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CONTINUOUS-TIME SEARCH MODEL WITH IMPROVEMENTS OF DETECTION RATES

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Abstract We consider a continuous-time search model for a stationary object in which a searcher allocates a limited time not only to search the object but also to improve detection rates. The objective is to maximize the probability that the searcher detcts the object by a given time T. We show that there is an optimal policy which indicates us to improve detection rates until a certain time $T_1(<T)$ and to search the object thereafter. The necessary condition for an allocation of time ot be optimal is given. We propose a method for solving the case of increasing concave detection rate functions and give a numerical example. Furthermore we derive a necessary condition for a policy to be optimal for a modified model in which the objective is to minimize the expected time before detection.

1. Introduction

Search models for a stationary object, in which detection probabilities remain unchanged during the search processes, have been studied well enough ([1]∿[8]). Nakai [9] treats a search problem in which detection probabilities vary according to a given rule but cannot be improved by a searcher. In this paper we consider a continuous-time search model for a stationary object in which a searcher can improve detection rates as well as search for an object during the search process. Though such a search phase can be often seen in real life (for example, search for a lost ball in a field), it has not been studied yet.

A stationary object exists in one of n boxes with a distribution $p_1=\langle p_1, \cdots, p_n\rangle$ where p_i is the probability that the object is in box i : $p_i>0$ $(i=1,\cdots, n)$, $\sum_{i=1}^{n} p_i=1$. At each time a searcher must decide rates of an allocation of effort not only to search the object but also to improve the detection rate in each box and therefore his policy can be denoted by $\{f, g\}$ where $f=\{f_1(t),\cdots, f_n(t)|t\geq 0\}$ is a search policy and $g=\{g_1(t),\cdots, g_n(t)|t\geq 0\}$ is an improvement policy. The function $f_i(t)(g_i(t))$ denotes a density of search effort (improvement effort) allocated in box i at time t. We suppose (i) $f_i(t)\geq 0$, $g_i(t)\geq 0$ for any $t(\geq 0)$ and any i $(i=1,\cdots, n)$ and (ii) $\sum_{i=1}^{n}\{f_i(t)+g_i(t)\}=1$ for any t. The assumption (i) is reasonable. The assumption (ii) means that the available effort at each time is limited to unity (that is, the effort is identified with the time), but this assumption does not spoil the generality of the model. Let $F_i(t)(G_i(t))$ be the accumlated search (improvement) effort in box i by time t under the policy $\{f, g\}$, that is, $F_i(t)=\int_0^t f_i(s)ds$ and $G_i(t)=\int_0^t g_i(s)ds$ $(i=1,\cdots,n)$. Let $\lambda_i(x)$ be the detection rate in box i given that the accumlated improvement effort in box i is x. Suppose (i) $0<\lambda_i(x)<\infty$ for any $x(\geq 0)$ and any i and (ii) $\lambda_i(x)$ is continuous, nondecreasing and piecewise differentiable in x for any i. The detection rate $\lambda_i(x)$ means that if the object is in box i and is not detected until time t by policy $\{f,g\}$, the probability that it is detected in the time interval $[t, t+\Delta t]$ is given by $\lambda_i[G_i(t)]f_i(t)\Delta t+o(\Delta t)$. We want to find the policy maximizing the probability that the object is detected by the given time $T(>0)$. For the purpose of excluding trivial cases, we suppose that there is at least one box i such that $\lambda_i(T)>0$. Let $Q_i(t)$ be the conditional probability that the object is not detected by time t by the policy $\{f, g\}$ given that it is in box i. By the definitions of $\lambda_i(x)$ and $Q_i(t)$, we obtain

$$Q_i(t+\Delta t) = Q_i(t)\{1-\lambda_i[G_i(t)]f_i(t)\Delta t+ (\Delta t)\}$$

from which we can derive a differential equation

$$Q_i'(t) = -\lambda_i[G_i(t)]f_i(t)Q_i(t) \ ; \ Q_i(0)=1.$$

Then we can obtain

$$(1)\quad Q_i(t) = \exp\{-\int_0^t \lambda_i[G_i(s)]f_i(s)ds\}.$$

Let $Q[t: f, g]$ be the probability that the object is not detected by time t by the policy $\{f, g\}$. Therefore

$$(2)\quad Q[t: f, g] = \sum_{i=1}^{n} p_iQ_i(t) = \sum_{i=1}^{n}p_i\exp\{-\int_0^t \lambda_i[G_i(s)]f_i(s)ds\}.$$

Our problem is to obtain a policy $\{f, g\}$ minimizing $Q[T: f, g]$.

2. The General Case

In this section we show the existence of the special type of optimal policy and obtain a necessary condition for a policy to be optimal.

Theorem 1. There exists an optimal policy $\{f^*, g^*\}$ which indicates us to improve detection rates until a certain time T_1 $(0 \leq T_1 < T)$ and to search the object thereafter, that is,

(3) $f_i^*(t)\left\{\begin{matrix}=\\ \geq\end{matrix}\right\}0$ and $g_i^*(t)\left\{\begin{matrix}\geq\\ =\end{matrix}\right\}0$ if $t\left\{\begin{matrix}\leq\\ >\end{matrix}\right\}T_1$ $(i=1, \cdots, n)$.

Proof: Suppose that the optimal policy $\{f^*, g^*\}$ satisfies the following property: For some time $t_1 (0<t_1<T)$ and some $\Delta(>0)$, there are box i and j such that $f_i^*(t)>0$ for any $t\in[t_1-\Delta, t_1]$ and $g_j^*(t)>0$ for any $t\in[t_1, t_1+\Delta]$. For a sufficiently small $\varepsilon(>0)$, we define a new policy $\{f, g\}$ as follows:

$$f_i(t) = \begin{cases} f_i^*(t)-\varepsilon & \text{if } t_1-\Delta\leq t\leq t_1 \\ f_i^*(t)+\varepsilon & \text{if } t_1\leq t\leq t_1+\Delta \\ f_i^*(t) & \text{otherwise} \end{cases}$$

$$f_k(t) = f_k^*(t) \qquad \text{for any } t(\geq 0), \text{any } k(\neq i)$$

$$g_j(t) = \begin{cases} g_j^*(t)+\varepsilon & \text{if } t_1-\Delta\leq t\leq t_1 \\ g_j^*(t)-\varepsilon & \text{if } t_1\leq t\leq t_1+\Delta \\ g_j^*(t) & \text{otherwise} \end{cases}$$

$$g_k(t) = g_k^*(t) \qquad \text{for any } t(\geq 0), \text{ any } k(\neq j).$$

That is to say, the new policy $\{f, g\}$ is obtained by exchanging the search effort ε in box i in the time interval $[t_1-\Delta, t_1]$ for the improvement effort ε in box j in $[t_1, t_1+\Delta]$. We will prove that

(4) $Q[T: f, g] \leq Q[T: f^*, g^*]$.

If so, the optimality holds even though the improvements in all boxes precede the searches in all boxes. Then the proof is completed. In order to prove (4),

(5) $$Q[T: f^*, g^*] - Q[T: f, g]$$
$$= p_i[\exp\{-\textstyle\int_0^T\lambda_i[G_i^*(s)]f_i^*(s)ds\} - \exp\{-\textstyle\int_0^T\lambda_i[G_i(s)]f_i(s)ds\}]$$
$$+ p_j[\exp\{-\textstyle\int_0^T\lambda_j[G_j^*(s)]f_j^*(s)ds\} - \exp\{-\textstyle\int_0^T\lambda_j[G_j(s)]f_j(s)ds\}].$$

Since $\lambda_i(\cdot)$ is nondecreasing,

$$\textstyle\int_0^T\lambda_i[G_i(s)]f_i(s)ds$$
$$= \textstyle\int_0^T \lambda_i[G_i^*(s)]f_i^*(s)ds + \varepsilon\{\textstyle\int_{t_1}^{t_1+\Delta} \lambda_i[G_i^*(s)]ds - \textstyle\int_{t_1-\Delta}^{t_1}\lambda_i[G_i^*(s)]ds\}$$
$$\geq \textstyle\int_0^T \lambda_i[G_i^*(s)]f_i^*(s)ds$$

and therefore the first terms of the right-hand side of (5) is nonnegative. On the other hand, noting that

$$\lambda_j[G_j(s)] = \begin{cases} \lambda_j[G_j^*(s)]+\varepsilon(s-t_1+\Delta)\lambda_j'[G_j^*(s)]+o(\varepsilon) & \text{if } t_1-\Delta\leq s\leq t_1 \\ \lambda_j[G_j^*(s)]+\varepsilon(t_1+\Delta-s)\lambda_j'[G_j^*(s)]+o(\varepsilon) & \text{if } t_1\leq s\leq t_1+\Delta \\ \lambda_j[G_j^*(s)] & \text{otherwise,} \end{cases}$$

we obtain

$$\begin{aligned} &\int_0^T\lambda_j[G_j(s)]f_j(s)ds \\ &= \int_0^T\lambda_j[G_j^*(s)]f_j^*(s)ds+\varepsilon\int_{t_{1-\Delta}}^{t_1}(s-t_1+\Delta)\lambda_j'[G_j^*(s)]f_j^*(s)ds \\ &+\varepsilon\int_{t_1}^{t_1+\Delta}(t_1+\Delta-s)\lambda_j'[G_j^*(s)]f_j^*(s)ds + o(\varepsilon) \\ &\geq\int_0^T\lambda_j[G_j^*(s)]f_j^*(s)ds \end{aligned}$$

since $\lambda_j(\cdot)$ is nondecreasing. Then the second term of the right-hand side of (5) is nonnegative and therefore the relation (4) is proved.

(q.e.d.)

Corollary 1. The optimal policy $\{f^*, g^*\}$ mentioned in Theorem 1 depends on values of $F_i^*(T)$ and $G_i^*(T)$ (i=1,•••, n), but is independent of the forms of the allocation functions $f_i^*(t)$ and $g_i^*(t)$ (i=1,•••,n; $0\leq t\leq T$), that is , the order of the allocation is arbitrary as long as the accu-accumulated total efforts $F_i^*(t)$ and $G_i^*(T)$ in each box are optimal.

Proof: If we put

$$\int_0^{T_1} g_i^*(t)dt = G_i^* \quad \text{and} \quad \int_{T_1}^T f_i^*(t)dt = F_i^* \qquad (i=1,\cdots, n),$$

then from (3) we obtain $F_i^*(T) = F_i^*$, $G_i^*(T) = G_i^*$ (i=1,•••,n) and $\lambda_i[G_i^*(t)] = \lambda_i(G_i^*)$ for any $t\in[T_1,T]$ and any i.

Therefore we obtain

$$Q[T : f^*,g^*] = \sum_{i=1}^n p_i\exp\{-\lambda_i(G_i^*)F_i^*\}$$

which is independent of the forms of f_i^* and g_i^* (i=1,•••,n) as long as F_i^* and G_i^* (i=1,•••,n) are fixed. (q.e.d.)

Hereafter we restrict our attention to the type of policy mentioned in Theorem 1, and call the process of the effort allocation until time T_1 by the first stage and the process thereafter by the second stage. In the first stage we concentrate in the improvement of detection rates and in the second stage we concentrate in the search for the object. Noting Corollary 1, a policy can be denoted by $\{F,G\} = \{F_1,\cdots,F_n;G_1,\cdots,G_n\}$ where $G_i(F_i)$ is the accumlated improvement (search) effort in box i in the first (second) stage. That is to say,

$$F_i = F_i(T) = \int_0^T f_i(s)ds \text{ and } G_i = G_i(t) = \int_0^t g_i(s)ds \quad (i=1,\cdots n).$$

Prof. Sakaguchi of Osaka University showed the following formulation of the model in the discussion with the author. When an improvement policy G is used, at the start of the second stage we face the classical detection search problem with the new detection rates $\lambda_i(G_i)$ $(i=1,\cdots n)$ and the total search effort $T - \sum_{i=1}^{n} G_i$. Charnes and Cooper [2] considers the following classical search problem :The prior distribution of the object is $p=<p_1,\cdots,p_n>$. Associated with box $i(=1,\cdots,n)$ is the exponential detection function with rate λ_i. The objective is to maximize the probability that the object is detected by the total search effort T.

Theorem 2. (Charnes and Cooper) The solution of the above detection search problem is given as follows: Without loss of generality, we can suppose that

$$p_1\lambda_1 \geq p_2\lambda_2 \geq \cdots\cdot \geq p_n\lambda_n .$$

Define

$$T_i = \sum_{j=1}^{i} \lambda_j^{-1} \log[p_j\lambda_j/(p_i\lambda_i)] \quad (i=1,\cdots,n;\ T_0 \equiv 0,\ T_{n+1} \equiv \infty)$$

and hence we can obtain $T_0 < T_1 \leq T_2 \leq \cdots \leq T_n < T_{n+1}$.

If $T_k < T \leq T_{k+1}$, then the optimal allocation of search effort is

$$F_i^* = \lambda_i^{-1} \mu_{ki} \ (i=1,\cdots,k) \ ; \ = 0 \ (i=k+1,\cdots,n)$$

where

$$\mu_{ki} = (T - \sum_{j=1}^{k} \lambda_j^{-1} \log [p_j\lambda_j/(p_i\lambda_i)] / \sum_{j=1}^{k} \lambda_j^{-1} .$$

The detection probability under the optimal search policy is given by

$$p^*[T;\ \lambda_1,\cdots,\ \lambda_n] = \sum_{j=1}^{k} p_i \ [1- \exp(-\mu_{ki})].$$

Therefore our problem in the first stage is to solve the allocation problem

$$p^*[T - \sum_{i=1}^{n} G_i \ ;\ \lambda_1(G_1),\ \cdots,\ \lambda_n(G_n)] \longrightarrow \max_{(G_1,\cdots\cdot,G_n)} .$$

But the set of the searched boxes in the second stage varies according to the values of $G_i(i=1,\cdots, n)$ and hence it is very difficult to obtain the optimal improvement policy G^* by this method. Then we treat the allocation problems in both stages simultaneously.

Theorem 3. A necesssary condition for a policy $\{F^*, G^*\}$ to be optimal is given as fllows : There exists a positive constant μ, which depends on p_i, $\lambda_i(\cdot)$ $(i=1,\cdots,n)$ and T but is indepedent of i, such that

(6) $$p_i\lambda_i(G_i^*)\exp[-\lambda_i(G_i^*)F_i^*]\begin{Bmatrix} = \\ \leq \end{Bmatrix}\mu \quad \text{if} \quad F_i^*\begin{Bmatrix} > \\ = \end{Bmatrix}0$$

(7) $$p_i\lambda_i'(G_i^*)F_i^*\exp[-\lambda_i(G_i^*)F_i^*]\begin{Bmatrix} = \\ \leq \end{Bmatrix}\mu \quad \text{if} \quad G_i^*\begin{Bmatrix} > \\ = \end{Bmatrix}0.$$

Proof: The problem becomes a nonlinear programming:

$$Q[T: F,G] = \sum_{i=1}^{n} p_i \exp[-\lambda_i(G_i)F_i] \longrightarrow \min_{\{F,G\}}$$

subject to

$$\sum_{i=1}^{n}(F_i + G_i) = T \ ; \ F_i,\ G_i \geq 0 \ (i=1,\cdots, n).$$

For a multiplier μ, the Lagrangian is given by

$$L(F,\ G: \mu) = -\sum_{i=1}^{n} p_i\exp[-\lambda_i(G_i)F_i] + \mu\{T-\sum_{i=1}^{n}(F_i+G_i)\}.$$

By the Kuhn-Tucker theorem, we can obtain the following relations for an optimal solution (F^*, G^*, μ):

(8) $$\left.\frac{\partial L}{\partial F_i}\right|_{(F^*,\ G^*,\ \mu)} = p_i\lambda_i(G_i^*)\ \exp[-\lambda_i(G_i^*)F_i^*] - \mu \leq 0 \quad \text{for any } i$$

(9) $$\left.\frac{\partial L}{\partial G_i}\right|_{(F^*,\ G^*,\ \mu)} = p_i\lambda'_i(G_i^*)F_i^*\ \exp[-\lambda_i(G_i^*)F_i^*] - \mu \leq 0 \quad \text{for any } i$$

$$\left.\frac{\partial L}{\partial \mu}\right|_{(F^*,\ G^*,\ \mu)} = T - \sum_{i=1}^{n}(F_i^* + G_i^*) = 0$$

(10) $$\sum_{i=1}^{n} F_i^*\{p_i\lambda_i(G_i^*)\ \exp[-\lambda_i(G_i^*)F_i^*] - \mu\}$$
$$+ \sum_{i=1}^{n} G_i^*\{p_i\lambda'_i(G_i^*)F_i^*\ \exp[-\lambda_i(G_i^*)F_i^*] - \mu\} = 0$$

(11) $F_i^*,\ G_i^* \geq 0$ for any i.

From (8), (9) and (11), all terms of the left hand side of (10) are zero and hence the relations (6) and (7) can be obtained. By the relation (6), the constant μ is positive since there is at least one box with $\lambda_i(G_i^*)>0$.
(q.e.d.)

RemarK. (i) When $\lambda_i(x) = \lambda_i (i=1,\cdots, n)$ for any x ($\geqq 0$), the optimal policy is to search (not to search) in box i with no improvement if $p_i\lambda_i > \mu$ ($p_i\lambda_i \leqq \mu$) where the constant μ can be obtained if necessary. This agrees with the well-known result.

(ii) In order to consider the meanings of relations (6) and (7), we define $P[T ; F, G]$ by the probability that the object is detected by time T under the policy $\{F, G\}$, that is,

$$P[T ; F, G] = 1-Q[T ; F, G] = \sum_{i=1}^{n} p_i \{1- \exp[-\lambda_i(G_i)F_i]\}.$$

Then the relations (6) and (7) become

$$\left.\frac{\partial P[T ; F, G]}{\partial F_i}\right|_{(F^*, G^*)} \begin{Bmatrix} = \\ \leqq \end{Bmatrix} \mu \qquad \text{if } F_i^* \begin{Bmatrix} > \\ = \end{Bmatrix} 0$$

and

$$\left.\frac{\partial P[T ; F, G]}{\partial G_i}\right|_{(F^*, G^*)} \begin{Bmatrix} = \\ \leqq \end{Bmatrix} \mu \qquad \text{if } G_i^* \begin{Bmatrix} > \\ = \end{Bmatrix} 0$$

respectively. That is to say, the relation (6) denoted that the optimal policy maximized the marginal detection probability with respect to the search effort in boxes being allocated the search effort. The meaning of the relation (7) is similar. These relations are popular in the search theory.

Corollary 2. An optimal policy $\{F^*, G^*\}$ has the following properties : (i) If $F_i^* = 0$, then $G_i^* = 0$, that is, a unsearched box should not be improved. (ii) For any i,

$$(12) \quad F_i^* = \begin{cases} 0 & \text{if } G_i^* = 0 \text{ and } p_i\lambda_i(0) \leqq \mu \\ \lambda_i^{-1}(0) \log[p_i\lambda_i(0)/\mu] & \text{if } G_i^* = 0 \text{ and } p_i\lambda_i(0) > \mu \\ \lambda_i(G_i^*)/\lambda_i'(G_i^*) & \text{if } G_i^* > 0 \end{cases}$$

and G_i^* (>0) is a positive root of an equation

$$(13) \quad p_i\lambda_i(x) \exp[-\lambda_i^2(x)/\lambda_i'(x)] = \mu.$$

Proof: (i) If $F_i^* = 0$ and $G_i^* > 0$ for some i, then $\mu = 0$ by the relation (7) which is contradictory. (ii) From (6) and (7), we know the followings : If $F_i^* = G_i^* = 0$, then $p_i\lambda_i(0) \leqq \mu$. If $F_i^* > 0$ and $G_i^* = 0$, then $p_i\lambda_i(0) > \mu$ and $F_i^* = \lambda_i^{-1}(0)\log[p_i\lambda_i(0)/\mu]$ by (6). If $G_i^* > 0$, then $F_i^* > 0$ by the assertion (i) and therefore the equations hold in (6) and (7). Hence we can obtain $F_i^* = \lambda_i(G_i^*)/\lambda_i'(G_i^*)$ and $p_i\lambda_i(G_i^*)\ \exp[-\lambda_i^2(G_i^*)/\lambda_i'(G_i^*)]=\mu$. Then the result is clear. (q.e.d.)

In conclusion of the discussion of the general case, we consider a single special case, that is, one-box case. [This example was showed by Prof. Sakaguchi of Osaka University.] In the case of n=1, our problem (20) becomes

$$\lambda(G)(T - G) \longrightarrow \max_{0 \leqq G \leqq T}.$$

When $\lambda(x)$ is increasing and concave, the optimal improvement effort G^* is as follows : If $T \leqq \lambda(0)/\lambda'(0)$, then $G^* = 0$ and if $T > \lambda(0)/\lambda'(0)$, then G^* is an unique root of an equation $\lambda'(G)(T-G) = \lambda(G)$ on the interval $(0,T)$. That is to say, if the total time T is smaller than $\lambda(0)/\lambda'(0)$, then we must search the object immediately without improvement of the detection rate. Otherwise, it is optimal to improve the rate somewhat.

3. The Case of Concave Detection Rate Functions

In this section we add the assumption that all $\lambda_i(x)$ $(i,\cdots,n)$ are concave in x, and propose a method for obtainning an optimal policy, that is, for deriving the value of μ in Corollary 2 explicitely. In particular, we explain the case of linear detection rate functions in detail.

For box i (=1,$\cdots$,n), we consider a curve C_i on the (x, y)-plane defined by $C_i = C_{i,1} \cup C_{i,2}$ where

$$C_{i,1} = \{(0,y) \mid y \geqq p_i\lambda_i(0)\exp[-\lambda_i^2(0)/\lambda_i'(0)]\}$$

$$C_{i,2} = \{(x,y) \mid x>0,\ \ y=p_i\lambda_i(x)\exp[-\lambda_i^2(x)/\lambda_i'(x)]\}.$$

Theorem 4. If all $\lambda_i(x)$ $(i=1,\cdots,n)$ are increasing and concave in x, the following policy $\{F^*, G^*\}$ is optimal: The improvement effort G_i^* in box i is given by the x-coordinate of one of intersections of the straight line $y=\mu$ and the curve C_i. The search effort F_i^* in box i is given by (12). Here the determination of the value of μ and a selection of an appropriate intersection must be carried out together with the satisfaction of the relation $\sum_{i=1}^{n}(F_i^* + G_i^*)=T$.

Proof: If all $\lambda_i(x)$ $(i=1,\cdots,n)$ are concave in x, conditions (6) and becomes a concave programming. Therefore it is sufficient to find μ satisfying the assertion (ii) of Corollary 2. Hence the result is clear.

(q.e.d)

In the following, we analyze the case of linear detection rate functions in detail. Suppose that $\lambda_i(x) = a_i x + b_i$ ($i=1,\cdots, n$; $a_i, b_i > 0$). By Corollary 2, we obtain

$$(14)\quad F_i^* = \begin{cases} 0 & \text{if } G_i^* = 0 \text{ and } p_i b_i \leq \mu \\ b_i^{-1} \log(p_i b_i/\mu) & \text{if } G_i^* = 0 \text{ and } p_i b_i > \mu \\ G_i^* + b_i/a_i & \text{of } G_i^* > 0 \end{cases}$$

where $G_i^*(>0)$ is a positive root of an equation

$$(15)\quad p_i(a_i x + b_i) \exp[-(a_i x + b_i)^2/a_i] = \mu.$$

When the improvement effort x in box i increases from zero, we shall investigate the move of a point $Z_i = (X_i, Y_i)$ where

$$(16)\quad X_i = (a_i x + b_i)^2/a_i, \qquad Y_i = p_i(a_i x + b_i).$$

Eliminating x from relations (16), we obtain

$$(17)\quad Y_i^2 = a_i p_i^2 X_i$$

along which the point Z_i moves. On the other hand, by (15) we obtain

$$(18)\quad Y_i = \mu \exp(X_i)$$

on which the point Z_i lies at $x=G_i^*$. Therefore when the improvement effort x in box i increases from zero, the point Z_i starts from the initial point $Z_i^0 = (b_i^2/a_i, p_i b_i)$, moves along the curve (17) and reaches the curve (18) at $x = G_i^*$ if possible. The condition, under which two curves (17) and (18) have at least one intersection, is given by $a_i p_i^2 \geq 2e\mu^2$. Thus we obtain Figure 1 in which each region is defined as follows:

$$\text{Region A} = \left\{ (X,Y) \;\middle|\; \begin{array}{ll} Y^2 < 2e\mu^2 X & \text{if } 0 < X \leq (2e)^{-1} \\ Y \leq \mu & \text{if } (2e)^{-1} < X \end{array} \right\}$$

$$\text{Region B} = \left\{ (X,Y) \;\middle|\; \begin{array}{ll} \mu < Y \text{ and } Y^2 < 2e\mu^2 X & \text{if } (2e)^{-1} < X < 1/2 \\ \mu < Y \leq \mu \exp(X) & \text{if } 1/2 \leq X \end{array} \right\}$$

$$\text{Region C} = \{ (X,Y) \mid Y > \mu \exp(X) \}$$

$$\text{Region D} = \{ (X,Y) \mid 0 < X \leq (2e)^{-1},\ Y \leq \mu,\ Y^2 \geq 2e\mu^2 X \}$$

$$\text{Region E} = \left\{ (X,Y) \;\middle|\; \begin{array}{ll} \mu < Y \leq \mu \exp(X) & \text{if } 0 < X < (2e)^{-1} \\ Y^2 \geq 2e\mu^2 X \text{ and } Y \leq \mu \exp(X) & \text{if } (2e)^{-1} < X < 1/2. \end{array} \right\}$$

For each i, consider three types of solution:

Type (i) : $F_i^* = G_i^* = 0$

Type (ii) : $F_i^* = b_i^{-1} \log (p_i b_i / \mu)$ and $G_i^* = 0$

Type (iii) : $F_i^* = G_i^* + b_i/a_i$ and $G_i^* (> 0)$ is a positive root of the equation (15).

Corollary 3. If $\lambda_i(x) = a_i x + b_i$ (i=1,···, n ; a_i, b_i > 0), then the following policy $\{F^*, G^*\}$ is optimal;

Case [A] : If $Z_i^0 \in$ Region A, then Type (i) occurs.

Case [B] : If $Z_i^0 \in$ Region B, then Type (ii) occurs.

Case [C] : If $Z_i^0 \in$ Region C, then either Type (ii) or (iii) occurs.

In this case, the equation (15) has a unique positive root.

Case [D] : If $Z_i^0 \in$ Region D, then either Type (i) or (iii) occurs.

Case [E] : If $Z_i^0 \in$ Region E, then either Type (ii) or (iii) occurs.

In Case [D] and [E], the equation (15) has two positive roots. When we select one of two types in Case [C], [D] and [E], and when we select one of two positive roots in Type (iii), all selections must be carried out under the condition that the relation $\sum_{i=1}^{n} (F_i^* + G_i^*) = T$ is satisfied.

Proof: If $Z_i^0 \in$ Region A, two curves (17) and (18) have no intersection, that is, the equation (15) has no positive root. Furthermore $p_i b_i \leq \mu$ in Region A and therefore by (12) we can obtain $F_i^* = G_i^* = 0$. The other case can be proved by the similar method. (q.e.d.)

From Corollary 3, we can know the following property of the optimal policy in the case of $\lambda_i(x) = a_i x + b_i$ (i=1,···n). When the initial detection rate $p_i b_i$ is sufficiently small and b_i^2/a_i is sufficiently large (that is, the improvement rate a_i is sufficiently small), the initial point $Z_i^0 = (b_i^2/a_i, p_i b_i)$ is contained in Region A and therefore no improvement and no search in box i is optimal. This result is consistent with our common sense that both to improve and to search are nonsense when both the initial detection rate $p_i b_i$ and the improvement rate a_i are sufficient small. When $p_i b_i$ is somewhat large and b_i^2/a_i is sufficiently large, the point Z_i^0 is contained in Region B and therefore the optimal policy indicates to search in box i with no improvement. When b_i^2/a_i is sufficiently small (that is, the improvement rate a_i is sufficiently large for the initial detection rate $p_i b_i$), the point Z_i^0 is contained in one of three regions C, D, and E and therefore it may occur that the optimal policy indicates to improve and search in box i. These results are also consistent with our common sense.

[Numerical Example]

We consider a two-box case in which $p=\langle 1/2,1/2\rangle$, $\lambda_1(x)=3x+1$ and $\lambda_2(x)=2x+2$. Note that in box 1, the initial detection rate is smaller but the improvement rate is large than in box 2. Two curves C_1 and C_2 are described in the right-half of Figure 2. Let $\alpha_1(\mu)$ and $\beta_1(\mu)$ [We suppose that $\alpha_1(\mu)\leq\beta_1(\mu)$] be two positive roots of the equation $(1/2)(3x+1)\exp[-(3x+1)^2/3]=\mu$, and let $\beta_2(\mu)$ be a unique positive root of the equation $(x+1)\exp[-2(x+1)^2]=\mu$. When the value of μ decreases continuously from infinity to zero, we want to calculate the value of $T=\sum_{i=1}^{2}(F_i^*+G_i^*)$ for any value of μ by Theorem 4. However for any $\mu\in(0.358, 0.371)$ there are three intersections of the curve C_1 and the straight line $y=\mu$, and the value of T is different for each intersection. In order to guarantee the continuous change of the value of T, we force the value of μ vary overlappingly such that the intersection Z_i of the curve C_i (i=1, 2) and the line $y=\mu$ moves continuously. If we denote such a situation in Figure 2 concretely, μ and Z_i (i=1, 2) moves along the following routes:

μ : A $\longrightarrow$ C $\longrightarrow$ B $\longrightarrow$ D $\longrightarrow$ 0

Z_1 : A $\longrightarrow$ C $\longrightarrow$ E $\longrightarrow$ F $\longrightarrow$ G

Z_2 : A $\longrightarrow$ C $\longrightarrow$ B $\longrightarrow$ D $\longrightarrow$ H

where letters A, •••, H indicate marks on the routes in Figure 2. We can calculate the value of T for each case. For example, when μ varies from C to B, $Z_1(\alpha_1(\mu),\mu)$ and $Z_2=(0,\mu)$, and therefore $G_1^*=\alpha_1(\mu)$, $G_2^*=0$, $F_1^*=\alpha_1(\mu)+1/3$ and $F_2^*=(1/2)\log(1/\mu)$ from (14). Hence we can obtain

$$T=\sum_{i=1}^{2}(F_i^*+G_i^*) = 2\alpha_1(\mu)-(1/2)\log\mu + 1/3.$$

For other cases, the similar discussion can be developed. Values of T are given in Figure 2. Using Figure 2, we can obtain an optimal policy. For given $T(>0)$, we obtain the corresponding value of μ in the left-half of Figure 2 and the corresponding value of G_i^*(i=1,2) in the right-half. The values of F_i^*(i=1,2) are given by (14). The maximal detection probability

$P^*(T)$ is given by $P^*(T) = 1 - (1/2)\ \exp[-(3G_1^* + 1)F_1^*] - (1/2)\exp[-(2G_2^* + 2)F_2^*]$. For comparison we consider the case without improvement, in which $p = \langle 1/2, 1/2 \rangle$, $\lambda_1 = 1$ and $\lambda_2 = 2$. The solution for this ordinary detection search model is given by Theorem 2. The optimal allocation effort $\tilde{F}_i$ in box i $(=1, 2)$ is given as follows : If $0 \leqq T \leqq \tau_1 [=(1/2)\log 2 \fallingdotseq 0.347]$, then $\tilde{F}_1 = 0$ and $\tilde{F}_2 = T$. If $\tau_1 < T < \infty$, then $\tilde{F}_1 = (1/3)(2T - \log 2)$ and $\tilde{F}_2 = (1/3)(T + \log 2)$. The maximal detection probability is given by $\tilde{P}(T) = 1-(1/2)\exp(-\tilde{F}_1) - (1/2)\exp(-2\tilde{F}_2)$.

For values of T, we select 0.3, 0.6, 0.9, 1.0, 3.0 as representative values in five regions of T in Figure 2. For these values, the above-mentioned quantities are calculated in Table 1. Furthermore Table 1 contains values of $d(T) = P^*(T) - \tilde{P}(T)$ which is the increase of the detection probability by the improvement. From Table 1, we know the followings :

[i] If T is sufficiently small (T=0.3, 0.6), there is no time to improve. If T is somewhat large (T=0.9, 1.0), it is optimal to improve box 1 rather than box 2. If T is sufficiently large (T=3.0), there is time enough to improve both boxes. Note that in this example box 1 is more effective for the improvement than box 2 since it has a smaller initial detection probability and a large improvement rate.

[ii] A remarkable point is that the optimal search effort allocation F_2^* in box 2 is not necessarily increasing in T, for example, $F_2^* = 0.501$ for T= 0.9 and $F_2^* = 0.495$ for T= 1.0. The reason is as follows : Since the detection rate in box 1 is improved enough in the case of $T = 0.1$ rather than in the case of T=0.9, it is profittable to allocate much search effort in box 1 even if the search effort in box 2 decreases (note that $p_1 = p_2 = 1/2$).

In general, when $G_i^* = 0$ and $\mu < p_i b_i$ in (14), F_i^* is decreasing in μ and hence if μ is nondecreasing in T, then F_i^* is decrasing in T.

[iii] We observe that $F_i^* \leqq \tilde{F}_i$ (i=1,2) for five values of T. But it is not necessarily clear that this property holds in general since it is possible that the improvement of the detection rate in box i stimulates us to allocate much search effort in box i.

[iv] Since $P^*(T) \geqq \tilde{P}(T)$ for any $T(>0)$ evidently, the difference d(T),which indicates the effect of the improvement, is nonnegative. Though in Table 1 d(T) is nondecreasing in T, this property is not true in general because $\lim_{T\to 0} d(T) = \lim_{T\to\infty} d(T) = 0$. we are interested in $T = T^*$ attaining $\max_{0<T<\infty} d(T)$ but it is difficult to obtain T^* explicitly since the function d(T) is not necessarily unimodal.

Table 1. The solution of numerical example for

T = 0.3, 0.6, 0.9, 1.0, 3.0.

T	0.3	0.6	0.9	1.0	3.0
G_1^*	0	0	0.033	0.086	0.663
G_2^*	0	0	0	0	0.170
F_1^*	0	0.169	0.366	0.419	0.997
F_2^*	0.300	0.431	0.501	0.495	1.170
$P^*(T)$	0.226	0.367	0.482	0.519	0.942
$\tilde{F}_1$	0	0.169	0.369	0.436	1.769
$\tilde{F}_2$	0.300	0.431	0.531	0.564	1.231
$\tilde{P}(T)$	0.226	0.367	0.481	0.515	0.872
d(T)	0	0	0.001	0.004	0.070

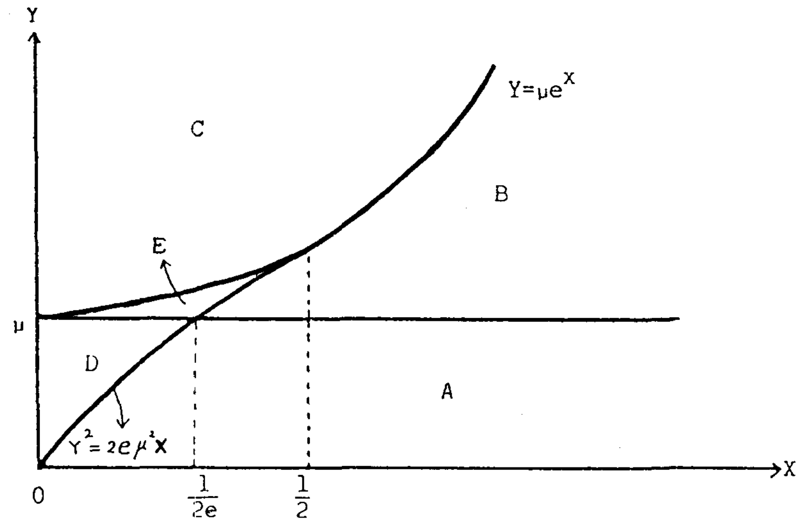

Fig. 1. Optimal policy region in the case of linear detection rate functions.

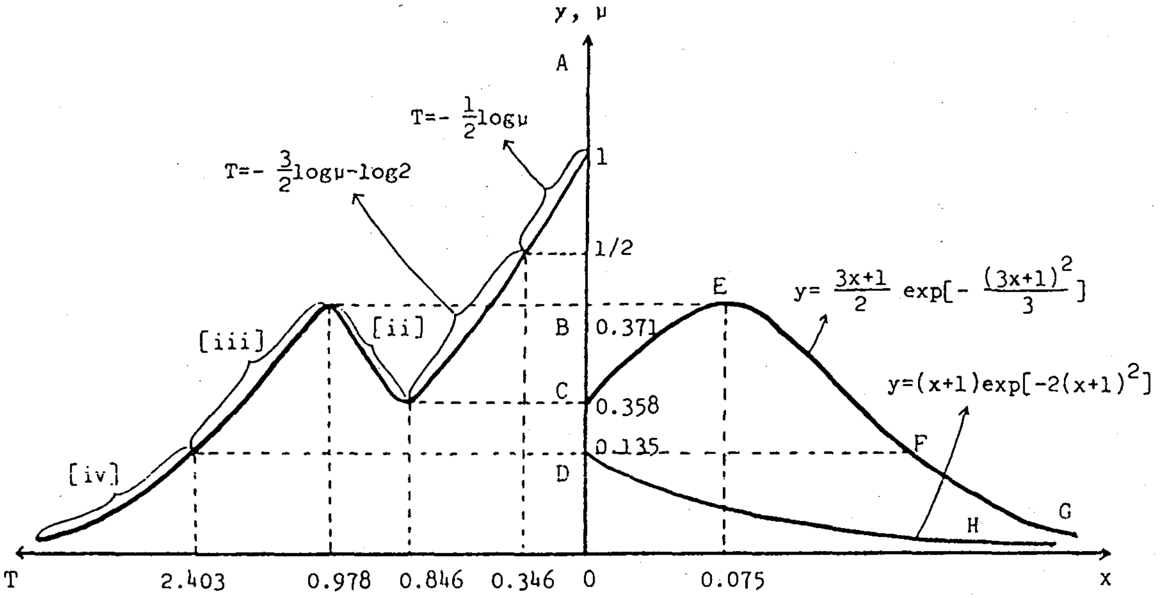

Fig. 2. The solution of the numerical example in which

$p = \langle 1/2, 1/2 \rangle$, $\lambda_1(x)=3x+1$, $\lambda_2(x)=2x+2$.

[ii] $T=2\alpha_1(\mu)-(1/2)\log\mu + 1/3$

[iii] $T=2\beta_1(\mu)-(1/2)\log\mu + 1/3$

[iv] $T=2\beta_1(\mu)+2\beta_2(\mu)+4/3$

4. Minimizing Problem of Expected Effort

In this section we shall turn our attention to another objective under which we minimize the expected effort until detection. For this modified model, a policy, which indicates us to improve detection rates until a certain time and to search the object thereafter, seems to be no longer optimal since this policy wastes much time in the early stage. But since the optimal policy in the privious section is uniformly optimal with respect to the total time T, it seems that the following conjecture holds.

Conjecture : The policy, which satisfies the following property, is optimal : At any time t, the accumlated search effort and the accumlated improvement effort in box i (=1,$\cdots$, n) until time t are equal to the optimal allocations F_i^* and G_i^* respectively in the previous model with time period $T=t$.

If the optimal allocation F_i^* and G_i^* (i=1,$\cdots$,n) of the previous model are nondecreasing in T, then the above conjecture is true and therefore the optimal policy in the modified model can be constructed by the sequential allocation method ; otherwise , the policy in the above conjecture can not be constructed. Hence the above conjecture can not necessarily hold. For example, the numerical example in Section 3 gives a counterexample for this conjecture since F_2^* is not nondecreasing in T.

Let T_i be the time at which the object is detected by a policy {f,g}, given that it is in box i. The distribution function of T_i is given by $H_i(t) = \mathrm{P}\{T_i \leq t\} = 1-Q_i(t)$ where $Q_i(t)$ is given in Section 1. Let C[f,g] be the expected effort required to detect the object by a policy {f, g}. Then the modified problem is represented by

$$\begin{aligned} \mathrm{C}[f, g] &= \sum_{i=1}^{n} p_i \int_0^\infty t\, dH_i(t) \\ &= \sum_{i=1}^{n} p_i \int_0^\infty \exp\{-\int_0^t \lambda_i[G_i(s)]f_i(s)ds dt \longrightarrow \min_{\{f,g\}} \end{aligned}$$

Theorem 5. A necessary condition for a policy $\{f^*, g^*\}$ to be optimal for the modified model is given as follows : There exist two nonnegative functions $\xi_1(t)$ and $\xi_2(t)$ such that

$$(19)\quad p_i\lambda_i[G_i^*(t)] \int_t^\infty L_i^*(s)\,ds \ \{\begin{smallmatrix} = \\ \leq \end{smallmatrix}\}\ \xi_1(t) \qquad \text{if} \qquad f_i^*(t)\ \{\begin{smallmatrix} > \\ = \end{smallmatrix}\}\ 0$$

$$(20)\quad p_i\int_t^\infty \{\int_t^s \lambda_i'[G_i^*(u)]f_i^*(u)du\}L_i^*(s)ds\ \{\begin{smallmatrix} = \\ \leq \end{smallmatrix}\}\ \xi_2(t) \quad \text{if}\ g_i^*(t)\{\begin{smallmatrix} > \\ = \end{smallmatrix}\}\ 0$$

$L_i^*(s) = \exp\{-\int_0^s \lambda_i[G_i^*(u)]f_i^*(u)du\}.$

Proof: First we prove the relation (19). Suppose that a policy $\{f^*, g^*\}$ is optimal. For any fixed $t_1(\geqq 0)$, there is a box i such that $f_i^*(t)>0$, on the interval $[t_1, t_1+\Delta]$ where Δ is sufficiently small. For $\varepsilon=o(\Delta)>0$ and any $j(\neq i)$, we difine a new search policy $f=\{f_1(t),\cdots,f_n(t)\,|\,t\geqq 0\}$ as follows :

$$f_i(t) = \begin{cases} f_i^*(t)-\varepsilon & \text{on } [t_1, t_1+\Delta] \\ f_i^*(t) & \text{otherwise} \end{cases}$$

$$f_j(t) = \begin{cases} f_j^*(t)+\varepsilon & \text{on } [t_1, t_1+\Delta] \\ f_j^*(t) & \text{otherwise} \end{cases}$$

$$f_k(t) = f_k^*(t) \qquad \text{on } [0,\infty] \quad (k\neq i,j).$$

Using a relation $\exp(x)=1+x+o(x)$, we can obtain

(21) $C[f,g^*] - C[f^*, g^*]$

$$= p_i\int_{t_1}^{t_1+\Delta} L_i^*(t)\{\varepsilon\int_{t_1}^{t} \lambda_i[G_i^*(s)]ds + o(\varepsilon)\}dt$$

$$+ p_i\int_{t_1+\Delta}^{\infty} L_i^*(t)\{\varepsilon\int_{t_1}^{t_1+\Delta} \lambda_i[G_i^*(s)]ds + o(\varepsilon)\}dt$$

$$+ p_j\int_{t_1}^{t_1+\Delta} L_j^*(t)\{-\varepsilon\int_{t_1}^{t} \lambda_j[G_j^*(s)]ds + o(\varepsilon)\}dt$$

$$+ p_j\int_{t_1+\Delta}^{\infty} L_j^*(t)\{-\varepsilon\int_{t_1}^{t_1+\Delta} \lambda_j[G_j^*(s)]ds + o(\varepsilon)\}dt \quad \geqq 0$$

where the last inequality follows from the optimality of f^*. Dividing both sides of the inequality (21) by $\varepsilon\Delta$ (>0) and letting Δ approach to zero, the first and third terms converge to zero since for any continuous function $z(s)$,

$$\lim_{\Delta\to 0} \frac{1}{\Delta} \int_{t_1}^{t_1+\Delta} z(s)ds = z(t_1).$$

Therefore we obtain

(22) $$p_i\lambda_i[G_i^*(t_1)]\int_{t_1}^{\infty} L_i^*(t)dt \geqq p_j\lambda_j[G_j^*(t_1)]\int_{t_1}^{\infty} L_j^*(t)dt.$$

If we select box $j(\neq i)$ such that $f_j^*(t) > 0$ on $[t_1, t_1+\Delta]$, the discussion obtained by exchanging i for j in the above discussion can be developed and therefore the opposite inequality holds in (22). Hence if $f_i^*(t) > 0$ and $f_j^*(t) > 0$ on $[t_1, t_1+\Delta]$, the equality holds in (22). In other words, if $f_i^*(t) > 0$, the left-hand side of (22) is independent of i. Hence if $f_i^*(t) > 0$, the left-hand side of (22) is independent of i. Hence there is a nonnegative function $\xi_1(t)$ such that

$$(23)\quad p_i\lambda_i[G_i^*(t)]\int_t^\infty L_i^*(s)ds = \xi_1(t) \qquad \text{if } f_i^*(t) > 0.$$

On the other hand, if $f_j^*(t_1) = 0$, the opposite inequality cannot hold in (22) and therefore

$$(24)\quad p_j\lambda_j[G_j^*(t)]\int_t^\infty L_j^*(s)ds \leq \xi_1(t) \qquad \text{if } f_j^*(t) = 0.$$

The relation (19) is derived from (23) and (24). The relation (20) can be proved by the same method as (19) and therefore its proof is omitted.

(q. e. d.)

We consider the case of $\lambda_i(x)=\lambda_i$ ($i=1,\cdots, n$) for any $x(\geq 0)$. By (20) it is evident that $g_i^*(t)=0(i=1,\cdots, n)$ for any $t(\geq 0)$, that is, no improvement is optimal. From (19), we obtain that

$$p_i\lambda_i\int_t^\infty e^{-\lambda_i F_i^*(s)}ds\{\begin{smallmatrix}=\\ \leq\end{smallmatrix}\}\xi_1(t) \text{ if } f_i^*(t) \{\begin{smallmatrix}>\\ =\end{smallmatrix}\} 0.$$

By the well-known method, the optimal policy f^* can be obtained as follows

$$(25)\quad f_i^*(t) = \begin{cases} \lambda_i^{-1}/(\sum_{j\in I(t)} \lambda_j^{-1}) & \text{If } i \in I(t) \\ 0 & \text{If } i \notin I(t) \end{cases}$$

where

$$I(t) = \{i \mid p_i\lambda_i\exp[-\lambda_i F_i^*(t)] = \max_{1\leq j\leq n} p_j\lambda_j\exp[-\lambda_j F_j^*(t)]\}.$$

The relation (25) denotes tht at any time it is optimal to allocate search effort in boxes having the maximum posterior detection probability in proportion to the inverse of the detection rate. This result is well-known.

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Reference

[1] Black, W.L. : Discrete Seqential Search. *Information and Control*, Vol.8 (1965), 159-162.

[2] Charnes, A. and Cooper, W.W. : The Theory of Search : Optimum Distribution of Search Effort. *Management Science*, Vol.5, No.1 (1958), 44-50.

[3] Chew, M.C. : A Sequential Search Procedure. *Annals of Mathematical Statistics*, Vol.38 (1967), 494-502.

[4] DeGuenin, J. ; Optimum Distribution of Effort : An Extension of the Koopman Basic Theory. *Operations Research*,Vol.9,No.1 (1961),1-7.

[5] Dobbie, J. M. : Search Theory : A Sequential Approach. *Naval Research Logistic Quarterly*, Vol.4 (1963), 323-334.

[6] Kadane, J.B. : Discrete Search and the Neyman-Pearson Lemma. *Journal of Mathematical Analysis and Applications*, Vol.22 (1968), 156-171.

[7] Koopman, B.O. : The Theory of Search, Part III. The Optimum Distribution of Searching Effort. *Operations Research*, Vol.5, No.5 (1957), 613 626.

[8] Matula, D. : A Periodic Optimal Search. *American Mathematical Monthly* Vol.71, No.1 (1964), 15-21.

[9] Nakai, T. : Search Problem with Time-Dependent Detection Probabilities.*Mathematical Japonica*,Vol.26,No.5 (1981),499-505.

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