Journal of the Operations Research
Society of Japan
Vol. 30, No. 2, June 1987

CONFIDENCE REGION METHOD FOR A STOCHASTIC PROGRAMMING PROBLEM

Hiroshi Morita, Hiroaki Ishii and Toshio Nishida
Osaka University

(Received June 12, 1986; Revised November 21, 1986)

Abstract We propose a minimax model with a "quadratic" recourse. In stochastic linear programming models, a decision maker has been assumed to know the probability distribution of random variables. Here we consider the case that the parameters of distribution are unknown. We impose the restrictions on the unknown parameters from the view point of a confidence region, and then seek a minimax solution that minimizes the worst case of the parameters. This model reflects the situation minimizing the maximal possible damage. Especially, the independent normal distribution model is discussed in detail. The analysis for a sufficiently large sample size and a numerical result are given.

1. Introduction

The elements of coefficient matrices of linear programming problems have been assumed to be exactly known. However, in a real life, we never know these values with the full conviction, so we have to include the uncertainty in the formulation of the practical problems. In a usual stochastic programming problem, this uncertainty is viewed as a random model characterized by a probability distribution, and in order to solve the problems under uncertainty, several approaches have been proposed, e.g., two-stage stochastic programs with recourse [6], chance-constrained stochastic programs [1] and so on. In most of these approaches it is assumed that the probability distributions of the random variables are perfectly known. But when the parameters of distributions are unknown, we can not apply these approaches as they are. In this situation we need to obtain some knowledge about the parameters of distributions in statistical ways. Jagannathan [3] proposed an approach with sample informations for a given prior distribution of unknown parameters using a Bayesian approach.

Here we propose a so-called game theoretic minimax approach in section 2. First we impose the restrictions on the unknown parameters which are estimated by a confidence region of them based on random samples drawn from a parent population, and then we seek a minimax solution that minimizes the worst case of the parameters among those restrictions, that is, we minimize the maximal possible damage for a given significance level. In section 3, we investigate the independent normal distribution model in detail. The minimax model is solved numerically using some properties, and the discussion of limiting property is added. Finally we show a numerical example in section 4.

2. Formulation of a Minimax Model

A linear stochastic programming problem is given as follows:

$$\text{(2.1)}\quad \text{LP:}\quad \underset{x}{\text{Minimize}}\quad c^t x \quad \text{subject to}\quad Ax = b,$$

where c and x are n real vectors, b is an m real vector and A is an m by n real matrix with a full rank. Here we assume that A and c are exactly known and b is a random vector having a distribution function $F(\cdot;\theta)$ where θ is an unknown parameter vector. We consider the following two-stage stochastic programming problem P_0 with a "quadratic" recourse:

$$\text{(2.2)}\quad P_0:\quad \underset{x}{\text{Minimize}}\quad c^t x + E\left(\sum_{i=1}^{m} d_i y_i^2\right) \quad \text{subject to}\quad Ax + y = b,$$

where $d=(d_1,d_2,\ldots,d_m)$, weight of each constraint infeasibility, is an m-vector with positive elements, and A_i is an i-th row of matrix A. The quadratic recourse is more tractable than a simple recourse in this model and reflects the situation that the infeasibility of constraint has a critical meaning. P_0 is rewritten to the following problem P_0':

$$\text{(2.3)}\quad P_0':\quad \underset{x}{\text{Minimize}}\quad c^t x + E\left[\sum_{i=1}^{m} d_i (A_i x - b_i)^2\right]$$

$$= c^t x + \int\cdots\int \sum_{i=1}^{m} d_i (A_i x - t_i)^2 dF(t_1,t_2,\ldots,t_m;\theta)$$

The unknown parameter θ is imposed the restrictions estimated by the confidence region S with a certain significance level. We consider the worst case of the parameter θ among S, and then minimize the objective function of

P'_0. That is, we propose a following minimax model P:

$$\text{(2.4)} \quad \text{P:} \quad \underset{x}{\text{Minimize}} \quad L = c^t x + \max_{\theta \in S} \int \cdots \int \sum_{i=1}^{m} d_i (A_i x - t_i)^2 \times dF(t_1, t_2, \ldots, t_m; \theta).$$

This model reflects on the situation that we should make a decision minimizing the maximal possible damage if the correct value of the parameter θ is not known perfectly.

3. Normal Distribution Model

Let b_i, $i=1,\ldots,m$, be an independently, normally distributed random variable, respectively. First we construct the confidence region S of parameters (i.e., mean and variance). For this purpose, we define the following notations.

μ_i : mean of b_i, $i=1,\ldots,m$

σ_i^2 : variance of b_i, $i=1,\ldots,m$

$\bar{\mu}_i$: sample mean of b_i, $i=1,\ldots,m$

s_i^2 : sample variance of b_i, $i=1,\ldots,m$

N : sample size

α : significance level (%)

$\phi(\cdot)$: probability density function of independent multivariate standard normal distribution

3.1. Confidence region of parameters

(i) Confidence region of mean μ_i, $i=1,\ldots,m$

Note that the distribution of Hotelling statistics, F,

$$\text{(3.1)} \quad F = \frac{N(N-m)}{m(N-1)} \sum_{i=1}^{m} \frac{(\mu_i - \bar{\mu}_i)^2}{s_i^2}$$

is the F-distribution with (m,N-m) degrees of freedom. Using this statistics F, the confidence region of μ_i, $i=1,\ldots,m$, is given by

$$\text{(3.2)} \quad \sum_{i=1}^{m} \frac{(\mu_i - \bar{\mu}_i)^2}{s_i^2} \le \frac{m(N-1)}{N(N-m)} F_\alpha(m, N-m),$$

where $F_\alpha(\cdot,\cdot)$ is an α percintile of F-distribution.

(ii) Confidence region of variance σ_i^2, $i=1,\ldots,m$

Since, in general, it is difficult to find the confidence region of the variance-covariance matrix, we find it approximately by that of each variance for an independent case. The distribution of $(N-1)s_i^2/\sigma_i^2$ is the chi-square distribution with (N-1) degrees of freedom. Therefore the confidence region of σ_i^2, $i=1,\dots,m$, is given by

$$(3.3)\qquad \frac{(N-1)s_i^2}{\chi_\beta^2(N-1)} \leqq \sigma_i^2 \leqq \frac{(N-1)s_i^2}{\chi_{1-\beta}^2(N-1)},\quad i=1,\dots,m,$$

where $\beta = \frac{1}{2}\alpha^{1/m}$ and $\chi_\beta^2(\cdot)$ is a β percentile of chi-square distribution. Then this region is a rectangular region.

From (i) and (ii), the confidence region S is

$$(3.4)\qquad S = \left\{(\mu_i,\sigma_i^2),\ i=1,\dots,m \;\middle|\; \sum_{i=1}^{m}\frac{(\mu_i-\bar{\mu}_i)^2}{s_i^2} \leqq \frac{m(N-1)}{N(N-m)}F_\alpha(m,N-m),\right.$$
$$\left.\frac{(N-1)s_i^2}{\chi_\beta^2(N-1)} \leqq \sigma_i^2 \leqq \frac{(N-1)s_i^2}{\chi_{1-\beta}^2(N-1)},\ i=1,\dots,m\right\}.$$

3.2. Minimax model

We consider the maximizing part of minimax model P under the above setting, i.e.,

$$(3.5)\qquad P':\ \underset{(\mu_i,\sigma_i^2)\in S}{\text{Maximize}} \int_{\infty}^{\infty}\cdots\int_{\infty}^{\infty}\sum_{i=1}^{m} d_i(A_ix-t_i)^2\phi(t_1,\dots,t_m;\mu_i,\sigma_i^2)dt_1\cdots dt_m$$
$$= \sum_{i=1}^{m}\{d_i(A_ix-\mu_i)^2 + d_i\sigma_i^2\}.$$

Moreover this problem P' is divided into the following two problems:

$$(3.6)\qquad P_1:\ \underset{\sigma_i}{\text{Maximize}}\quad L_1 = \sum_{i=1}^{m} d_i\sigma_i^2$$
$$\text{subject to}\quad \frac{(N-1)s_i^2}{\chi_\beta^2(N-1)} \leqq \sigma_i^2 \leqq \frac{(N-1)s_i^2}{\chi_{1-\beta}^2(N-1)},\quad i=1,\dots,m,$$

$$(3.7)\qquad P_2:\ \underset{\mu_i}{\text{Maximize}}\quad L_2 = \sum_{i=1}^{m} d_i(A_ix-\mu_i)^2$$
$$\text{subject to}\quad \sum_{i=1}^{m}\frac{(\mu_i-\bar{\mu}_i)^2}{s_i^2} \leqq \frac{m(N-1)}{N(N-m)}F_\alpha(m,N-m) \triangleq K$$

Then P_1 corresponds to the variances and P_2 corresponds to the means. Since maximum of P_1 is independent of x, we may regard it as a constant in a minimizing of P with respect to x. Hence we consider only P_2.

In order to solve P, we show the several properties.

Property 1. L_2 is maximized on the boundary of the feasible region, and further the sign of $(\mu_i^*-\bar{\mu}_i)$ is opposite to that of $(A_i x-\bar{\mu}_i)$, where μ_i^* is an optimum of μ_i, $i=1,\ldots,m$.

Proof: It is easily shown that L_2 is a convex function in every μ_i, $i=1,\ldots,m$, and so the first part of property 1 is proved directly by the theory of convexity. Actually, L_2 is maximized when

$$(3.8) \quad \frac{(\mu_i^*-\bar{\mu}_i)^2}{s_i^2} = K_i^2, \quad i=1,\ldots,m,$$

that is,

$$(3.9) \quad \mu_i^* = \bar{\mu}_i \pm s_i K_i, \quad i=1,\ldots,m,$$

where $K_i \geqq 0$ and $\sum_{i=1}^{m} K_i^2 = K$. Since $(A_i x-\mu_i^*)^2 = (A_i x-\bar{\mu}_i \mp s_i K_i)^2$, μ_i^* which maximizes L_2 is given as follows:

$$(3.10) \quad \begin{aligned} \mu_i^* &= \bar{\mu}_i - s_i K_i \qquad \text{if } A_i x-\bar{\mu}_i \geqq 0,\\ \mu_i^* &= \bar{\mu}_i + s_i K_i \qquad \text{if } A_i x-\bar{\mu}_i < 0. \end{aligned}$$

□

Property 2. P_2 is transformed into a concave programming problem P_2' by making a variable transformation:

$$(3.11) \quad z_i = \frac{(\mu_i-\bar{\mu}_i)^2}{s_i^2}, \quad i=1,\ldots,m.$$

Proof: From

$$(3.13) \quad \frac{\partial^2 L_2'}{\partial z_i{}^2} = -\frac{d_i s_i |A_i x-\bar{\mu}_i|}{2\, z_i^{3/2}} < 0,$$

L_2' is a concave function. Therefore property 2 follows from property 1. □

Solving P_2', the optimand z_i^*, $i=1,\ldots,m$, is expressed as follows:

$$(3.14) \quad z_i^* = \frac{d_i^2 s_i^2 (A_i x-\bar{\mu}_i)^2}{(\lambda-d_i s_i^2)^2},$$

where λ is the Lagrange multiplier of L_2'.

Property 3. λ is the largest solution of the equation

$$(3.15) \quad \sum_{i=1}^{m} \frac{d_i^2 s_i^2 (A_i x-\bar{\mu}_i)^2}{(\lambda-d_i s_i^2)^2} = K.$$

Proof: Denoting the maximum value of L_2' with $L_2^*(\lambda)$, (3.16) $$L_2^*(\lambda) = \sum_{i=1}^{m} d_i(A_i x-\bar{\mu}_i)^2\left(\frac{\lambda}{\lambda-d_i s_i^2}\right)^2 .$$

Let λ_1, λ_2 ($\lambda_1 < \lambda_2$) be solutions of eq.(3.15), then

(3.17) $$\sum_{i=1}^{m} d_i(A_i x-\bar{\mu}_i)^2 \frac{d_i s_i^2}{(\lambda_1-d_i s_i^2)^2} = \sum_{i=1}^{m} d_i(A_i x-\bar{\mu}_i)^2 \frac{d_i s_i^2}{(\lambda_2-d_i s_i^2)^2} .$$

Therefore, from eq.(3.17),

(3.18) $$(\lambda_1+\lambda_2)\sum_{i=1}^{m} \frac{d_i^2 s_i^2(A_i x-\bar{\mu}_i)^2}{(\lambda_1-d_i s_i^2)^2(\lambda_2-d_i s_i^2)^2} = 2\sum_{i=1}^{m} \frac{d_i^3 s_i^4(A_i x-\bar{\mu}_i)^2}{(\lambda_1-d_i s_i^2)^2(\lambda_2-d_i s_i^2)^2} .$$

Using eq.(3.16) and (3.18),

(3.19) $$L_2^*(\lambda_2) - L_2^*(\lambda_1) = \frac{(\lambda_2-\lambda_1)^3}{2}\sum_{i=1}^{m} \frac{d_i^2 s_i^2(A_i x-\bar{\mu}_i)^2}{(\lambda_1-d_i s_i^2)^2(\lambda_2-d_i s_i^2)^2} > 0 .$$

Hence λ is a largest solution of eq.(3.15). □

Property 4. $\lambda > \lambda_0 \triangleq \max_{i\in I} d_i s_i^2$, where I is the set of indices such that $A_i x-\bar{\mu}_i \neq 0$.

Proof: From eq.(3.15), we define

(3.20) $$G(\lambda) \triangleq \sum_{i=1}^{m} \frac{d_i^2 s_i^2(A_i x-\bar{\mu}_i)^2}{(\lambda-d_i s_i^2)^2} - K.$$

Since

(3.21) $$\lim_{\lambda\to\lambda_0+0} G(\lambda) = +\infty > 0 ,$$

(3.22) $$\lim_{\lambda\to+\infty} G(\lambda) = -K < 0 ,$$

and

(3.23) $$G'(\lambda) = -2\sum_{i\in I} \frac{d_i^2 s_i^2(A_i x-\bar{\mu}_i)^2}{(\lambda-d_i s_i^2)^3} > 0,$$

the unique solution exists in (λ_0,∞). □

Consequently, the problem P is expressed as follows:

(3.24) P': $$\underset{x,\lambda}{\text{Minimize}} \quad L = c^t x + \sum_{i=1}^{m} d_i(A_i x-\bar{\mu}_i)^2\left(\frac{\lambda}{\lambda-d_i s_i^2}\right)^2$$

$$\text{subject to} \quad \sum_{i=1}^{m} \frac{d_i^2 s_i^2(A_i x-\bar{\mu}_i)^2}{(\lambda-d_i s_i^2)^2} = K,\ \lambda > \lambda_0,$$

where we neglect the constant associated with P_1.† In order to solve this problem P', we utilize the following property.

Property 5. For $\lambda \geq \frac{3}{2}\lambda_0$, P' is a convex programming problem.

Proof: It is sufficient to show that the Hessian matrix of the objective function is positive semi-definite for $\lambda \geq \frac{3}{2}\lambda_0$. The elements of the Hessian matrix, $H=(h_{ij})$, $i,j=1,\ldots,n+1$, are given as follows:

$$h_{ij} = \frac{\partial^2 L}{\partial x_i \partial x_j} = 2\sum_{k=1}^{m} d_k a_{ki} a_{kj} \frac{\lambda^2}{(\lambda - d_k s_k^2)^2}, \quad i,j=1,\ldots,n,$$

$$(3.25)\quad h_{n+1,i} = h_{i,n+1} = \frac{\partial^2 L}{\partial x_i \partial\lambda} = -4\sum_{k=1}^{m} d_k a_{ki}(A_k x - \bar{\mu}_k)\frac{\lambda d_k s_k^2}{(\lambda - d_k s_k^2)^3}, \quad k=1,\ldots,n,$$

$$h_{n+1,n+1} = \frac{\partial^2 L}{\partial\lambda^2} = 2\sum_{k=1}^{m} d_k (A_k x - \bar{\mu}_k)^2 \frac{d_k s_k^2(2\lambda + d_k s_k^2)}{(\lambda - d_k s_k^2)^4},$$

where a_{ij} denotes the (i,j)-th element of matrix A. Note that H is divided into two matrices, that is,

$$(3.26)\quad H = [\vec{u}_1,\ldots,\vec{u}_n,\vec{v}]^t[\vec{u}_1,\ldots,\vec{u}_n,\vec{v}] + \begin{bmatrix} 0 & \\ & w \end{bmatrix},$$

where $\vec{u}_i$, $i=1,\ldots,n$, and $\vec{v}$ are m-vectors with $\frac{\sqrt{2d_j}\,\lambda}{\lambda - d_j s_j^2} a_{ji}$, and $-\frac{2\sqrt{2d_j}(A_j x - \bar{\mu}_j) d_j s_j^2}{(\lambda - d_j s_j^2)^2}$ as j-th element, $j=1,\ldots,m$, respectively, and

$$(3.27)\quad w = 2\sum_{k=1}^{m} \frac{(A_k x - \bar{\mu}_k)^2 d_k^2 s_k^2}{(\lambda - d_k s_k^2)^4}(2\lambda - 3d_k s_k^2).$$

Since for $\lambda \geq \frac{3}{2}\lambda_0$, w is non-negative, H is expressed as the sum of two positive semi-definite matrices. Therefore H is positive semi-definite. □

In order to solve this problem P', we should consider several cases according to the value of λ because of the dependency on x. Define

$$(3.28)\quad \lambda_{max} = \max_{1 \leq i \leq n} d_i s_i^2 .$$

<u>Case 1.</u> $\lambda \geq \frac{3}{2}\lambda_{max}$:

Since it is clear that $\lambda_0 \leq \lambda_{max}$, from property 5, the problem P' is a convex program. Then we use some approaches for a constrained non-linear programming problem, e.g., the steepest descent method with a penalty func-

† It is pointed out by the referee that the problem (3.24) which is equivalent to the problem (3.7) is also found by a duslity theorem [2, Theorem 2.1.]. We summarize it in appendix.

tion.

Case 2. $\lambda_0 < \lambda < \frac{3}{2}\lambda_{max}$:

For fixed λ, L is convex with respect to x. Then, for several fixed λ between λ_0 and $\frac{3}{2}\lambda_{max}$, we seek optimal solutions and choose the best ones among them. But we must divide λ into the following some intervals because the constraints differ on each interval. For a sake of convenience, we rearrange $d_i s_i^2$, $i=1,\ldots,m$, in an increasing order with its constraint. By the definition of λ_0,

$$(3.29)\quad \lambda_0 = d_m s_m^2$$

means

$$(3.30)\quad A_i x \neq \bar{\mu}_i,\ i=1,\ldots,m,$$

and

$$(3.31)\quad \lambda_0 = d_p s_p^2,\ \text{for } 1\leqq p<m,$$

means

$$(3.32)\quad \begin{aligned} &A_i x \neq \bar{\mu}_i,\ i=1,\ldots,p,\\ &A_i x = \bar{\mu}_i,\ i=p+1,\ldots,m. \end{aligned}$$

Thus p denotes the largest index such that $A_i x \neq \bar{\mu}_i$. For $\lambda_0 = d_p s_p^2$, $1\leqq p<m$, the first constraint of P' is relaxed as follows by a way of compensation that the equality constraints $A_i x = \bar{\mu}_i$, $i=p+1,\ldots,m$, are imposed. Moreover, since rank A = n, the constraints eq.(3.32) should be infeasible for $p<m-n$. That is, the constraints of problem P' are partitioned into n cases as follows.

$$(3.33)\quad \begin{aligned} &\sum_{i=1}^{p} \frac{d_i^2 s_i^2 (A_i x-\bar{\mu}_i)^2}{(\lambda-d_i s_i^2)^2} = K,\\ &A_i x = \bar{\mu}_i,\ i=p+1,\ldots,m,\\ &d_p s_p^2 < \lambda \leqq d_{p+1} s_{p+1}^2,\quad \text{for } m-n \leqq p < m, \end{aligned}$$

and

$$(3.34)\quad \begin{aligned} &\sum_{i=1}^{m} \frac{d_i^2 s_i^2 (A_i x-\bar{\mu}_i)^2}{(\lambda-d_i s_i^2)^2} = K,\\ &d_m s_m^2 < \lambda < \frac{3}{2}\lambda_{max},\quad \text{for } p = m. \end{aligned}$$

Hence we need to obtain an optimal solution for every p.

Choosing the best optimal solution among optimal ones of above two cases, we obtain the global optimal solutions. Next we show the existance of this best optimal solution.

Lemma 1. The objective function L is bounded subject to eq.(3.33) or eq.(3.34).

Proof: Assum that the optimal solution is unbounded, then since the denominator of the first constraint is bounded, the left hand side of it must be unbounded. This contradicts boundness of K. □

Lemma 2. The solution od P' is continuous with respect to λ.

Proof: Let x_1 and x_2 be optimal solutions when $\lambda = \lambda_1$ and $\lambda = \lambda_2$, respectively. It is easily shown that $L(x_1) \to L(x_2)$ as $\lambda_2 \to \lambda_1$ and the solution is unique because this problem has a strictly convex objective function and a convex feasible region for fixed λ. From this convexity the continuity of the solution is shown. □

We can find the solution of this problem approximately with discretizing value of λ.

Theorem. We can obtain the approximately global optimal solution by means of choosing the best optimal solution among optimal ones for each discretized λ.

3.3. Limiting property

We discuss the problem with sufficiently large sample size. We see, from a consistency of sample mean and variance, that the correct value of parameters can be known as $N\to\infty$. With these parameters, P_0 is rewritten to a problem PL with perfect information as follows:

(3.35) PL: $$\underset{x}{\text{Minimize}} \quad c^t x + \sum_{i=1}^{m} \{d_i(A_i x-\mu_i)^2 + d_i\sigma_i^2\}.$$

Here we show that our proposed model with finite sampling with size N, say P, tends to PL as $N\to\infty$.

Lemma 3. Let M and N be positive integers, then

(3.36) $$\lim_{N\to\infty} \frac{\chi_\alpha^2(N)}{N} = 1$$

and

(3.37) $$\lim_{N\to\infty} \frac{F_\alpha(M,N)}{N} = 0,$$

where $0 < \alpha < 1$.

Proof: As in [4], with the Cornish-Fisher expansion,

(3.38) $$\sqrt{2\chi_\alpha^2(N)} - \sqrt{2N-1} = u_\alpha + O(\frac{1}{\sqrt{N}}),$$

where u_α is α percentile of a standard normal distribution. Since $|u_\alpha| < \infty$, for $0 < \alpha < 1$, eq.(3.36) is proved.

Denoting the probability density function of F-distribution with (M,∞) degrees of freedom with $f_{M,\infty}(t)$,

$$(3.39) \quad f_{M,\infty}(t) = \frac{1}{\Gamma(\frac{M}{2})} (\frac{M}{2} te^{-t})^{\frac{M}{2}} ,$$

it is clear that an α percentile of this distribution, $F_\alpha(M,\infty)$, finitely exists. □

Remark 1. From lemma 3, the problem P with unknown parameters tends to the problem PL with known parameters as a sample size N tends to infinity.

Remark 2. In the problem P', λ tends to infinity as $N \to \infty$.

Remark 3. Note that sample variances are not concerned with a decision making for a sufficiently large sample size. They are concerned with only the magnitude of the recourse.

Remark 4. This proposed minimax model would be useful to the other stochastic programming problems which the parameters of their random variables are unknown.

4. Numerical Example

Consider a following problem:

$$(4.1) \quad \text{PE:} \quad \underset{x_1,x_2}{\text{Minimize}} \quad 2x_1 + x_2$$
$$\text{subject to} \quad x_1 + x_2 = b_1$$
$$2x_1 - x_2 = b_2$$
$$x_2 = b_3$$

where b_1, b_2 and b_3 are independently, normally distributed random variables with unknown parameters, and let a weight vector $d = (10,5,10)$. Using an artificial result of sampling of a computer simulation, we give the sample means and variances in Table 1.

Table 1. Sample means and variances (N=11)

	$\bar{\mu}$	s^2	$d \cdot s^2$
b_1	2.979	0.007	0.070
b_2	0.056	0.360	1.800
b_3	1.020	0.043	0.430

In this case, the problem PE is trnsformed into the following our model:

(4.2) PE': $\underset{x_1, x_2, \lambda}{\text{Minimize}} \quad 2x_1 + x_2 + 10(x_1 + x_2 - 2.979)^2\lambda^2/(\lambda - 0.070)^2$

$$+ 5(2x_1 - x_2 - 0.056)^2\lambda^2/(\lambda - 1.800)^2$$

$$+ 10(x_2 - 1.020)^2\lambda^2/(\lambda - 0.430)^2 ,$$

subject to $10(x_1 + x_2 - 2.979)^2\cdot 0.070/(\lambda - 0.070)^2$

$$+ 5(2x_1 - x_2 - 0.056)^2\cdot 1.800/(\lambda - 1.800)^2$$

$$+ 10(x_2 - 1.020)^2\cdot 0.430/(\lambda - 0.430)^2 = 1.388,$$

where a significance level is set to 0.05. The optimal solutions for each case are given in Table 2.

Table 2. The optimal solutions (1)

		range of λ	λ^*	x_1^*	x_2^*	L^*
Case 1	---	$\lambda \geqq 2.70$	2.700	0.736	1.710	16.809
Case 2	p=3	$1.80<\lambda\leqq 2.70$	1.802	0.803	1.551	12.244
	p=2	$0.43<\lambda\leqq 1.80$	1.471	0.821	1.587	13.232
	p=1	$0.07<\lambda\leqq 0.43$	0.430	0.538	1.020	31.386

The best optimal solution among these ones is $(\lambda^*, x_1^*, x_2^*) = (1.802, 0.803, 1.551)$ and $L^* = 12.244$. Fourthermore, in Table 3, the best optimal solutions for some sample sizes are given.

Table 3. The optimal solutions (2)

N	λ^*	x_1^*	x_2^*	L^*
11	1.802	0.803	1.551	12.244
100	5.870	0.825	1.710	11.355
300	8.642	0.820	1.649	10.640
1000	16.270	0.840	1.651	10.271
∞	-	0.967	1.695	9.887

5. Conclusion

We have proposed a minimax model. When the distribution of parameters of random variables are assumed to be unknown, our approach is seemed to be not only reasonable but useful. It notes that the maximizing process with respect to variances does not affect upon an optimal decision of our model though they are unknown. Fourthermore we show that, for a sufficiently large sample size, our model tends to one with a perfect information. Hereafter the better solving algorithm should be considered. But it is ascertained the non-convexity of the objective function by a simulation, so it is remained a difficulty on solving our model.

Appendix. Derivation of Problem P' by a Duality Theorem

The problem (3.34) P' which is equivalent to the problem (2.4) P for a normal model is also derived by a duality theorem [2]. To solve the maximizing process of minimax problem P, it is sufficient to consider only the problem (3.7) P_2.

From [2, Theorem 2.1],

$$\text{(A.1)}\quad \sup_{\mu_i}\left\{L_2 = \sum_{i=1}^{m} d_i(A_i x-\mu_i)^2 \,\middle|\, \sum_{i=1}^{m}\frac{(\mu_i-\bar{\mu}_i)^2}{s_i^2} \leqq K\right\}$$

$$= \inf_{\lambda}\left\{\sup_{\mu_i}\{L(\mu,\lambda) = \sum_{i=1}^{m} d_i(A_i x-\mu_i)^2 + \lambda\Big(K - \sum_{i=1}^{m}\frac{(\mu_i-\bar{\mu}_i)^2}{s_i^2}\Big)\}\,\middle|\, \lambda \geqq 0\right\}$$

Then for $\lambda > \lambda_0$, $L(\mu,\lambda)$ is maximized at $\mu_i = \hat{\mu}_i$, $i=1,\ldots,m$, where

$$\text{(A.2)}\quad \hat{\mu}_i = \frac{\lambda\bar{\mu}_i - d_i s_i^2 A_i x}{\lambda - d_i s_i^2},$$

and

$$\text{(A.3)}\quad L(\hat{\mu},\lambda) = \lambda K + \sum_{i=1}^{m}\frac{\lambda d_i(A_i x-\bar{\mu}_i)^2}{\lambda - d_i s_i^2}.$$

From $dL(\hat{\mu},\lambda)/d\lambda = 0$,

$$\text{(A.4)}\quad K = \sum_{i=1}^{m}\frac{d_i^2 s_i^2(A_i x-\bar{\mu}_i)^2}{(\lambda - d_i s_i^2)^2}$$

and

$$\text{(A.5)}\quad L(\hat{\mu},\lambda) = \sum_{i=1}^{m} d_i(A_i x-\bar{\mu}_i)^2\left(\frac{\lambda}{\lambda - d_i s_i^2}\right)^2.$$

Therefore the problem (3.24) P' is obtained, where the constraint $\lambda > \lambda_0$ is found from property 4.

Acknowledgements

The authors would like to thank the referees for their valuable comments made on this paper.

References

[1] Charnes, A. and Cooper, W. W.: Chance constrained programming. *Management Science*, Vol.6, No.1 (1960), 73-79.

[2] Isii, K.: Inequalities of the types of Chebyshev and Cramér-Rao and mathematical programming, *Annals of Institute Statistical Mathematics*, Vol. 16 (1964), 277-293.

[3] Jagannathan, J.: Use of sample information in stochastic recourse and chance-constrained programming models, *Management Science*, Vol.31, No.1 (1985), 96-108.

[4] Takeuchi, K.: *Approximation of probability distribution*, Kyoiku-shuppan, 1973, (in Japanease).

[5] Vajda, S.: *Probabilistic programming*, Academic Press, 1972.

[6] Walkup, D. W. and Wets, R.: Stochastic programs with recourse, *SIAM Journal of Applied Mathematics*, Vol.15 (1967), 1299-1314.

Hiroshi MORITA: Department of Applied Physics, Faculty of Engineering, Osaka University, Yamada-Oka, Suita, Osaka, 565, Japan.