. We will show the reverse relation. Let

$\tau \cup \gamma \in \delta K_i^{\#(n-1)} \cap \delta(X_{i+1} \cdot Y_{i+1})^{\#(n-1)}$. Then $\tau \cup \gamma \in \delta(X_j \cdot Y_j)^{\#(n-1)} \cap \delta(X_{i+1} \cdot Y_{i+1})^{\#(n-1)}$ for some $j \leqq i$. Since $\delta(X_j \cdot Y_j)^{\#(n-1)} \cap \delta(X_{i+1} \cdot Y_{i+1})^{\#(n-1)} = (X_{i+1} \cdot Y_j)^{\#(n-1)}$. (see the proof of Lemma 6.1), there is $\sigma \cup \eta \in X_{i+1} \cdot Y_j$ having $\tau \cup \gamma$. By the shelling order chosen and that $X_{i+1} \cdot Y_j \neq \emptyset$ we have $Y_j \subset \delta Y_{i+1}$. Therefore there is k such that $j < k \leqq i+1$ and $Y_j \subset \delta Y_k$. Thus we have found a simplex $\sigma \cup \eta' \in X_{i+1} \cdot Y_k = \delta(X_{i+1} \cdot Y_{i+1}) \cap \delta(X_k \cdot Y_k) \subset L_i$ which contains $\sigma \cup \eta$, and hence $\tau \cup \eta$.

By these two lemmas we have seen that M_1 and M_2 are homogeneous and orientable #(n+1)-pm's. The pm M_1 will play a central role in generalizing Sperner's lemma in the following section.

Next we introduce a pdpm derived from face lattice $F(C)$ of C and a triangulation of another convex polytope B. Let B be an n-dimensional convex polytope of R^n and let S be a finite triangulation of B. Then S is an #(n+1)-pm. Let $Q_{n+1} = \{S\}$, Q_n be a finite partition of δS into n-pm's, Q_{n-1} be a finite partion of $\bigcup\{\ \delta Z \mid Z \in Q_n\ \}$ into #(n-1)-pm's, such that $\delta Z_1 \cap \delta Z_2$ is partition by Q_{n-1} for any Z_1, $Z_2 \in Q_n$. In general let Q_k be a finite partition of $\bigcup\{\ \delta Z \mid Z \in Q_{k+1}\ \}$ into #k-pm's such that $\delta Z_1 \cap \delta Z_2$ is partition by Q_k for any Z_1, $Z_2 \in Q_{k+1}$ (see the definition of pdpm in Section 4). Then we have a lattice $Q = \bigcup_{k=0}^{n+1} Q_k$ with the partial order $\prec$. Suppose the lattice Q is isomorphic to the face lattice $F(C)$ of C. Then Q forms an #(n+1)-pdpm together with D derived from the polar C^o, $Q \cup D \setminus \{\emptyset\}$ and a natural

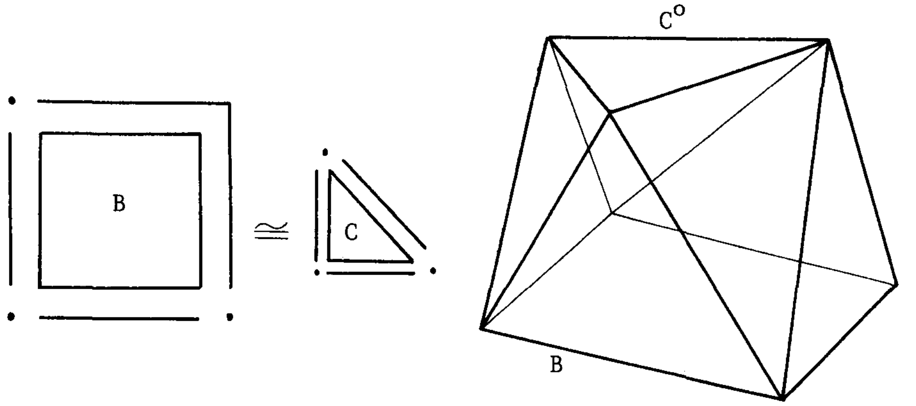

Fig. 1 M_3 derived from a square and the face lattice of a triangle definition of the operator $*$. Let us denote the #(n+1)-pm derived from this pdpm by M_3. Since M_3 is essentially the same as M_1, it is homogeneous and orientable. We omit the proof. Figure 1 shows Q derived from a square B and the face lattice of a triangle. The #3-pm M_3 is also shown. In this way we could treat the polytope B as another polytope C. This maneuverability will be used to prove combinatorial lemmas in the following section.

The final pdpm is also derived from B and F(C). Let S' be a finite triangulation of bd.B, where bd.B is the boundary of B with respect to the topology of R^n. Let $R'_n, R'_{n-1}, \ldots, R'_0$ be the set of pm's constructed from S' as in the definition of pdpm. Suppose $R' = \bigcup_{k=0}^{n} R'_k$ is isomorphic to the lattice of proper faces of polytope C. Let us denote the pm of R' corresponding to proper face F of C by Z'_F. Choose an arbitrary interior point b of B and let $b|Z'_F| = \{ x \mid x = \lambda b + (1-\lambda)y,\ y \in |Z'_F|,\ 0 \leq \lambda \leq 1 \}$, where we use the convention that $b\emptyset = \{b\}$. Let S be a finite triangulation of B such that it coincides with S' when restricted on bd.B and it triangulates $b|Z'_F|$ for each $Z'_F \in R'$ when restriced. Now let $Z_F = \{ \sigma \mid \sigma \in \bar{S},\ \sigma \subset b|Z'_F|,\ \dim \sigma = \dim F + 1 \}$, then Z_F is a #(dim F + 2)-pm. Let $R = \{ Z_F \mid F$ is a proper face of $C \} \cup \{\emptyset\}$ and $(Z_F)^* = Y_{\psi(F)}$,

$$
\begin{aligned}
(Y_G)^* &= Z_\emptyset = \{b\} && \text{if } G = C^o \\
&= Z_{\psi^{-1}(F)} && \text{if } G \neq C^o,
\end{aligned}
$$

then (R,D,R $\cup$ D \ $\{\emptyset\}$,*,n+2) is an #(n+2)-pdpm. Let us denote the #(n+2)-pm derived from this pdpm by M_4. We will not use M_4 in the

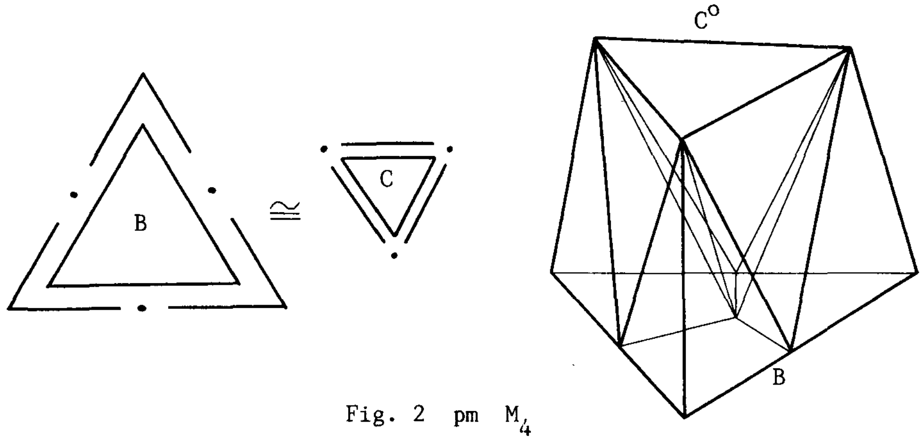

Fig. 2 pm M_4

following section because it requires a special triangulation of B that triangulates each $b|Z'_F|$. But this pm is very important for the interpretation of variable dimension algorithms working on B. Figure 2 shows R derived from triangle B and the face lattice of another triangle C, and also the pm M_4.

7. Combinatorial Lemmas in Topology

In this section we will show some combinatorial lemmas in topology. Here we require neither the boundary conditions on the labelling function nor the polytope to be a simlpex and give generalizations of Sperner's lemma. We will see that the theorem in this section implies the generalizations of Sperner's lemma on a simplex (Fan [2], Freund [4]) and on the cross product of simplices (van der Laan-Talman-Van der Heyden [15], Freund [5]).

Now let C be an n-dimensional convex polytope with m facets $F_1,\ldots,F_m$ in R^n. Let S be a finite triangulation of C. For a simplex σ of $\bar{S}$ we denote by $I(\sigma)$ the set of indices of facets containing σ, i.e., $I(\sigma) = \{ i \mid i \in I, \ \sigma \subset F_i \}$, where $I = \{1,\ldots,m\}$. For a given labelling function $\ell : S^{\#1} \to I$ we have the following generalization of Sperner's lemma.

Theorem 7.1. Let C be an n-dimensional convex polytope of R^n and let S be a finite triangulation of C. Then given a labelling function $\ell : S^{\#1} \to I$ and a nongenerate vertex v of C there is an odd number of simplices σ of $\bar{S}$ such that $\ell(\sigma) \cup I(\sigma)$ contains $I(\{v\})$ properly.

Proof. Let C^o be the polar of C and let T be a finite triangulation of C^o that introduces no new vertices (see, for example, Proposition 2.9 in Rourke-Sanderson [18] for the existence of T). Since there is a one-to-one correspondence ψ from the set of facets of C to the set of vertices of C^o, we extend the labelling function ℓ so that

$$\ell(u) = i \quad \text{if} \quad \psi^{-1}(\{u\}) = F_i.$$

Since vertex v is nongenerate, the corresponding facet $\psi(\{v\})$ of C^o is an #n-simplex. Therefore, by construction, we have seen that T has only one #(n+1)-simplex, say γ, such that $\ell(\gamma)$ contains $I(\{v\})$ properly. Now consider the #(n+1)-pm M_1 in the preceding section. Since M_1 has no boundary, we see that M_1 has an even number of simplices $\sigma \cup \eta$ such that $\ell(\sigma \cup \eta)$ contains $I(\{v\})$ properly by the usual path-following argument (see for example Eaves [1], Todd [19]). Since $T \subset M_1$, we have already found one simplex $\emptyset \cup \gamma = \gamma$ with the desired labellings. Hence M_1 has an odd number of simplices $\sigma \cup \eta$ such that $\ell(\sigma \cup \eta)$ contains $I(\{v\})$ properly and $\sigma \neq \emptyset$. It is readily seen from the construction of M_1 that $\ell(\eta) \subset I(\sigma)$ for $\sigma \cup \eta \in M_1$ with $\sigma \neq \emptyset$. Thus σ has the desired property.

To show the oddness we suppose that σ of $\bar{S}$ has the desired property. First we consider the case where $I(\sigma)$ has a label not in $I(\{v\})$, i.e., $I(\sigma) \not\subset I(\{v\})$. Let

$$H = I(\sigma) \cap I(\{v\})$$

$$K = I(\{v\}) \setminus H.$$

Since H is a subset of $I(\{v\})$, $F = \bigcap\{ F_i \mid i \in H \}$ is a face of C. Furthermore since v is a nondegenerate vertex of C, $\dim F = n - \#H$, so that

(7.1) $\dim F^* = \#H - 1$

and F^* has $\#H$ vertices. Consequently

(7.2) F^* is a $\#H$-simplex.

By the definition of H and K, we have

(7.3) $\#\sigma \geq \#\ell(\sigma) \geq \#K = \#I(\{v\}) - \#H = n - \#H.$

Let $G = \bigcap\{ F_i \mid i \in I(\sigma) \}$. Since $H \subset I(\sigma)$, G is a face of F, which implies that

(7.4) F^* is a face of G^*.

Since G contains σ, $\dim G \geq n - \#H - 1$ by (7.3). On the other hand, since $I(\sigma)$ has a label not in H,

(7.5) $\dim G \leq n - \#H - 1.$

In fact, the contrary of (7.5) would imply that vertex v were degenerate. Therefore we have $\dim G = n - \#H - 1$ and

(7.6) $\dim G^* = \#H.$

By (7.1), (7.4) and (7.6) we obtain that

(7.7) F^* is a facet of G^*.

By (7.2),(7.7) and that triangulation T introduces no new vertices, there exists a unique simplex η of $\bar{T}$ such taht η has F^* as a facet and

$\eta \subset G^*$. Let w be the unique vertex of $\eta \setminus F^*$. Then $\ell(w) \in I(\sigma)$. If $\ell(w) \in I(\{v\})$, it should be in H, which is contrary to (7.2) and that $\ell(F^*) = H$. Therefore $\ell(w) \notin I(\{v\})$, and hence we see that $\ell(\sigma \cup \eta)$ contains $I(\{v\})$ properly.

Next, we consider the case where $I(\sigma) \subset I(\{v\})$. By the similar argument to the above we see that G^* is an $\#I(\sigma)$-simplex of $\bar{T}$ with labels $I(\sigma)$. Therefore $\eta = G^*$ is the desired simplex.

The uniqueness of simplex η might be clear from the above argument, however, we show the uniqueness. Suppose that there are two distinct simplices η and η' of $\bar{T}$ whose label sets together with $\ell(\sigma)$ contain $I(\{v\})$ properly. Since $\#\sigma \cup \eta = \#\sigma \cup \eta' = n + 1$, $\ell(\sigma \cup \eta) = I(\{v\}) \cup \{j\}$ and $\ell(\sigma \cup \eta') = I(\{v\}) \cup \{j'\}$ for some labels j and j'. Therefore $\ell(\eta) \cap \ell(\eta') \subset I(\{v\})$. Note that vertex v is a nondegenerate vertex. Then there is a simplicial face of C^o such that whose vertices are $\ell(\eta) \cap \ell(\eta') = \ell(\eta \cap \eta')$. This contradicts the fact that both η and η' are the simplices of G^*. □

We give an illustration of the theorem in Figure 3, where polytope C is a pentagon with facets $F_1,\ldots,F_5$, vertex v lies on the intersection of F_1 and F_2. The odd number of simplices in the theorem are circled.

When vertex v is degenerate, facet $\psi(\{v\})$ is not an $\#n$-simplex but an $(n-1)$-dimensional convex polytope. Choose arbitrarily n indices out of $I(\{v\})$ and let $u_1,\ldots,u_n$ denote the corresponding vertices of $\psi(\{v\})$. Suppose we have a finite triangulation T' of $\psi(\{v\})$ which has

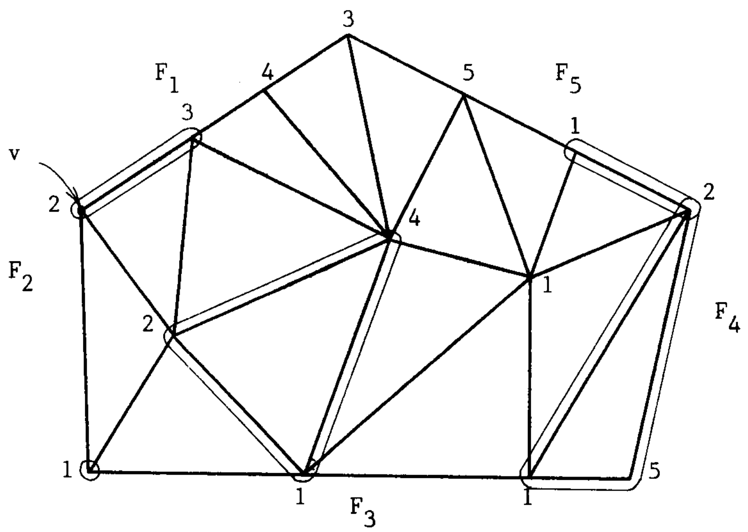

Fig. 3 Example of Theorem 7.1

$\{u_1,\ldots,u_n\}$ as its simplex and introduces no new vertices. Then we can make a triangulation T of C^o such that some simplex has $\{u_1,\ldots,u_n\}$ as a facet and it introduces no new vertices. Since T has only one simplex having $\{u_1,\ldots,u_n\}$, we see by the same argument as in the above proof that $\bar{S}$ has a simplex σ such that $\ell(\sigma) \cup I(\sigma)$ properly contains the previously chosen n indices. The author did not know whether the triangulation T' exists for n indices arbitrarily chosen out of $I(\{v\})$. But he received a letter [7] from Freund which presented a method of constructing such a triangulation. It will be outlined in Appendix.

In van der Laan-Talman-Van der Heyden [15] and in Freund [5] Sperner's lemma is generalized on the cross product of simplices, that is named simplotope by Freund. In the followings we show that their generalized lemma is derived from Theorem 7.1. Let C_i be an n_i-dimensional simplex of in R^{n_i} for $i = 1,\ldots,h$ and let $C = C_1 \times\ldots\times C_h$. Let S be a finite triangulation of C. Let the facets of C_i be indexed by $(i,1),\ldots, (i,n_i+1)$ for $i = 1,\ldots,h$ and consider the labelling function $\ell : S^{\#1} \to \{ (i,j) \mid 1 \leq i \leq h,\ 1 \leq j \leq n_i + 1 \}$. Choose an arbitrary vertex v of C. Then it lies in exactly n_i facets of C_i for each i $= 1,\ldots,h$. By Theorem 7.1 we see that there is at least one simplex σ of $\bar{S}$ such that $\ell(\sigma) \cup I(\sigma)$ contains $I(\{v\})$ properly. Then $\ell(\sigma) \cup I(\sigma)$ contins labels $(i,1),\ldots,(i,n_i+1)$ for some i. Thus we have the following corollary.

Corrollary 7.2. (Lemma 2.3 in van der Laan-Talman-Van der Heyden [15], Theorem 1 in Freund [5])
Let C_i be an n_i-dimensional simplex of R^{n_i} and let $F_{(i,1)},\ldots, F_{(i,n_i+1)}$ be its facets for $i = 1,\ldots,h$. Let $C = C_1 \times\ldots\times C_h$ be the cross product of the simplices and let S be a finite triangulation of C. Given a labelling function $\ell : S^{\#1} \to \{ (i,j) \mid 1 \leq i \leq h,\ 1 \leq j \leq n_i+1 \}$, there is at least one simplex σ of $\bar{S}$ such that $\{(i,1),\ldots,(i,n_i+1)\} \subset \ell(\sigma) \cup I(\sigma)$ for some i, where $I(\sigma) = \{ (i,j) \mid 1 \leq i \leq h,\ 1 \leq j \leq n_i+1,\ \sigma \subset F_{(i,j)} \}$.

By applying Theorem 7.1 to a simplex, we obtain the generalization of Sperner's lemma in Fan [2].

Corollary 7.3. Let C be an n-dimensional simplex of R^n and let S be a finite triangulation of C. Given a labelling function $\ell : S^{\#1} \to$

$\{1,\ldots,n+1\}$, there is an odd number of simplices σ of $\bar{S}$ such that $\ell(\sigma) \cup I(\sigma) = \{1,\ldots,n+1\}$.

Note that a simplex is indeed a simplotope but the oddness property of Corollary 7.3 is not obtained from Corollary 7.2. For the sake of consistency we will also prove Sperner's lemma by Corollary 7.3 (see Freund [4] for an inductive proof).

Corollary 7.4. Let C be an n-dimensional simplex and let S be a finite triangulation of C. Given a labelling function $\ell : S^{\#1} \to \{1,\ldots,n+1\}$ such that $\ell(v) \cap I(v) = \emptyset$ for any $v \in S^{\#1}$, there is an odd number, say $2h + 1$, of simplices of S having all labels.

Let $Or(.,.)$ be the orientation of S. We suppose that the vertices $v_1,\ldots,v_{n+1}$ of each of the above simplices are arranged so that $\ell(v_i) = i$ for $i = 1,\ldots,n+1$. Then

$$\begin{aligned} Or(\sigma,(v_1,\ldots,v_{n+1})) &= +1 \text{ (or } -1 \text{) for } h \text{ simplices of them,} \\ &= -1 \text{ (or } +1 \text{) for the other } h + 1 \text{ simplices.} \end{aligned}$$

Proof. Let π be a cyclic permutation of $\{1,\ldots,n+1\}$ such that

$$\begin{aligned} \pi(i) &= i + 1 \quad \text{if } 1 \leqq i \leqq n \\ &= 1 \qquad\quad \text{if } i = n + 1, \end{aligned}$$

and consider the labelling function $\pi\ell$. By Corollary 7.3 we obtain an odd number of simplices σ of $\bar{S}$ such that $(\pi\ell)(\sigma) \cup I(\sigma) = \{1,\ldots,n+1\}$. We show that $\ell(\sigma) \neq \{1,\ldots,n+1\}$ implies that $\sigma = \emptyset$ and hence a contradiction. Suppose that $k \notin \ell(\sigma)$. Then $k+1$ (1 when $k = n+1$) $\notin (\pi\ell)(\sigma)$. Since $(\pi\ell)(\sigma) \cup I(\sigma) = \{1,\ldots,n+1\}$, we have $k+1 \in I(\sigma) = \bigcup\{ I(\{v\}) \mid v \in \sigma \}$. Then by the condition of the labelling function $k+1 \notin \ell(\sigma)$. Repeating this argument we obtaind $\ell(\sigma) = \emptyset$, and hence $\sigma = \emptyset$.

Since M_1 is homogeneous and orientable, we have the latter half of the corollary by the usual path-following argument (see Gould-Tolle [8] and Eaves [1]).

In the followings we will give a sketch that the Theorem 7.1 is obtained by using the pm M_3. Let B be an n-dimensional convex polytope of R^n, S be a finite triangulation of B and v be a nongenerate vertex of B. Then there are exactly n facets, say $F_1,\ldots,F_n$, having vertex v. Let F_{n+1} be the union of all the other facets of B. Construcing Q as shown in Section 6, it is isomorphic to the face lattice of an n-dimensional simplex, which we will denote by C (see Figure 1 in Section 6). Since the polar C^o of C is a simplex, we do not need to subdivide it to have a finite triangulation T, i.e., T consists of a single #(n+1)-simplex C^o. Thus taking the set of all faces of C^o as D we have an #(n+1)-pdpm ($Q,D,Q \cup D \setminus \{\emptyset\},*,n+1$) and an #(n+1)-pm M_3. Now we extend the labelling function ℓ so that

$$\ell(u) = i \quad \text{if} \quad u \text{ is a vertex of } C^o \text{ and } \{u\}^* = R_{F_i} .$$

Then the simplex C^o has labels $1,\ldots,n+1$ and consequently M_3 has an odd number of simplices $\sigma \cup \eta$ such that $\ell(\sigma \cup \eta) = \{1,\ldots,n+1\}$ and $\sigma \neq \emptyset$. If we note that F_{n+1} is the union of several facets and hence the label $n + 1$ represents several labels, we have Theorem 7.1. Corollary 7.2 could be also proved in this manner.

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Appendix

Here we will sketch the proof of the existence of the triangulation T' referred after Theorem 7.1. This proof is due to Freund [7].

For simplicity we assume that the n-dimensional convex polytope C has the origin in its interior. Then C can be written as

$$C = \{ x \in R^n \mid Ax \leqq e \},$$

for some $m \times n$ matrix A, and e, the vector of ones. We assume here that each inequality of $Ax \leqq e$ defines a facet of C. The polar C^o of C can be written as

$$C^o = \{ y \in R^n \mid y = A^t\lambda,\ \lambda \geqq 0,\ e^t\lambda = 1 \},$$

i.e., C^o is the convex hull of the vectors of rows of A, because C is assumed to be bounded. We perturb C to a simple polytope, which induces a perturbation of C^o to a simplicial polytope and yields a simplicial subdivision of the boundary of C^o. Let B be any nonsingular $m \times m$ matrix and let $\varepsilon = (r^0, r^1, \ldots, r^m)$ for a sufficiently small positive number r. Define the perturbed polytope C as

$$C(B,\varepsilon) = \{ x \in R^n \mid Ax \leqq [e,B]\varepsilon \}.$$

Then $C(B,\varepsilon)$ is a simple polytope and its polar $C^o(B,\varepsilon)$ is simplicial. A subset β of the index set $\{1,\ldots,m\}$ of inequalities defining $C(B,\varepsilon)$ is called a basis for $C(B,\varepsilon)$ if

$$F(B,\varepsilon,\beta) = \{ x \in C(B,\varepsilon) \mid A_\beta = [e,B]_\beta\varepsilon \}$$

is a vertex of $C(B,\varepsilon)$, where A_β is the submatrix of A made up of rows of A indexed by β. Defining

$$G(B,\varepsilon,\beta) = \{ y \in R^n \mid y = A_\beta{}^t\lambda,\ \lambda \geqq 0,\ e^t\lambda = 1 \},$$

we obtain the following proposition.

Proposition 1. The collection of $G(B,\varepsilon,\beta)$, as β ranges over all bases for $C(B,\varepsilon)$, forms a triangulation $T'(B,\varepsilon)$ of the boundary of C^{o}.

Then the problem is now reduced to how to choose the matrix B so that the triangulation $T'(B,\varepsilon)$ has the simplex with vertices $u^{1},\ldots, u^{n}$ corresponding to the n indices chosen out of $I(\{v\})$. Note that

Proposition 2. $G(B,\varepsilon,\beta)$ is a simplex of $T'(B,\varepsilon)$ if and only if

$|\beta| = n$,

A_{β} has rank n, and

$A_{\alpha}A_{\beta}^{-1}[e,B]_{\beta} <_{\ell} [e,B]_{\alpha}$,

where $\alpha = \{1,\ldots,m\} \setminus \beta$ and $<_{\ell}$ denotes lexicographic ordering of a matrix.

We will define the matrix B as

$$B = \left[\begin{array}{c} B_{\beta} \\ \hline B_{\alpha} \end{array}\right] = \left[\begin{array}{c|c} A_{\beta} & 0 \\ \hline D & I \end{array}\right],$$

where D contains as its first column a vector of very large positive numbers and I is an identity matrix. Then B has clearly full rank. Since $v = A_{\beta}^{-1}e_{\beta}$, we have

$$A_{\alpha}A_{\beta}^{-1}[e,B]_{\beta} = A_{\alpha}A_{\beta}^{-1}[e_{\beta},A_{\beta},0] = [A_{\alpha}v,\ A_{\alpha},\ 0]$$

and also

$$[e,B]_{\alpha} = [e_{\alpha},\ D,\ I].$$

By the construction of D we see that

$$[A_{\alpha}v,\ A_{\alpha},\ 0] <_{\ell} [e_{\alpha},\ D,\ I].$$

Therefore $G(B,\varepsilon,\beta)$ is a simplex of $T'(B,\varepsilon)$.

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