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NONPREEMPTIVE SCHEDULING OF A FINITE-SOURCE QUEUE WITH TWO CUSTOMER CLASSES

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Abstract We consider a finite-source queueing system with two distinct customer classes. The problem is to obtain a nonpreemptive service assignment policy which maximizes the expected discounted value of rewards received over an infinite planning horizon. In all policies, there are particular policies which simply enforce priority rankings. We call these static policies. When the expected "thinking times" of all customers are the same, it is shown that there is an optimal static policy.

1. Introduction

We consider in this paper an optimization problem associated with a finite-source single server queueing system whose customers are of two distinct classes. The system is as follows. Before requiring service at the single server "service station", each class k customer (k=1,2) stays in the "source" for a time T_k which is an independently, identically and exponentially distributed random variable with mean $1/\lambda_k$. We call T_k a thinking time of class k customers. The service time S_k of class k customers is an independently and identically distributed random variable with mean $1/\mu_k>0$. Total number of class k customers in the system (the source and the service station) is N_k (k=1,2). A reward of r_k is received upon completion of each class k service. All rewards are continuously discounted with a factor $\beta>0$.

Our objective is to solve the following dynamic scheduling (service assignment) problem. Our decision points are those epochs at which either a service is completed or a customer arrives at the service station to find the server idle. At each decision point, the state of the system is the number of customers from each class present in the service station; and one of the customers is selected for service. That is, the service assignment is nonpreemptive.

A policy for this problem is a rule which specifies the action of the customer selection taken at each decision point. We seek a policy that, regardless of the initial state, maximizes the expected present value of rewards received over an infinite planning horizon. We are interested in policies which appropriately give a priority order of the customer classes and select customers for service according to this order. For example, if class 1 is given a higher priority, then class 1 customers are selected for service ahead of class 2 customers, whenever class 1 customers are present in the service station. These we call static policies. Our main result is that if the expected thinking times are the same and the service times are exponentially distributed, there is an optimal static policy.

The finite-source queueing system is used to analyze the time-shared computer systems [16]. In the time-shared system, the user may be thought of as being in one of two states: either the user is waiting for the system to respond, or the system is waiting for the user. The users in the former state are modeled as customers in the service station and the users in the latter state are modeled as customers in the source. It is important to schedule the order of customers' service under these circumstances. These kinds of scheduling problems have been solved for the infinite-source model and are often called the "bandit problems". See [5],[7],[8],[12],[13],[17] and [18]. Our model is different from their models in the following important point: the arrival process of our model depends on the number of customers in the service station, whereas their arrival processes are independent of it. In this case, the busy period process also depends on the service assignment policy. But it has been shown that some invariant property of the busy period holds for the finite-source M/M/1 queueing system, if the following is assumed: the mean thinking times $1/\lambda_k$ are all the same. In [10], it was shown that the average length of the busy period at the service station is independent of the service assignment policy. In [3], it was shown that if the service assignment is preemptive, then the busy period distribution is independent of the assignment policy. Further in [2], [3] and [10], some optimization problems are solved. Considering these properties, we show some invariance which holds for the non-preemptive finite-source M/M/1 queue with two classes and same thinking times. Then by using it, we solve the above optimization problem.

In section 2, we consider a finite-source queueing system whose customers are of M distinct classes. The reward functions of the system are expressed by the Laplace-Stieltjes transforms (LSTs) of the busy period distributions. Section 3 obtains the explicit forms of the LSTs of the busy period distributions for the finite-source queue with two customer classes. In sections 2 and 3, we need not assume that the service times are exponentially distributed and that the mean thinking times are all the same. In section 4, we prove the optimality of the static policy.

2. A Finite-Source Queue with M Customer Classes

We consider in this section a finite-source single server queueing system whose customers are of M distinct classes. Total number of class k customers in the system is N_k ($1 \leq k \leq M$). Each class k customer stays in the source for a time T_k which is an independently, identically and exponentially distributed random variable with mean $1/\lambda_k > 0$ ($1 \leq k \leq M$). The service time for class k customers is independently and identically distributed as a nonnegative random variable S_k with distribution function $F_k(\cdot)$, mean $1/\mu_k$ and LST

$$\psi_k(\theta) = \mathrm{E}[e^{-\theta S_k}] = \int_0^\infty e^{-\theta t}\, dF_k(t), \qquad (\theta > 0,\ 1 \leq k \leq M).$$

We assume that admission for service is granted on a nonpreemptive priority basis, class 1 customers having highest priority, ..., and class M customers having lowest priority.

We suppose that a reward r_k is received upon the completion of each class k service ($1 \leq k \leq M$) and that these rewards are continuously discounted with a factor $\beta > 0$. Our only assumption regarding the service time distributions is that $F_k(0) < 1$ ($1 \leq k \leq M$). Let $n = (n_1, \ldots, n_M)$ denote the initial state of the system where n_i is the number of the class i customers in the service station. Assuming that operation of the system begins at time 0, we first obtain an expression for

$V_k(n)$=the expected present value of rewards received over the infinite planning horizon, given that the system starts with state n and that only the customers belonging to classes 1 through k are admitted for service.

Let $B_0(n)=0$ and $B_k(n)$ be the first epoch at which the service station is cleared of customers from classes 1 through k. We define

$$\pi_k(n,\theta) = \mathrm{E}[e^{-\theta B_k(n)}], \qquad (0 \leq k \leq M).$$

Now letting $U_k(n)$ be the expected present value of rewards received during $B_k(n)$, $1 \leq k \leq M$, we see that

$$(2.1) \qquad V_k(n) = U_k(n) + \pi_k(n,\beta)V_k(0)$$

where 0 is an M-vector whose components are all equal to 0.

To express $U_k(n)$ as a function of $\pi_k(n,\beta)$, we initially develop a recursive expression for $\pi_k(n,\beta)$. We define $\alpha_k^k=\pi_k(\delta_k,\beta)$ where δ_k is a k-th unit vector whose k-th component is unity and whose other components are zero ($1\leqq k\leqq M$).

Lemma 1. If $N_i=1$ for all i ($1\leqq i\leqq M$),

$$\pi_k(n,\beta) = \pi_{k-1}(n,\beta)\alpha_k^k + \pi_{k-1}(n,\beta+\lambda_k)\mathrm{I}\{n_k=0\}(1 - \alpha_k^k) \tag{2.2}$$

where $\mathrm{I}\{\bullet\}$ is an indicator function of the event $\{\bullet\}$.

Proof: Note that only one customer is present in each class and that a class k customer is never admitted for service until the service station is cleared of customers from classes 1 through k-1. If $n_k=1$, the result directly follows from the above discussion and the independence between $B_{k-1}(n)$ and $B_k(\delta_k)$. If $n_k=0$, let T_k be the thinking time of the class k customer which starts simultaneously with $B_{k-1}(n)$. By conditioning on $B_{k-1}(n)$ and T_k, we obtain the following expressions

$$\pi_k(n,\beta|B_{k-1}(n)=t,\ T_k>t) = e^{-\beta t},$$

$$\pi_k(n,\beta|B_{k-1}(n)=t,\ T_k\leqq t) = e^{-\beta t}\alpha_k^k.$$

Because $B_{k-1}(n)$ and T_k are mutually independent, we have

$$\Pr\{T_k>t|B_{k-1}(n)=t\} = e^{-\lambda_k t},$$

$$\Pr\{T_k\leqq t|B_{k-1}(n)=t\} = 1 - e^{-\lambda_k t}.$$

Hence, we obtain

$$\pi_k(n,\beta) = \pi_{k-1}(n,\beta+\lambda_k) + [\pi_{k-1}(n,\beta) - \pi_{k-1}(n,\beta+\lambda_k)]\alpha_k^k. \qquad \text{(Q.E.D.)}$$

Now we define $C_i = U_i(\delta_i)/(1 - \alpha_i^i)$, $1\leqq i\leqq M$. These constants are very important because these become the "indices" for ranking the customer classes. (See Theorem 1.)

Lemma 2. If $N_i=1$ for all i ($1\leqq i\leqq M$),

$$U_k(n) = \textstyle\sum_{i=1}^{k}[\pi_{i-1}(n,\beta) - \pi_i(n,\beta)]C_i, \qquad (1 \leqq k \leqq M). \tag{2.3}$$

Proof: We first prove the following expression

$$U_k(n) = U_{k-1}(n) + [\pi_{k-1}(n,\beta) - \pi_{k-1}(n,\beta+\lambda_k)\mathrm{I}\{n_k=0\}]U_k(\delta_k) \tag{2.4}$$

where $U_0(n)$ is defined to be 0. We use the same discussion in Lemma 1. For $n_k=1$, by conditioning on $B_{k-1}(n)$, we obtain

$$U_k(n|B_{k-1}(n)=t) = U_{k-1}(n|B_{k-1}(n)=t) + e^{-\beta t}U_k(\delta_k).$$

Hence, we obtain

$$U_k(n) = U_{k-1}(n) + \pi_{k-1}(n,\beta)U_k(\delta_k).$$

For $n_k=0$, by conditioning on $B_{k-1}(n)$ and T_k,

$$U_k(n|B_{k-1}(n)=t,\ T_k>t) = U_{k-1}(n|B_{k-1}(n)=t),$$

$$U_k(n|B_{k-1}(n)=t,\ T_k\leq t) = U_{k-1}(n|B_{k-1}(n)=t) + e^{-\beta t}U_k(\delta_k).$$

Hence, we obtain

$$U_k(n) = U_{k-1}(n) + [\pi_{k-1}(n,\beta) - \pi_{k-1}(n,\beta+\lambda_k)]U_k(\delta_k).$$

Therefore, (2.4) follows. Now, from (2.2),

$$\pi_{k-1}(n,\beta) - \pi_k(n,\beta) = [\pi_{k-1}(n,\beta) - \pi_{k-1}(n,\beta+\lambda_k)\mathrm{I}\{n_k=0\}](1 - \alpha_k^k).$$

From the above expression and (2.4), we have

$$U_k(n) = U_{k-1}(n) + [\pi_{k-1}(n,\beta) - \pi_k(n,\beta)]C_k.$$

Since $U_0(n)=0$, it follows that

$$U_k(n) = \textstyle\sum_{i=1}^{k}[\pi_{i-1}(n,\beta) - \pi_i(n,\beta)]C_i. \qquad \text{(Q.E.D.)}$$

In order to determine C_k, we follow the discussion in [6]. Let j be fixed ($1\leq j\leq M$) and assume that the initial state of the system is $n=(n_1,\ldots,n_M)$ with $n_j>0$. Suppose that a class j customer is admitted for service at epoch zero and that thereafter admission for service is granted on a priority basis as usual. If $n_1=\ldots=n_{j-1}=0$, then this is the usual service discipline; otherwise this discipline represents a slight deviation from normal procedures. Let S_j denote the service time of the initial class j customer and define $B_0^j(n)=S_j$, $B_k^j(n)$=the first epoch after S_j at which the service station is cleared of customers from classes 1 through k, $1\leq k\leq M$, $\pi_k^j(n,\theta)$ = $\mathrm{E}[\exp(-\theta B_k^j(n))]$, $0\leq k\leq M$ and $U_k^j(n)$ = the expected present value of rewards received during $B_k^j(n)$, $0\leq k\leq M$. We can express $\pi_k^j(n,\theta)$ and $U_k^j(n)$ in another way:

$$\text{(2.5)} \qquad \pi_k^j(n,\theta) = \mathrm{E}[e^{-\theta(S_j+B_k(n'))}|n],$$

$$\text{(2.6)} \qquad U_k^j(n) = \mathrm{E}[e^{-(\beta S_j)}\{r_j + U_k(n')\}|n],$$

where n' is the state of the system directly after the completion of the class

j customer's service. It is of course immediately clear that $\pi_0^j(n,\beta)=\psi_j$ and $U_0^j(n)=\psi_j r_j$ where $\psi_j=\psi_j(\beta)$.

Lemma 3. If $N_i=1$ for all i $(1 \le i \le M)$,

$$U_k^j(n) = \psi_j r_j + \textstyle\sum_{i=1}^{k}[\pi_{i-1}^{j}(n,\beta) - \pi_i^j(n,\beta)]C_i, \qquad (1 \le k \le M). \tag{2.7}$$

Proof: The equation immediately follows from (2.3), (2.5) and (2.6). (Q.E.D.)

In order to show the optimality of the static policy, let us define the following reward function:

$V_k^j(n)$=the expected present value of rewards received over the infinite planning horizon, given that the system starts with state n and a class j customer is initially selected for service, and that only the customers belonging to classes 1 through k are admitted for service after the completion of the class j customer's service.

Then, we see that

$$V_k^j(n) = U_k^j(n) + \pi_k^j(n,\beta)V_k(0). \tag{2.8}$$

Now, we shall express C_k as a function of $\alpha_k^j=\pi_k^j(\delta_j,\beta)$. From the definition,

$$C_k = U_k(\delta_k)/(1 - \alpha_k^k).$$

Then, by using the relation $U_k^k(\delta_k)=U_k(\delta_k)$ and (2.7),

$$C_k = [\psi_k r_k + \textstyle\sum_{i=1}^{k}\{\pi_{i-1}^{k}(\delta_k,\beta) - \pi_i^k(\delta_k,\beta)\}C_i]/(1 - \alpha_k^k).$$

Therefore,

$$\begin{aligned} C_1 &= \psi_1 r_1/(1 - \psi_1), \\ C_k &= [\psi_k r_k + \textstyle\sum_{i=1}^{k-1}\{\alpha_{i-1}^{k} - \alpha_i^k\}C_i]/(1 - \alpha_{k-1}^{k}), \qquad (2 \le k \le M). \end{aligned} \tag{2.9}$$

The following Lemma will be used in the proof of the optimality of the static policy.

Lemma 4. If $N_i=1$ for all i, we obtain, for all n and j $(1 \le j \le M;\ n_j>0)$,

$$\begin{aligned} V_M(n) - V_M^j(n) = {} & (1 - \psi_j)C_1 - \psi_j r_j - \textstyle\sum_{i=1}^{M-1}[\pi_i(n,\beta) - \pi_i^j(n,\beta)](C_i - C_{i+1}) \\ & - [\pi_M(n,\beta) - \pi_M^j(n,\beta)](C_M - V(0)). \end{aligned} \tag{2.10}$$

Proof: The result immediately follows from (2.1), (2.3), (2.7) and (2.8). (Q.E.D.)

The results so far obtained can be easily extended to the general model in which the number of customers in each class is greater than 1. As defined at the beginning of this section, let N_m be the number of customers in class m ($1 \leq m \leq M$). Define $G_m = \{Z_{m-1}+1, Z_{m-1}+2, \ldots, Z_m\}$ ($1 \leq m \leq M$) where $Z_0 = 0$ and $Z_m = \sum_{k=1}^{m} N_k$. We number the customers from 1 through Z_M such that, for all i ($1 \leq i \leq Z_M$), if i customer belongs to class m ($1 \leq m \leq M$), then $i \varepsilon G_m$. To change the general model into the model in which the number of customers in each class is equal to 1, we must interpret the customer number in the general model as the class number in the other model. With this change, we can use Lemmas 1 through 4. Then it is obvious that if $k \varepsilon G_m$ and $k-1 \varepsilon G_m$ for some m ($1 \leq m \leq M$), $\psi_k r_k = \psi_{k-1} r_{k-1}$ and $\alpha_i^k = \alpha_i^{k-1}$ for $0 \leq i \leq k-2$. From this discussion and (2.9), it follows that $C_k = C_{k-1}$. Then the reward $U_{Z_m}(n)$ earned by the time the service station is cleared of customers from classes 1 through m is

$$\begin{aligned} U_{Z_m}(n) &= \sum_{i=1}^{Z_m} [\pi_{i-1}(n,\beta) - \pi_i(n,\beta)] C_i \\ &= \sum_{i=1}^{m} [\pi_{Z_{i-1}}(n,\beta) - \pi_{Z_i}(n,\beta)] C_{Z_i}. \end{aligned}$$

So if we appropriately change the subscript k of $U_k(n)$, we see that Lemma 2 holds also for the general model. In the same manner, we see that Lemmas 3 and 4 hold for the general model.

We summarize these results below.

Lemma 2'. For all n,

$$U_k(n) = \sum_{i=1}^{k} [\pi_{i-1}(n,\beta) - \pi_i(n,\beta)] C_i, \qquad (1 \leq k \leq M),$$

where

$$C_1 = \psi_1 r_1 / (1 - \psi_1),$$

$$C_i = [\psi_i r_i + \sum_{m=1}^{i-1} \{\alpha_{m-1}^i - \alpha_m^i\} C_m] / (1 - \alpha_{i-1}^i), \qquad (2 \leq i \leq M).$$

Lemma 3'. For all n and j ($1 \leq j \leq M$; $n_j > 0$),

$$U_k^j(n) = \psi_j r_j + \sum_{i=1}^{k} [\pi_{i-1}^j(n,\beta) - \pi_i^j(n,\beta)] C_i, \qquad (1 \leq k \leq M).$$

Lemma 4'. For all n and j ($1 \leq j \leq M$; $n_j > 0$),

$$\begin{aligned} V_M(n) - V_M^j(n) = &(1 - \psi_j) C_1 - \psi_j r_j - \sum_{i=1}^{M-1} [\pi_i(n,\beta) - \pi_i^j(n,\beta)](C_i - C_{i+1}) \\ &- [\pi_M(n,\beta) - \pi_M^j(n,\beta)](C_M - V(0)). \end{aligned}$$

3. Busy Periods

We obtain in this section the LSTs of the busy period distributions of the finite-source M/G/1 queue with two customer classes.

Lemma 5.

$$(3.1)\quad \pi_1(n,\theta) = A_1(\theta)\sum_{i=0}^{N_1-n_1}\binom{N_1-n_1}{i}\frac{1}{v(i-1,\theta)},$$

$$(3.2)\quad \pi_1^2(n,\theta) = A_1(\theta)\sum_{i=0}^{N_1-n_1}\binom{N_1-n_1}{i}\frac{\psi_2(i\lambda_1+\theta)}{v(i-1,\theta)},$$

where

$$A_1(\theta) = \left[\sum_{i=0}^{N_1}\binom{N_1}{i}\frac{1}{v(i-1,\theta)}\right]^{-1},$$

$$v(i-1,\theta) = \begin{cases}\prod_{k=0}^{i-1}[\psi_1(k\lambda_1+\theta)/\{1-\psi_1(k\lambda_1+\theta)\}] & \text{if } i>0,\\ 1 & \text{if } i=0.\end{cases}$$

Proof: $\pi_1(n,\theta)$ can be obtained directly from the result in [9] (expression (2.25) in chapter 2), because $\pi_1(n,\theta)$ is the LST of the distribution of the class 1 customer's busy period starting with n_1 customers. $\pi_1^2(n,\theta)$ can also be obtained by using the same manner in [9]. We first condition it on S_2 (which is the service time of the class 2 customer being serviced first) and J_1 (which denotes the number of class 1 customers arriving at the service station during S_2). Then,

$$\pi_1^2(n,\theta|S_2=t,\ J_1=i) = e^{-\theta t}\pi_1((n_1+i,n_2),\theta), \qquad (0 \le i \le N_1-n_1).$$

Hence,

$$\begin{aligned}\pi_1^2(n,\theta) &= \int_0^\infty \sum_{i=0}^{N_1-n_1} e^{-\theta t}\pi_1((n_1+i,n_2),\theta)\\ &\qquad \times\binom{N_1-n_1}{i}(1-e^{-\lambda_1 t})^i(e^{-\lambda_1 t})^{N_1-n_1-i}dF_2(t)\\ &= \int_0^\infty \sum_{i=0}^{N_1-n_1}\binom{N_1-n_1}{i}e^{-t\{\theta+(N_1-n_1-i)\lambda_1\}}\\ &\qquad \times \pi_1((n_1+i,n_2),\theta)\sum_{k=0}^{i}(-1)^k(e^{-\lambda_1 t})^k\binom{i}{k}\,dF_2(t)\\ &= \sum_{i=0}^{N_1-n_1}\binom{N_1-n_1}{i}\pi_1((n_1+i,n_2),\theta)\sum_{k=0}^{i}(-1)^k\binom{i}{k}\psi_2(\theta+(N_1-n_1-i+k)\lambda_1).\end{aligned}$$

By using (3.1), we obtain

$$\pi_1^2(n,\theta) = A_1(\theta) \sum_{i=0}^{N_1-n_1} \binom{N_1-n_1}{i} \sum_{j=0}^{N_1-n_1-i} \binom{N_1-n_1-i}{j} \frac{1}{v(j-1,\theta)}$$
$$\times \sum_{k=N_1-n_1-i}^{N_1-n_1} \binom{i}{k-N_1+n_1+i} (-1)^{k-N_1+n_1+i} \psi_2(k\lambda_1+\theta).$$

Changing the order of summation, we obtain

$$\pi_1^2(n,\theta) = A_1(\theta) \sum_{k=0}^{N_1-n_1} \sum_{i=N_1-n_1-k}^{N_1-n_1} \binom{N_1-n_1}{i} \binom{i}{k-N_1+n_1+i} (-1)^{k-N_1+n_1+i}$$
$$\times \psi_2(k\lambda_1+\theta) \sum_{j=0}^{N_1-n_1-i} \binom{N_1-n_1-i}{j} \frac{1}{v(j-1,\theta)}$$

which, on simplification and change of the order of summation, becomes

$$\pi_1^2(n,\theta) = A_1(\theta) \sum_{k=0}^{N_1-n_1} \psi_2(k\lambda_1+\theta) \sum_{j=0}^{k} \frac{1}{v(j-1,\theta)} \binom{N_1-n_1}{k} \binom{k}{j} \sum_{i=0}^{k-j} \binom{k-j}{i} (-1)^i$$

so that, by using the relation

$$\sum_{i=0}^{k-j} \binom{k-j}{i} (-1)^i = 0 \qquad \text{if } k>j$$
$$= 1 \qquad \text{if } k=j,$$

we obtain

$$\pi_1^2(n,\theta) = A_1(\theta) \sum_{k=0}^{N_1-n_1} \binom{N_1-n_1}{k} \frac{\psi_2(k\lambda_1+\theta)}{v(k-1,\theta)} . \qquad \text{(Q.E.D.)}$$

Lemma 6.

(3.3) $\pi_1(n,\theta) - \pi_1^2(n,\theta) \leqq \pi_1(\delta_2,\theta) - \pi_1^2(\delta_2,\theta).$

Proof: From Lemma 5,

$$\pi_1(n,\theta) - \pi_1^2(n,\theta) = A_1(\theta) \sum_{i=0}^{N_1-n_1} \binom{N_1-n_1}{i} \frac{1-\psi_2(i\lambda_1+\theta)}{v(i-1,\theta)} .$$

So that,

$$[\pi_1(n,\theta) - \pi_1^2(n,\theta)] - [\pi_1(n-\delta_1,\theta) - \pi_1^2(n-\delta_1,\theta)]$$
$$= A_1(\theta) \left[\sum_{i=1}^{N_1-n_1} \left\{ \binom{N_1-n_1}{i} - \binom{N_1-n_1+1}{i} \right\} \frac{1-\psi_2(i\lambda_1+\theta)}{v(i-1,\theta)} - \frac{1-\psi_2((N_1-n_1+1)\lambda_1+\theta)}{v(N_1-n_1,\theta)} \right]$$
$$= -A_1(\theta) \sum_{i=1}^{N_1-n_1+1} \binom{N_1-n_1}{i-1} \frac{1-\psi_2(i\lambda_1+\theta)}{v(i-1,\theta)}$$
$$\leqq 0.$$

Note that as far as $n_2>0$, the above expression does not depend on n_2. So by induction on n_1 and n_2, we obtain (3.3). (Q.E.D.)

Next, we shall obtain $\pi_2(n,\theta)$ and $\pi_2^2(n,\theta)$.

Lemma 7.

$$(3.4)\quad \pi_2(n,\theta) = A_2(\theta)\sum_{i=0}^{N_1-n_1}\sum_{j=0}^{N_2-n_2}\binom{N_1-n_1}{i}\binom{N_2-n_2}{j}\frac{A_1(j\lambda_2+\theta)}{v(i-1,j\lambda_2+\theta)h(j\lambda_2+\theta)w(j-1,\theta)},$$

$$(3.5)\quad \pi_2^2(n,\theta) = A_2(\theta)\sum_{i=0}^{N_1-n_1}\sum_{j=0}^{N_2-n_2}\binom{N_1-n_1}{i}\binom{N_2-n_2}{j}\psi_2(i\lambda_1+j\lambda_2+\theta)$$
$$\times\left[\frac{A_1(j\lambda_2+\theta)}{v(i-1,j\lambda_2+\theta)h(j\lambda_2+\theta)w(j-1,\theta)} + \frac{A_1((j+1)\lambda_2+\theta)}{v(i-1,(j+1)\lambda_2+\theta)h((j+1)\lambda_2+\theta)w(j,\theta)}\right],$$

where

$$A_2(\theta) = \left[\sum_{j=0}^{N_2}\binom{N_2}{j}\frac{1}{h(j\lambda_2+\theta)w(j-1,\theta)}\right]^{-1},$$

$$w(j-1,\theta) = \begin{cases}\prod_{k=0}^{j-1}[h(k\lambda_2+\theta)/\{1 - c(k\lambda_2+\theta)\}] & \text{if } j>0,\\ 1 & \text{if } j=0,\end{cases}$$

$$h(\theta) = A_1(\theta)\sum_{i=0}^{N_1}\binom{N_1}{i}\frac{\psi_2(i\lambda_1-\lambda_2+\theta)}{v(i-1,\theta)},$$

$$c(\theta) = A_1(\theta)\sum_{i=0}^{N_1}\binom{N_1}{i}\frac{\psi_2(i\lambda_1+\theta)}{v(i-1,\theta)}.$$

Proof: From [9] (expression (2.72) in chapter 2), we obtain

$$(3.6)\quad \pi_2((0,n_2),\theta) = A_2(\theta)\sum_{j=0}^{N_2-n_2}\binom{N_2-n_2}{j}\frac{1}{h(j\lambda_2+\theta)w(j-1,\theta)}.$$

(Because the busy period of class 1 customers which starts upon completion of a class 2 customer's service becomes the "restoration time.") By conditioning $\pi_2(n,\theta)$ on $B_1(n)$ and J_2 (which denotes the number of class 2 customers arriving during $B_1(n)$), we obtain

$$\pi_2(n,\theta|B_1(n)=t,\ J_2=j) = e^{-\theta t}\pi_2((0,n_2+j),\theta), \qquad (0 \leq j \leq N_2-n_2).$$

Hence,

$$\pi_2(n,\theta) = \int_0^\infty \sum_{j=0}^{N_2-n_2} e^{-\theta t}\pi_2((0,n_2+j),\theta)$$
$$\times\binom{N_2-n_2}{j}(1 - e^{-\lambda_2 t})^j(e^{-\lambda_2 t})^{N_2-n_2-j}dP\{B_1(n)\leq t\}.$$

Similarly, as in the proof of Lemma 5, we can obtain (3.4) from (3.6) and the above expression. Next, we shall prove (3.5). By conditioning $\pi_2^2(n,\theta)$ on S_2 (which is the service time of the class 2 customer being serviced first) and J_k (k=1,2) (which denotes the number of class k customers arriving during S_2), we obtain

$$\pi_2^2(n,\theta|S_2=t,\ J_1=i,\ J_2=j) = e^{-\theta t}\pi_2(n+(i,j-1),\theta).$$

Hence,

$$\pi_2^2(n,\theta) = \int_0^\infty \sum_{i=0}^{N_1-n_1}\sum_{j=0}^{N_2-n_2}\binom{N_1-n_1}{i}(1-e^{-\lambda_1 t})^i(e^{-\lambda_1 t})^{N_1-n_1-i}$$
$$\times\binom{N_2-n_2}{j}(1-e^{-\lambda_2 t})^j(e^{-\lambda_2 t})^{N_2-n_2-j}e^{-\theta t}\pi_2(n+(i,j-1),\theta)dF_2(t).$$

Similarly, as in the proof of Lemma 5, the above expression and (3.4) can lead to (3.5). (The detailed proof is given in the appendix.) (Q.E.D.)

4. Optimal Policy of the Model

In this section, we shall obtain the optimal policy of the model. We maximize the following reward function:

$$V(n;f) = E_f\left[\sum_{k=1}^{2} r_k \int_0^\infty e^{-\beta t}dD_k(t)\middle|\ n\right]$$

where $D_k(t)$ denotes the number of class k customers departing from the service station by time t and where f denotes a policy. In this section, we further assume the following things:

1) $\lambda_1 = \lambda_2 = \lambda$.
2) S_1 and S_2 are the exponentially distributed random variables.

As we stated in the introduction, we will try to show the optimality of a static policy. We define g as a static policy which preferentially selects class 1 customers. (Class 2 customers are selected for service only when no class 1 customers are present at the service station.) Further, we define g_2 as a policy which selects a class 2 customer at the first decision point and thereafter follows the same customer selection rule as g.

Lemma 8.

(4.1) $\pi_2(n,\theta) = \pi_2^2(n,\theta)$.

Proof: From the definition of v, the exponential assumption of S_i and the assumption that $\lambda=\lambda_1=\lambda_2$, we obtain

$$v(i-1,(j+1)\lambda+\theta) = v(i-1,j\lambda+\theta)\,\frac{j\lambda+\theta}{(i+j)\lambda+\theta}\,.$$

By using these,

$$h((j+1)\lambda+\theta) = A_1((j+1)\lambda+\theta)\sum_{k=0}^{N_1}\binom{N_1}{k}\frac{\psi_2(k\lambda-\lambda+(j+1)\lambda+\theta)}{v(k-1,(j+1)\lambda+\theta)}$$

$$= \frac{\mu_2}{j\lambda+\theta}\,A_1((j+1)\lambda+\theta)\sum_{k=0}^{N_1}\binom{N_1}{k}\frac{1}{v(k-1,j\lambda+\theta)}\cdot\frac{(k+j)\lambda+\theta}{\mu_2+(k+j)\lambda+\theta}$$

$$= \frac{\mu_2}{j\lambda+\theta}\,A_1((j+1)\lambda+\theta)\sum_{k=0}^{N_1}\binom{N_1}{k}\frac{1-\psi_2((k+j)\lambda+\theta)}{v(k-1,j\lambda+\theta)}$$

$$= \frac{\mu_2}{j\lambda+\theta}\,A_1((j+1)\lambda+\theta)\,\frac{1-c(j\lambda+\theta)}{A_1(j\lambda+\theta)}\,.$$

From the definitions of w, h and c,

$$w(j,\theta) = w(j-1,\theta)\,\frac{h(j\lambda+\theta)}{1-c(j\lambda+\theta)}\,.$$

From these expressions, we obtain

$$\frac{A_1((j+1)\lambda+\theta)\psi_2((i+j)\lambda+\theta)}{v(i-1,(j+1)\lambda+\theta)h((j+1)\lambda+\theta)w(j,\theta)} = \frac{A_1(j\lambda+\theta)[1-\psi_2((i+j)\lambda+\theta)]}{v(i-1,j\lambda+\theta)h(j\lambda+\theta)w(j-1,\theta)}\,.$$

The result (4.1) follows from the above expression, (3.4) and (3.5). (Q.E.D.)

This lemma means that the distributions of the busy periods are invariant whether a class 1 customer is initially selected for service or not.

Theorem 1. If $C_1 \geqq C_2$, then the static policy g is optimal.

Proof: Note that the rewards which can be received at decision points are bounded. Then we can apply the results presented in Chapter 7 in [14]. So we only need to show that $V(n;g)\geqq V(n;g_2)$ for all n ($n_2>0$). From Lemmas 4', 6 and 8,

$$\begin{aligned} V(n;g) - V(n;g_2) &= V_2(n) - V_2^2(n) \\ &= (1-\psi_2)C_1 - \psi_2 r_2 - [\pi_1(n,\beta) - \pi_1^2(n,\beta)](C_1 - C_2) \\ &\qquad - [\pi_2(n,\beta) - \pi_2^2(n,\beta)](C_2 - V(0)) \\ &\geqq (1-\psi_2)C_1 - \psi_2 r_2 - [\pi_1(\delta_2,\beta) - \pi_1^2(\delta_2,\beta)](C_1 - C_2) \\ &= V_2(\delta_2) - V_2^2(\delta_2) = 0. \end{aligned}$$

This completes the proof.

Corollary. The optimal policy is a static policy in which the class k that has a higher value of $\mu_k r_k$ than another class has a higher priority.

Proof: From (2.9), we can show that $C_1 \geqq C_2$ if and only if $\mu_1 r_1 \geqq \mu_2 r_2$. (Q.E.D.)

5. Conclusion

We have shown the optimality of the static policy for the finite-source M/M/1 queueing system with two customer classes and same thinking times. Further, we have shown that the optimal policy assigns a higher priority to the class k which has a higher value of $\mu_k r_k$. This assignment rule is similar to the well known "cμ rule", although the rule minimizes the expected waiting costs ([1] and [4]). The optimality of the static policy may be proved for the model with more than two customer classes. In this case, the problem is how to obtain LSTs of busy period distributions. These LSTs can be obtained in principle by using the similar manner in section 3. But the procedures become more complicated, so we will need a sophisticated method.

Appendix

We shall prove (3.5). We have shown that

$$\pi_2^2(n,\theta) = \int_0^\infty \sum_{i=0}^{N_1-n_1} \sum_{j=0}^{N_2-n_2} \binom{N_1-n_1}{i}(1 - e^{-\lambda_1 t})^i (e^{-\lambda_1 t})^{N_1-n_1-i}$$
$$\times \binom{N_2-n_2}{j}(1 - e^{-\lambda_2 t})^j (e^{-\lambda_2 t})^{N_2-n_2-j} e^{-\theta t} \pi_2(n+(i,j-1),\theta) dF_2(t).$$

We interchange the integration and the summations, and calculate the integral part as follows:

$$\int_0^\infty e^{-\theta t}(1 - e^{-\lambda_1 t})^i (e^{-\lambda_1 t})^{N_1-n_1-i}(1 - e^{-\lambda_2 t})^j (e^{-\lambda_2 t})^{N_2-n_2-j} dF_2(t)$$
$$= \int_0^\infty \sum_{k=0}^{i}\binom{i}{k}(-1)^k (e^{-\lambda_1 t})^k \sum_{m=0}^{j}\binom{j}{m}(-1)^m (e^{-\lambda_2 t})^m$$
$$\times e^{-t[\theta+(N_1-n_1-i)\lambda_1+(N_2-n_2-j)\lambda_2]} dF_2(t)$$
$$= \sum_{k=0}^{i} \sum_{m=0}^{j}\binom{i}{k}\binom{j}{m}(-1)^k(-1)^m \psi_2[\theta+(N_1-n_1-i+k)\lambda_1+(N_2-n_2-j+m)\lambda_2]$$

$$= \sum_{k=N_1-n_1-i}^{N_1-n_1} \sum_{m=N_2-n_2-j}^{N_2-n_2} \binom{i}{k-(N_1-n_1-i)}\binom{j}{m-(N_2-n_2-j)}$$
$$\times (-1)^{k-(N_1-n_1-i)}(-1)^{m-(N_2-n_2-j)}\psi_2(k\lambda_1+m\lambda_2+\theta).$$

By applying the above expression and changing the order of the summations, we have

$$\pi_2^2(n,\theta) = \sum_{k=0}^{N_1-n_1}\sum_{m=0}^{N_2-n_2}\psi_2(k\lambda_1+m\lambda_2+\theta)$$
$$\times \sum_{i=N_1-n_1-k}^{N_1-n_1}\sum_{j=N_2-n_2-m}^{N_2-n_2}(-1)^{k-(N_1-n_1-i)}(-1)^{m-(N_2-n_2-j)}$$
$$\times\binom{i}{k-(N_1-n_1-i)}\binom{j}{m-(N_2-n_2-j)}\binom{N_1-n_1}{i}\binom{N_2-n_2}{j}\pi_2(n+(i,j-1),\theta).$$

By substituting (3.4) into π_2 of the above expression,

$$\pi_2^2(n,\theta)$$
$$= A_2(\theta)\sum_{k=0}^{N_1-n_1}\sum_{m=0}^{N_2-n_2}\psi_2(k\lambda_1+m\lambda_2+\theta)$$
$$\times \sum_{i=N_1-n_1-k}^{N_1-n_1}\sum_{j=N_2-n_2-m}^{N_2-n_2}(-1)^{k-(N_1-n_1-i)}(-1)^{m-(N_2-n_2-j)}\binom{i}{k-(N_1-n_1-i)}$$
$$\times\binom{j}{m-(N_2-n_2-j)}\binom{N_1-n_1}{i}\binom{N_2-n_2}{j}\sum_{a=0}^{N_1-n_1-i}\sum_{b=0}^{N_2-n_2-j+1}\binom{N_1-n_1-i}{a}\binom{N_2-n_2-j+1}{b}$$
$$\times\frac{A_1(b\lambda_2+\theta)}{v(a-1,b\lambda_2+\theta)h(b\lambda_2+\theta)w(b-1,\theta)}$$

$$= A_2(\theta)\sum_{k=0}^{N_1-n_1}\sum_{m=0}^{N_2-n_2}\psi_2(k\lambda_1+m\lambda_2+\theta)\sum_{a=0}^{k}\sum_{b=0}^{m+1}\frac{A_1(b\lambda_2+\theta)}{v(a-1,b\lambda_2+\theta)h(b\lambda_2+\theta)w(b-1,\theta)}$$
$$\times\sum_{i=N_1-n_1-k}^{N_1-n_1-a}(-1)^{k-(N_1-n_1-i)}\binom{i}{k-(N_1-n_1-i)}\binom{N_1-n_1}{i}\binom{N_1-n_1-i}{a}$$
$$\times\sum_{j=N_2-n_2-m}^{\min\{N_2-n_2-b+1,N_2-n_2\}}(-1)^{m-(N_2-n_2-j)}\binom{j}{m-(N_2-n_2-j)}\binom{N_2-n_2}{j}\binom{N_2-n_2-j+1}{b}.$$

Because

$$\sum_{i=N_1-n_1-k}^{N_1-n_1-a}(-1)^{k-(N_1-n_1-i)}\binom{i}{k-(N_1-n_1-i)}\binom{N_1-n_1}{i}\binom{N_1-n_1-i}{a}$$

$$= \begin{cases} 0 & \text{if } a \neq k, \\ \binom{N_1-n_1}{k} & \text{if } a = k, \end{cases}$$

we have

$$\pi_2^2(n,\theta)$$
$$= A_2(\theta) \sum_{k=0}^{N_1-n_1} \sum_{m=0}^{N_2-n_2} \binom{N_1-n_1}{k} \psi_2(k\lambda_1+m\lambda_2+\theta) \sum_{b=0}^{m+1} \frac{A_1(b\lambda_2+\theta)}{v(k-1,b\lambda_2+\theta)h(b\lambda_2+\theta)w(b-1,\theta)}$$
$$\times \sum_{j=N_2-n_2-m}^{\min\{N_2-n_2-b+1,N_2-n_2\}} (-1)^{m-(N_2-n_2-j)} \binom{j}{m-(N_2-n_2-j)} \binom{N_2-n_2}{j} \binom{N_2-n_2-j+1}{b}$$

$$= A_2(\theta) \sum_{k=0}^{N_1-n_1} \sum_{m=0}^{N_2-n_2} \binom{N_1-n_1}{k} \psi_2(k\lambda_1+m\lambda_2+\theta)$$
$$\times \left[\frac{A_1(\theta)}{v(k-1,\theta)h(\theta)w(-1,\theta)} \sum_{j=N_2-n_2-m}^{N_2-n_2} (-1)^{m-(N_2-n_2-j)} \binom{j}{m-(N_2-n_2-j)} \binom{N_2-n_2}{j} \right.$$
$$+ \sum_{b=1}^{m+1} \frac{A_1(b\lambda_2+\theta)}{v(k-1,b\lambda_2+\theta)h(b\lambda_2+\theta)w(b-1,\theta)}$$
$$\left. \times \sum_{j=N_2-n_2-m}^{N_2-n_2-b+1} (-1)^{m-(N_2-n_2-j)} \binom{j}{m-(N_2-n_2-j)} \binom{N_2-n_2}{j} \binom{N_2-n_2-j+1}{b} \right].$$

Further we can show that

$$\sum_{j=N_2-n_2-m}^{N_2-n_2} (-1)^{m-(N_2-n_2-j)} \binom{j}{m-(N_2-n_2-j)} \binom{N_2-n_2}{j}$$
$$= \begin{cases} 0 & \text{if } m \neq 0 \\ 1 & \text{if } m = 0 \end{cases}$$

and that

$$\sum_{j=N_2-n_2-m}^{N_2-n_2-b+1} (-1)^{m-(N_2-n_2-j)} \binom{j}{m-(N_2-n_2-j)} \binom{N_2-n_2}{j} \binom{N_2-n_2-j+1}{b}$$
$$= \begin{cases} 1 & \text{if } m = 0 \\ \binom{N_2-n_2}{m} & \text{if } b = m \text{ or } m+1 \ (m > 0) \\ 0 & \text{if } 1 \leq b \leq m-1 \ \ (m > 0). \end{cases}$$

Then, we have

$$\frac{A_1(\theta)}{v(k-1,\theta)h(\theta)w(-1,\theta)} \sum_{j=N_2-n_2-m}^{N_2-n_2} (-1)^{m-(N_2-n_2-j)} \binom{j}{m-(N_2-n_2-j)} \binom{N_2-n_2}{j}$$

$$+ \sum_{b=1}^{m+1} \frac{A_1(b\lambda_2+\theta)}{v(k-1,b\lambda_2+\theta)h(b\lambda_2+\theta)w(b-1,\theta)}$$

$$\times \sum_{j=N_2-n_2-m}^{N_2-n_2-b+1} (-1)^{m-(N_2-n_2-j)} \binom{j}{m-(N_2-n_2-j)} \binom{N_2-n_2}{j} \binom{N_2-n_2-j+1}{b}$$

$$= \binom{N_2-n_2}{m} \left[\frac{A_1(m\lambda_2+\theta)}{v(k-1,m\lambda_2+\theta)h(m\lambda_2+\theta)w(m-1,\theta)} + \frac{A_1((m+1)\lambda_2+\theta)}{v(k-1,(m+1)\lambda_2+\theta)h((m+1)\lambda_2+\theta)w(m,\theta)} \right].$$

From the above expressions, we can obtain the desired result. (Q.E.D.)

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