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EXTENSIONS OF THE MULTIPLICATIVE PENALTY FUNCTION METHOD FOR LINEAR PROGRAMMING

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Abstract We shall extend Iri's multiplicative penalty function method for linear programming [4] so that it can handle the problem of unknown optimum value of the objective function, without solving both primal and dual problems simultaneously, and generate convergent dual solutions. By making use of these dual variables, lower bounds of the optimum objective function value are updated efficiently, which makes the total number of iterations required in the extended algorithm small. In doing so, a new duality on the multiplicative penalty function is discussed. A sufficient condition for a constraint to be inactive at all optimum solutions is given, which can be checked in the extended algorithm. Several computational techniques for enhancing the efficiency of the algorithm are also discussed. Some connection of the proposed algorithm with Sonnevend's and Renegar's methods [10, 11] is touched upon. Furthermore a method of estimating the optimum objective function value is given. Preliminary computational results on the random linear programming problem are finally shown.

Introduction

Since Khachian's work [7], attempts have been made to develop fast algorithms for linear programming, different from the simplex method [2]. In 1984, Karmarkar [6] gave a new polynomial-time algorithm, which is an interior iterative method, and, can be viewed as a gradient projection method for minimizing the potential function, introduced in [6], in a projectively transformed space. Karmarkar's algorithm has been extended in several ways subsequently, among which we here refer to two papers having some connection with this paper on how to extend algorithms (these two extend Karmarkar's algorithm, while this paper does Iri's multiplicative penalty function method). Todd and Burrell [12] describe an extension of Karmarkar's algorithm that handles problems with unknown optimum value and generates convergent dual solutions. Kojima [8] gives an update formula for a lower bound of the objective function, which corresponds to the update formula in [12] using the dual solutions, and shows a sufficient condition for a variable to be positive at all optimum solutions in Karmarkar's algorithm.

In [4] (see also [5]), Iri has introduced the multiplicative penalty function for linear programming, and proposed a Newton-like descent algorithm for minimizing it. It is shown that the multiplicative penalty function is convex, and, the proposed algorithm converges superlinearly when the optimum value of the objective function is known in advance. Also, under some assumption, global linear convergence of the algorithm is shown. Preliminary computational experiments, which evidence the effectiveness of the algorithm, have also been given in that paper. In those experiments, a given linear programming problem is paired with its dual problem in order to make the optimum value of an objective function zero.

In this paper, we shall extend Iri's multiplicative penalty function method for linear programming so that it can handle the problem of unknown optimum value, without solving both primal and dual problems simultaneously as in the experiments in [4]. The extended algorithm generates convergent dual solutions, by making full use of which good lower bounds of the optimum value can be found. The way of obtaining dual solutions proposed here is different from that in Todd and Burrell's extension [12] of Karmarkar's algorithm. A new duality on the multiplicative function is shown, which seems interesting since it is the duality on interior points of both primal and dual problems, not on extreme points.

We also show that, in the process of the algorithm, an ellipsoid can be easily constructed which contains all the optimum solutions and is determined by the current solution, the Hessian of the multiplicative penalty function at that solution and the number of constraints. Using this ellipsoid, we give a sufficient condition for a constraint to be inactive at all optimum solutions. This condition seems to have little to do with Kojima's condition [8] in Karmarkar's algorithm.

The algorithm proposed in this paper has strong connection with Sonnevend's "analytical centre" method [11], and Renegar's method [10]. Especially, the idea of considering the ellipsoid containing all the optimum solutions is due to Sonnevend [11]. Some of relations between these methods are discussed.

A method of estimating the optimum objective function value from a solution for the current lower bound of the optimum objective function value is given, which effectively works when the lower bound is close to the optimum value.

Finally, preliminary computational results on the random linear programming problem are shown.

1. The Multiplicative Penalty Function for Linear Programming

In this section we provide several assumptions on the linear programming problem treated in this paper, and describe some of the results on the multiplicative penalty function

[4] relevant to this paper.

We shall consider the following linear programming problem (P):

$$\begin{aligned} &\text{minimize} \quad \sum_{\kappa=1}^{n} c_\kappa x^\kappa \\ &\text{s.t.} \quad a^i(x) \equiv \sum_{\kappa=1}^{n} a^i_\kappa x^\kappa - a^i_0 \geq 0 \qquad (i = 1, \dots, m) \end{aligned} \tag{P}$$

where c_κ, a^i_0 and a^i_κ ($\kappa = 1, \dots, n$; $i = 1, \dots, m$) are given constants. (We basically adopt the tensor notation in order to maintain the geometrical meanings of the relevant expressions as clear as possible.)

Concerning (P), we assume the following (i)~(v):

Let X be the feasible region of (P):

$$X \equiv \{ x \in \mathbf{R}^n \mid a^i(x) \geq 0 \ (i = 1, \dots, m) \}$$

(i) The interior Int X of X is nonempty, and there exists a strictly interior point $x^{(0)} \in$ Int X such that $a^i(x^{(0)}) > 0$ ($i = 1, \dots, m$).

(ii) An optimum solution exists.

Let $\hat{c}_0$ be the optimum value of the objective function and $\widehat{X}$ the set of optimum solutions:

$$\hat{c}_0 = \min\{ \sum_{\kappa=1}^{n} c_\kappa x^\kappa \mid x \in X \},$$

$$\widehat{X} = \{ x \in X \mid \sum_{\kappa=1}^{n} c_\kappa x^\kappa = \hat{c}_0 \}.$$

(iii) $\widehat{X} \neq X$.

(iv) $\widehat{X}$ is bounded.

(v) At a basic optimum solution, there is at least one inactive constraint.

In the algorithm we propose in this paper, we further assume the following:

(vi) A strictly interior point $x^{(0)} \in$ Int X and a constant $c_0^{(0)}$ with $c_0^{(0)} < \hat{c}_0$ are given, while $\hat{c}_0$ is unknown in advance.

For a real number c_0, we define $c(x, c_0)$ by

$$c(x, c_0) = \sum_{\kappa=1}^{n} c_\kappa x^\kappa - c_0.$$

The multiplicative penalty function $F(x, c_0)$ introduced by Iri in [4] is

$$F(x, c_0) \equiv c(x, c_0)^{m+1} \Big/ \prod_{i=1}^{m} a^i(x)$$

in the interior Int X of the feasible region X for $c_0 \leq \hat{c}_0$. Define the "gradient" $\eta(x, c_0)$ of $F(x, c_0)$, which is the gradient of $F(x, c_0)$ divided by $F(x, c_0)$, by

$$\begin{aligned} \eta_\kappa(x, c_0) &\equiv \frac{\partial}{\partial x^\kappa} \log F(x, c_0) = \frac{1}{F(x, c_0)} \frac{\partial}{\partial x^\kappa} F(x, c_0) \\ &= (m+1) \frac{c_\kappa}{c(x, c_0)} - \sum_{i=1}^{m} \frac{a^i_\kappa}{a^i(x)}. \end{aligned} \tag{1.1}$$

Also, define the "Hessian" $H(\boldsymbol{x}, c_0)$ of $F(\boldsymbol{x}, c_0)$, which is the Hessian of $F(\boldsymbol{x}, c_0)$ divided by $F(\boldsymbol{x}, c_0)$, by

$$\begin{aligned} H_{\lambda\kappa}(\boldsymbol{x}, c_0) &\equiv \frac{1}{F(\boldsymbol{x}, c_0)} \frac{\partial^2}{\partial x^\lambda \partial x^\kappa} F(\boldsymbol{x}, c_0) \\ &= -(m+1)\frac{c_\lambda}{c(\boldsymbol{x}, c_0)}\frac{c_\kappa}{c(\boldsymbol{x}, c_0)} + \sum_{i=1}^{m} \frac{a^i_\lambda}{a^i(\boldsymbol{x})}\frac{a^i_\kappa}{a^i(\boldsymbol{x})} + \eta_\lambda(\boldsymbol{x}, c_0)\eta_\kappa(\boldsymbol{x}, c_0) \\ &= \Big[m\frac{c_\lambda}{c(\boldsymbol{x}, c_0)} - \sum_{i=1}^{m}\frac{a^i_\lambda}{a^i(\boldsymbol{x})}\Big]\Big[m\frac{c_\kappa}{c(\boldsymbol{x}, c_0)} - \sum_{i=1}^{m}\frac{a^i_\kappa}{a^i(\boldsymbol{x})}\Big] \\ &\qquad + \sum_{i=1}^{m}\Big[\frac{c_\lambda}{c(\boldsymbol{x}, c_0)} - \frac{a^i_\lambda}{a^i(\boldsymbol{x})}\Big]\Big[\frac{c_\kappa}{c(\boldsymbol{x}, c_0)} - \frac{a^i_\kappa}{a^i(\boldsymbol{x})}\Big] \end{aligned} \tag{1.2}$$

By expressing the Hessian as in (1.2) and by means of the assumptions, especially (v), Iri shows, in [4], the following.

Lemma 1.1. (Iri [4]) $H(\boldsymbol{x}, c_0)$ is positive definite so that $F(\boldsymbol{x}, c_0)$ is strictly convex in Int X. □

The following lemma is fundamental.

Lemma 1.2. For any $\tilde{c}_0 \geq \hat{c}_0$, $X \bigcap \{\boldsymbol{x} \mid \sum_{\kappa=1}^{n} c_\kappa x^\kappa = \tilde{c}_0\}$ is bounded.

Proof: If, for some $\tilde{c}_0 \geq \hat{c}_0$, $X \bigcap \{\boldsymbol{x} \mid \sum_{\kappa=1}^{n} c_\kappa x^\kappa = \tilde{c}_0\}$ is not bounded, there exists a vector $\boldsymbol{\xi} \neq \boldsymbol{0}$ such that $\sum_{\kappa=1}^{n} c_\kappa \xi^\kappa = 0$ and $\sum_{\kappa=1}^{n} a^i_\kappa \xi^\kappa \geq 0$. Let $\hat{\boldsymbol{x}}$ be an optimum solution, which exists by the assumption (ii). Then, $\hat{\boldsymbol{x}} + t\boldsymbol{\xi}$ is contained in $\widehat{X}$ for any $t \geq 0$, which implies that $\widehat{X}$ is unbounded, contradicting the assumption (iv). □

2. Minima of the Penalty Functions and Duality

Considering the problem of minimizing $F(\boldsymbol{x}, c_0)$ in $\boldsymbol{x} \in$ Int X for a constant $c_0 < \hat{c}_0$, we have the following.

Lemma 2.1. For $c_0 < \hat{c}_0$, there exists a unique optimum solution, which will be denoted by $\hat{\boldsymbol{x}}(c_0)$, in minimizing $F(\boldsymbol{x}, c_0)$ in $\boldsymbol{x} \in$ Int X. At $\hat{\boldsymbol{x}}(c_0)$, we have $\eta_\kappa(\hat{\boldsymbol{x}}(c_0), c_0) = 0$.

Proof: Concerning $F(\boldsymbol{x}, c_0)$, there are barriers at the boundary. Even when X is not bounded, it is seen from Lemma 1.2 that $F(\boldsymbol{x}, c_0)$ diverges to the infinity along any rays in Int X. Hence, the lemma follows from the strict convexity of $F(\boldsymbol{x}, c_0)$ (Lemma 1.1). □

$\sum_{\kappa=1}^{n} c_\kappa \hat{x}^\kappa(c_0)$ is decreasing when $c_0 \uparrow \hat{c}_0$, as is shown in the following lemma.

Lemma 2.2. For $c_0 < c_0' < \hat{c}_0$, we have $\sum\limits_{\kappa=1}^{n} c_\kappa \hat{x}^\kappa(c_0) > \sum\limits_{\kappa=1}^{n} c_\kappa \hat{x}^\kappa(c_0') > \hat{c}_0$.

Proof: Since $\eta_\kappa(\hat{\boldsymbol{x}}(c_0), c_0) = 0$ $(\kappa = 1, \ldots, n)$,

$$\eta_\kappa(\hat{\boldsymbol{x}}(c_0), c_0') = \frac{(m+1)c_\kappa}{c(\hat{\boldsymbol{x}}(c_0), c_0')} - \frac{(m+1)c_\kappa}{c(\hat{\boldsymbol{x}}(c_0), c_0)} = \frac{(m+1)(c_0' - c_0)}{c(\hat{\boldsymbol{x}}(c_0), c_0')c(\hat{\boldsymbol{x}}(c_0), c_0)} c_\kappa.$$

$F(\boldsymbol{x}, c_0)$ is strictly convex in Int X, so that, for $c_0 < c_0' < \hat{c}_0$,

$$0 > \sum_{\kappa=1}^{n} \eta_\kappa(\hat{\boldsymbol{x}}(c_0), c_0')(\hat{x}^\kappa(c_0') - \hat{x}^\kappa(c_0))$$
$$= \frac{(m+1)(c_0' - c_0)}{c(\hat{\boldsymbol{x}}(c_0), c_0')c(\hat{\boldsymbol{x}}(c_0), c_0)}\left(\sum_{\kappa=1}^{n} c_\kappa \hat{x}^\kappa(c_0') - \sum_{\kappa=1}^{n} c_\kappa \hat{x}^\kappa(c_0)\right).$$

Hence, $\sum_{\kappa=1}^{n} c_\kappa \hat{x}^\kappa(c_0') < \sum_{\kappa=1}^{n} c_\kappa \hat{x}^\kappa(c_0)$. □

The linear programming problem (D) dual to (P) is as follows:

$$\begin{aligned} \text{maximize} \quad & \sum_{i=1}^{m} a_0^i y_i \\ \text{s.t.} \quad & \sum_{i=1}^{m} a_\kappa^i y_i = c_\kappa \qquad (\kappa = 1, \ldots, n) \\ & y_i \geq 0 \qquad (i = 1, \ldots, m) \end{aligned} \tag{D}$$

For $c_0 < \hat{c}_0$, define $\boldsymbol{y}(c_0)$ by

$$y_i(c_0) = \frac{1}{m+1}\frac{c(\hat{\boldsymbol{x}}(c_0), c_0)}{a^i(\hat{\boldsymbol{x}}(c_0))} \geq 0 \qquad (i = 1, \ldots, m).$$

Theorem 2.1. For $c_0 < \hat{c}_0$, $\boldsymbol{y}(c_0)$ is a feasible solution to (D), and

$$\sum_{i=1}^{m} a_0^i y_i(c_0) = c_0 + \frac{1}{m+1}c(\hat{\boldsymbol{x}}(c_0), c_0) \leq \hat{c}_0 \leq c_0 + c(\hat{\boldsymbol{x}}(c_0), c_0) = \sum_{\kappa=1}^{n} c_\kappa \hat{x}^\kappa(c_0), \tag{2.1}$$

and so

$$\sum_{\kappa=1}^{n} c_\kappa \hat{x}^\kappa(c_0) - \hat{c}_0 \leq m(\hat{c}_0 - c_0). \tag{2.2}$$

Proof: Since $\eta_\kappa(\hat{\boldsymbol{x}}(c_0), c_0) = 0$, $\boldsymbol{y}(c_0)$ is a feasible solution to (D).

For $\boldsymbol{x} = \hat{\boldsymbol{x}}(c_0)$ and $\boldsymbol{y} = \boldsymbol{y}(c_0)$,

$$\sum_{i=1}^{m} a_0^i y_i = \sum_{i=1}^{m}\left(\sum_{\kappa=1}^{n} a_\kappa^i x^\kappa - a^i(\boldsymbol{x})\right)y_i = \sum_{\kappa=1}^{n}\left(\sum_{i=1}^{m} a_\kappa^i y_i\right)x^\kappa - \sum_{i=1}^{m}\frac{c(\boldsymbol{x}, c_0)}{m+1}$$
$$= \sum_{\kappa=1}^{n} c_\kappa x^\kappa - \frac{m}{m+1}c(\boldsymbol{x}, c_0)$$
$$= c_0 + \frac{c(\boldsymbol{x}, c_0)}{m+1}.$$

Then, the theorem follows from the duality theorem.

The theorem can directly be proved without explicitly using dual variables as follows. Since $\eta_\kappa(\hat{\boldsymbol{x}}(c_0), c_0) = 0$, for any $\boldsymbol{x} \in$ Int X,

$$0 = \sum_{\kappa=1}^{n} \eta_\kappa(\hat{\boldsymbol{x}}(c_0), c_0)(\hat{x}^\kappa(c_0) - x^\kappa)$$
$$= \frac{(m+1)[c(\hat{\boldsymbol{x}}(c_0), c_0) - c(\boldsymbol{x}, c_0)]}{c(\hat{\boldsymbol{x}}(c_0), c_0)} - \sum_{i=1}^{m}\frac{a^i(\hat{\boldsymbol{x}}(c_0)) - a^i(\boldsymbol{x})}{a^i(\hat{\boldsymbol{x}}(c_0))}$$
$$= 1 - (m+1)\frac{c(\boldsymbol{x}, c_0)}{c(\hat{\boldsymbol{x}}(c_0), c_0)} + \sum_{i=1}^{m}\frac{a^i(\boldsymbol{x})}{a^i(\hat{\boldsymbol{x}}(c_0))}. \tag{2.3}$$

Since $\dfrac{a^i(x)}{a^i(\hat{x}(c_0))} \geq 0$, we have

$$\sum_{\kappa=1}^{n} c_\kappa x^\kappa = c_0 + \frac{c(\hat{x}(c_0), c_0)}{m+1}\left(1 + \sum_{i=1}^{m} \frac{a^i(x)}{a^i(\hat{x}(c_0))}\right) \geq c_0 + \frac{c(\hat{x}(c_0), c_0)}{m+1}. \qquad \square$$

Lemma 2.3. As $c_0 \uparrow \hat{c}_0$, $\hat{x}(c_0)$ converges to an optimum solution $\hat{x}(\hat{c}_0) \in \widehat{X}$ to (P).

Proof: For $c_0 < \hat{c}_0$, $\hat{x}(c_0)$ is a unique solution of the system of equations $\dfrac{\partial F(x, c_0)}{\partial x^\kappa} = 0$ $(\kappa = 1, \ldots, n)$ and $\dfrac{\partial^2 F(x, c_0)}{\partial x^\kappa x^\lambda}$ is nonsingular and continuous, so that $\hat{x}(c_0)$ is continuous for $c_0 < \hat{c}_0$.

From (2.2), it is readily seen that the distance between $\widehat{X}$ and $\hat{x}(c_0)$ converges to 0 (hence, in case there is the unique optimum solution to (P), $\hat{x}(c_0)$ converges to the optimum).

Define I by $I = \{ i \mid a^i(x) = 0 \text{ for any } x \in \widehat{X} \ (i = 1, \ldots, m)\}$. Also, define $\tilde{c}(c_0)$ and $\tilde{a}^i(c_0)$ $(i \in I)$ by

$$\tilde{c}(c_0) = c(\hat{x}(c_0), c_0) \quad (c_0 < \hat{c}_0) \quad \text{and} \quad \tilde{c}(\hat{c}_0) \equiv \lim_{c_0 \uparrow \hat{c}_0} \tilde{c}(c_0) = 0,$$
$$\tilde{a}^i(c_0) = a^i(\hat{x}(c_0)) \quad (c_0 < \hat{c}_0) \quad \text{and} \quad \tilde{a}^i(\hat{c}_0) \equiv \lim_{c_0 \uparrow \hat{c}_0} \tilde{a}^i(c_0) = 0 \quad (i \in I).$$

Consider the following problem:

$$\begin{aligned} \text{minimize} \quad & \Psi_I(x) \equiv 1 \Big/ \prod_{i \notin I} a^i(x) \\ \text{s.t.} \quad & a^i(x) > 0 \qquad (i \notin I) \\ & a^i(x) = \tilde{a}^i(c_0) \qquad (i \in I) \\ & c(x, c_0) = \tilde{c}(c_0) \end{aligned} \tag{2.4}$$

$\Psi_I(x)$ is strictly convex in the relative interior of the nonempty and bounded feasible region of (2.4), and so there exists a unique minimum solution for each $c_0 \leq \hat{c}_0$, which will be denoted by $x(c_0)$. $x(c_0)$ satisfies

$$\begin{aligned} & c_\kappa z - \sum_{i \in I} a^i_\kappa z_i - \sum_{i \notin I} \frac{a^i_\kappa}{a^i(x)} = 0 \quad (\kappa = 1, \ldots, n) \\ & a^i(x) = \tilde{a}^i(c_0) \qquad (i \in I) \\ & c(x, c_0) = \tilde{c}(c_0) \end{aligned} \tag{2.5}$$

(z, z_i's $(i \in I)$ are variables and $a^i(x) > 0$ $(i \notin I)$). For $c_0 < \hat{c}_0$, setting $z = (m+1)/\tilde{c}(c_0)$ and $z_i = 1/\tilde{a}^i(c_0)$, we see that $\hat{x}(c_0)$ is a solution to (2.5), and so $\hat{x}(c_0) = x(c_0)$. We will show that $x(c_0)$ is continuous for $c_0 \leq \hat{c}_0$, and hence, as $c_0 \uparrow \hat{c}_0$, $\hat{x}(c_0) \to x(\hat{c}_0) \equiv \hat{x}(\hat{c}_0) \in \widehat{X}$.

Among $|I|$ vectors $(a^i_\kappa \mid \kappa = 1, \ldots, n)$ $(i \in I)$, take a maximum independent set of vectors $(a^i_\kappa \mid \kappa = 1, \ldots, n)$ $(i \in I')$. By the definitions of I and I', vector $(c_\kappa \mid \kappa = 1, \ldots, n)$

can be expressed by a linear combination of $(a^i_\kappa \mid \kappa = 1,\ldots,n)$ $(i \in I')$, and hence (2.5) is equivalent to the following system of equations:

$$\begin{aligned} &\sum_{i\notin I} \frac{a^i_\kappa}{a^i(x)} + \sum_{i\in I'} a^i_\kappa z'_i = 0 \qquad (\kappa = 1,\ldots,n) \\ &a^i(x) - \tilde{a}^i(c_0) = 0 \qquad (i \in I') \end{aligned} \tag{2.6}$$

where $a^i(x) > 0$ $(i \notin I)$. By virtue of the invariance of the system under the group of affine transformations in $\mathbf{R}^n$, we can assume without loss of generality that $I' = \{1, 2, \ldots, |I'|\}$ and $a^i_\kappa = \delta^i_\kappa$ for $i \in I'$ where $\delta^i_\kappa = 1$ if $i = \kappa$ and, otherwise, $\delta^i_\kappa = 0$. Define X_I by $X_I \equiv \{x \in \mathbf{R}^n \mid a^i(x) = \tilde{a}^i(c_0)\ (i \in I'),\ a^i(x) > 0\ (i \notin I)\}$. Since X_I is nonempty and bounded, for the Hessian $\widetilde{H}_{\lambda\kappa}$ of $\log \Psi_I$ where $\widetilde{H}_{\lambda\kappa} = \sum_{i\notin I} \frac{a^i_\lambda a^i_\kappa}{a^i(x)^2}$, its square submatrix $(\widetilde{H}_{\lambda\kappa} \mid \lambda, \kappa = |I'|+1,\ldots,n)$ is positive definite, and so we see that the Jacobian matrix of (2.6) is nonsingular and continuous in X_I. Hence, $x(c_0)$ is continuous for $c_0 \le \hat{c}_0$. □

By combining Lemma 2.3 and the following Theorem 2.3 (and restating the former) and by virtue of the invariance of the system under the group of affine transformations in $\mathbf{R}^n$, we have the following.

Theorem 2.2. As $c_0 \uparrow \hat{c}_0$, $\hat{x}(c_0)$ converges to the optimum solution $\hat{x}(\hat{c}_0) \in \widehat{X}$ to (P) and $y(c_0)$ converges to the optimum solution to (D). □

Theorem 2.3. Suppose that the problem (P) is in the so-called canonical form: $a^i_\kappa = \delta^i_\kappa$ and $a^i_0 = 0$ for $i = 1,\ldots,n$, where $\delta^i_\kappa = 1$ if $i = \kappa$ and, otherwise, $\delta^i_\kappa = 0$. Set $b^j = a^j_0$ for $j = n+1,\ldots,m$. In this case, the linear programming problem dual to (P) can be expressed as the following (D′):

$$\begin{aligned} \text{maximize} \quad & \sum_{j=n+1}^{m} b^j z_j \\ \text{s.t.} \quad & \sum_{j=n+1}^{m} a^j_\kappa z_j \le c_\kappa \qquad (\kappa = 1,\ldots,n) \\ & z_j \ge 0 \qquad (j = n+1,\ldots,m) \end{aligned} \tag{D$'$}$$

The problem (D′) is seen to satisfy the assumptions (i)∼(v) in section 1, and hence, for the problem of minimizing $G(z)$, defined by

$$G(z) = \frac{\left(b^0 - \sum_{j=n+1}^{m} b^j z_j\right)^{m+1}}{\left(\prod_{\kappa=1}^{n}\left(c_\kappa - \sum_{j=n+1}^{m} a^j_\kappa z_j\right)\right)\left(\prod_{j=n+1}^{m} z_j\right)},$$

in the interior of the feasible region to (D′) for $b^0 > \hat{c}_0$, there exists the unique optimum solution, which will be denoted by $\hat{z}(b^0)$.

Considering $\hat{x}(c_0)$ and $y(c_0)$ for this (P), we then have

$$\hat{z}\left(c_0 + \frac{m+2}{m+1} c(\hat{x}(c_0), c_0)\right) = y(c_0).$$

Proof: $\hat{z}(b^0)$ is the unique solution of the system of equations

$$\frac{-(m+1)b^j}{b^0-\sum_{i=n+1}^m b^i z_i}+\sum_{\kappa=1}^n \frac{a_\kappa^j}{c_\kappa-\sum_{i=n+1}^m a_\kappa^i z_i}-\frac{1}{z_j}=0 \qquad (j=n+1,\ldots,m).$$

We can easily see that $b^0 = c_0 + (m+2)c(\hat{x}(c_0),c_0)/(m+1)$ and $z_j = y_j(c_0)$ $(j = n+1,\ldots,m)$ is a solution, and hence obtain the theorem. □

If the problem (P) is in the so-called canonical form or in the basis form with respect to some basis, we may easily obtain a dual feasible solution from a general primal feasible solution, which is not necessarily $\hat{x}(c_0)$ for some c_0, as in the following.

Theorem 2.4. Suppose the problem (P) is such that, for $i = 1,\ldots,n$, $a_\kappa^i = \delta_\kappa^i$ where $\delta_\kappa^i = 1$ if $\kappa = i$ and $\delta_\kappa^i = 0$ if $\kappa \neq i$. Then, in case $\eta_i(x, c_0) + \dfrac{1}{a^i(x)} \geq 0$ $(i = 1,\ldots,n)$ for $x \in \operatorname{Int} X$, $\tilde{y}$ defined by

$$\tilde{y}_i = \frac{c(x,c_0)}{m+1}\Big(\eta_i(x,c_0)+\frac{1}{a^i(x)}\Big) \qquad (i=1,\ldots,n)$$
$$\tilde{y}_i = \frac{c(x,c_0)}{m+1}\frac{1}{a^i(x)} \qquad (i=n+1,\ldots,m)$$

is a feasible solution to (D).

Furthermore, the optimum value of the following simple linear programming problem:

$$\begin{aligned} \text{maximize} \quad & \sum_{i=1}^m a_0^i y_i \\ \text{s.t.} \quad & y_i = \tilde{y}_i t \geq 0 \qquad (i=n+1,\ldots,m) \\ & y_\kappa = c_\kappa - \sum_{j=n+1}^m a_\kappa^j y_j \geq 0 \qquad (\kappa=1,\ldots,n) \end{aligned}$$

is a lower bound for $\hat{c}_0$.

Proof: From (1.1), $\tilde{y}$ satisfies $\sum_{i=1}^m a_\kappa^i \tilde{y}_i = c_\kappa$ $(\kappa = 1,\ldots,n)$. Also, from the condition $\eta_i(x,c_0) + \dfrac{1}{a^i(x)} \geq 0$ $(i = 1,\ldots,n)$, $\tilde{y} \geq 0$. □

It should be noted that the condition of this theorem becomes more likely to be satisfied if x gets close to $\hat{x}(c_0)$ for some $c_0 < \hat{c}_0$.

If we have p solutions $y^{(\mu)}$ $(\mu = 1,\ldots,p)$ feasible to (D),

$$\begin{aligned} \text{maximize} \quad & \sum_{i=1}^m a_0^i y_i \\ \text{s.t.} \quad & y_i = \sum_{\mu=1}^p w(\mu) y_i^{(\mu)} \geq 0 \\ & \sum_{\mu=1}^p w(\mu) = 1 \end{aligned} \qquad (2.7)$$

gives a better lower bound for $\hat{c}_0$, since, in (2.7), y_i is a feasible solution to (D).

Note that, if a new lower bound c_0 is obtained from a solution $x^{(0)}$ by means of Theorem 2.1 and/or (2.7), it may be possible to obtain a new dual solution by using the new value c_0 again by Theorem 2.4.

3. An Algorithm

From the theorems established above, given an initial interior solution $x^{(0)}$ and a lower bound $c_0^{(0)} < \hat{c}_0$, we can consider the following algorithm:

1° Initialization: — $\nu := 0$; $c_0 := c_0^{(0)}$; $\mu := 0$;
2° Iteration: — while $c(x^{(\nu)}, c_0) \geq \epsilon$ do the following:
2.1° while the condition of Theorem 2.4 with respect to $x^{(\nu)}$ holds do the following:
$\mu := \mu + 1$;
Compute a dual solution $y^{(\mu)}$ from $x^{(\nu)}$ by the theorem;
Compute a lower bound $c_0^{(\mu)}$ from $y^{(\max\{1,\mu-p+1\})}, \ldots, y^{(\mu-1)}, y^{(\mu)}$ (p: a parameter) by (2.7);
$c_0 := \min\{c_0, c_0^{(\mu)}\}$;
(This while loop may be stopped within a few steps.)
2.2° Solve the system of linear equations at $x^{(\nu)}$

$$\sum_{\kappa=1}^{n} H_{\lambda\kappa}(x^{(\nu)}, c_0)\xi^{(\nu)\kappa} = -\eta_\lambda(x, c_0) \qquad (\lambda = 1, \ldots, n) \tag{3.1}$$

to determine the vector $\xi^{(\nu)\kappa}$;
Perform the line search in the direction of $\xi^{(\nu)}$ to find

$$\left[\frac{\mathrm{d}}{\mathrm{d}t} F(x^{(\nu)} + t\xi^{(\nu)}, c_0)\right]_{t=t^*} = 0$$

and set

$$x^{(\nu+1)} := x^{(\nu)} + t^*\xi^{(\nu)}.$$

2.3° $\nu := \nu + 1$;

Theoretically, we should set $\epsilon = O(2^{-L})$ as in [6], [7] where L is the number of bits in the input data representing a^i_κ, a^i_0, c_κ. In this algorithm, c_0 is correctly updated so that it converges to $\hat{c}_0$, since, as $x^{(\nu)}$ converges to $\hat{x}(c_0)$, a dual solution becomes more likely to be obtained in step 2.1, and, the dual solution corresponding to $\hat{x}(c_0)$ gives better lower bound than c_0 due to Theorem 2.1. In the Newton method in step 2.3, the line search can be performed also by the Newton method, as in [4]. Concerning the parameter p in (2.7), as p is set bigger, the lower bound $c_0^{(\mu)}$ becomes better but more computational efforts come to be needed.

4. Determining Constraints Inactive at All Optimum Solutions

For $x \in \mathbf{R}^n$ and c_0 with $c_0 < \hat{c}_0$, define $h(x, c_0)$ by

$$h(x,c_0) \equiv \sum_{\lambda=1}^{n} \sum_{\kappa=1}^{n} H_{\lambda\kappa}(\hat{x}(c_0),c_0)(x^\kappa - \hat{x}^\kappa(c_0))(x^\lambda - \hat{x}^\lambda(c_0)).$$

For c_0 and a nonnegative real number ρ, consider an ellipsoid $E(c_0, \rho)$ given by

$$E(c_0,\rho) \equiv \{x \in \mathbf{R}^n \mid h(x,c_0) \le \rho\}.$$

$h(x, c_0)$ has a nice expression as in the following lemma.

Lemma 4.1. $h(x,c_0) = 1 + \sum_{i=1}^{m} \left(\frac{a^i(x)}{a^i(\hat{x}(c_0))}\right)^2 - \frac{1}{m+1}\left(1 + \sum_{i=1}^{m} \frac{a^i(x)}{a^i(\hat{x}(c_0))}\right)^2.$

Proof: Denoting $\frac{c(x,c_0)}{c(\hat{x}(c_0),c_0)}$ and $\frac{a^i(x)}{a^i(\hat{x}(c_0))}$ by $\tilde{c}$ and $\tilde{a}^i$, respectively, for the sake of simplicity of notation, we have

$$\begin{aligned}
h(x,c_0) &= -(m+1)\Big[\sum_{\kappa=1}^{n} \frac{c_\kappa}{c(\hat{x}(c_0),c_0)}(x^\kappa - \hat{x}^\kappa(c_0))\Big]^2 \\
&\quad + \sum_{i=1}^{m}\Big[\sum_{\kappa=1}^{n} \frac{a^i_\kappa}{a^i(\hat{x}(c_0))}(x^\kappa - \hat{x}^\kappa(c_0))\Big]^2 \\
&= -(m+1)(1-\tilde{c})^2 + \sum_{i=1}^{m}(1-\tilde{a}^i)^2 \\
&= -(m+1)[1 - \frac{1}{m+1}(1+\sum_{i=1}^{m}\tilde{a}^i)]^2 + \sum_{i=1}^{m}(1-\tilde{a}^i)^2 \\
&= 1 + \sum_{i=1}^{m}(\tilde{a}^i)^2 - \frac{1}{m+1}(1+\sum_{i=1}^{m}\tilde{a}^i)^2,
\end{aligned}$$

where the third equality follows from (2.3). □

Theorem 4.1. For $\tilde{c}_0 \ge \hat{c}_0$,

$$X \bigcap \{x \in \mathbf{R}^n \mid \sum_{\kappa=1}^{n} c_\kappa x^\kappa \le \tilde{c}_0\} \subseteq E(c_0, \rho(\tilde{c}_0)),$$

where $\rho(\tilde{c}_0) = \max\{2,\ m(m+1)(\frac{\tilde{c}_0 - c_0}{c(\hat{x}(c_0),c_0)})^2 - 2(m+1)\frac{\tilde{c}_0 - c_0}{c(\hat{x}(c_0),c_0)} + 2\}$.

Proof: For $x \in \text{Int } X$, since $\sum_{i=1}^{m} \tilde{a}^i = (m+1)\tilde{c} - 1$ from (2.3), $\sum_{i=1}^{m}(\tilde{a}^i)^2 \le [(m+1)\tilde{c} - 1]^2$, and so

$$h(x,c_0) \le 1 + [(m+1)\tilde{c} - 1]^2 - (m+1)\tilde{c}^2 = m(m+1)\tilde{c}^2 - 2(m+1)\tilde{c} + 2.$$

For $x \in \text{Int } X \bigcap \{ x \mid \sum_{\kappa=1}^{n} c_\kappa x^\kappa \le \tilde{c}_0\}$, $0 \le \tilde{c} = \frac{c(x,c_0)}{c(\hat{x}(c_0),c_0)} \le \frac{\tilde{c}_0 - c_0}{c(\hat{x}(c_0),c_0)}$, hence we obtain the theorem. □

Corollary 4.1. $\widehat{X} \subseteq E(c_0, m(m-1))$.

Proof: From Theorem 4.1 and the fact that $\rho(\sum_{\kappa=1}^{n} c_\kappa \hat{x}^\kappa(c_0)) = m(m-1)$. □

Theorem 4.2. $E(c_0, 1/2) \subseteq X$.

Proof: Let $\hat{\rho} = \max\{\rho \mid \rho \geq 0,\ E(c_0,\rho) \subseteq X\}$. $\hat{\rho}$ satisfies

$$\hat{\rho} = 1 + \sum_{i=1}^{m} (\tilde{a}^i)^2 - \frac{1}{m+1}(1 + \sum_{i=1}^{m} \tilde{a}^i)^2$$

for $\tilde{a}^i \geq 0$ ($i = 1, \ldots, m$) and $\tilde{a}^i = 0$ for at least one i. As is readily seen,

$$\min\{1 + \sum_{i=1}^{m} (v^i)^2 - \frac{1}{m+1}(1 + \sum_{i=1}^{m} v^i)^2 \mid v^i \geq 0\ (i = 1, \ldots, m),\ v^i = 0 \text{ for at least one } i\} = \frac{1}{2},$$

and so $\hat{\rho} \geq 1/2$. □

Using Corollary 4.1, a sufficient condition for a constraint to be inactive at optimum solutions can be obtained as follows.

Theorem 4.3. If $a^i(\hat{x}(c_0)) > \sqrt{m(m-1)\sum_{\lambda=1}^{n}\sum_{\kappa=1}^{n}(H_{\lambda\kappa})^{-1} a^i_\kappa a^i_\lambda}$ where $(H_{\lambda\kappa})^{-1}$ is the inverse of $H_{\lambda\kappa}(\hat{x}(c_0), c_0)$, then the constraint $a^i(x) \geq 0$ is inactive at all optimum solutions (i.e., $a^i(\hat{x}) > 0$ for any $\hat{x} \in \widehat{X}$).

Proof: $\min\{\sum_{\kappa=1}^{n} a^i_\kappa x^\kappa \mid x \in E(c_0, m(m-1))\}$ is attained at $\tilde{x}$ given by

$$\tilde{x}^\lambda = \hat{x}^\lambda(c_0) - \sum_{\kappa=1}^{n}(H_{\lambda\kappa})^{-1} a^i_\kappa \sqrt{m(m-1) \Big/ \sum_{\mu=1}^{n}\sum_{\nu=1}^{n}(H_{\mu\nu})^{-1} a^i_\nu a^i_\mu}.$$

If $a^i(\tilde{x}) > 0$, then $a^i(x) > 0$ for any $x \in E(c_0, m(m-1))$; that is, $a^i(\hat{x}) > 0$ for any $\hat{x} \in \widehat{X}$. □

Also, we can obtain a lower bound by means of the ellipsoid, which follows immediately from Corollary 4.1.

Theorem 4.4. $\hat{c}_0 \geq \sum_{\kappa=1}^{n} c_\kappa \hat{x}^\kappa(c_0) - \sqrt{m(m-1)\sum_{\lambda=1}^{n}\sum_{\kappa=1}^{n}(H_{\lambda\kappa})^{-1} c_\kappa c_\lambda}$. □

Note that, in case we know $\tilde{c}_0$ such that $\hat{c}_0 \leq \tilde{c}_0 \leq \sum_{\kappa=1}^{n} c_\kappa \hat{x}^\kappa(c_0)$, we can replace $m(m-1)$ in Theorems 4.3 and 4.4 by $\rho(\tilde{c}_0)$.

Generally, as $c_0 \uparrow \hat{c}_0$, it would become more likely for the condition in Theorem 4.3 for inactive constraints to hold true. In the non-degenerate case, if $c(x, c_0)$ is sufficiently small, $H_{\lambda\kappa}(x, c_0)$ multiplied by $c(x, c_0)^2$ is a full-rank matrix with eigenvalues of magnitude $\Omega(1)$, as noted in [4] (see also [5]). In fact, in this case, we have the following.

Theorem 4.5. Suppose there are no primal and dual degeneracies. Let σ_j ($j = 1, \ldots, n$) be the eigenvalues of $H_{\lambda\kappa}(x, c_0)$ multiplied by $c(x, c_0)^2$ such that $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0$. Let $a^{i(j)}(x)$ be the jth smallest among $a^i(x)$ such that $a^i(\hat{x}) = 0$ at the unique basic optimum solution $\hat{x}$, and define $\tilde{\sigma}_j$ to be $\left(\frac{c(x,c_0)}{a^{i(j)}(x)}\right)^2$ $(j = 1,\ldots,n)$. If $c(x,c_0)$ is sufficiently small for $x \in \text{Int } X$, there exist positive constants α, β and γ independent of x such that

$$\gamma \leq \alpha\tilde{\sigma}_j \leq \sigma_j \leq \beta\tilde{\sigma}_j \quad (j = 1,\ldots,n)$$

Proof: For a subspace S of $\mathbf{R}^n$, define $\sigma(S)$ by

$$\sigma(S) = \max\Big\{\frac{\sum_{\lambda=1}^{n}\sum_{\kappa=1}^{n} c(x,c_0)^2 H_{\lambda\kappa}\xi^\kappa\xi^\lambda}{\sum_{\kappa=1}^{n}(\xi^\kappa)^2} \mid \xi \in S - \{0\}\Big\}.$$

Then, as is well know, we have

$$\sigma_{n-j+1} = \min\{\sigma(S') \mid S': \text{ any subset of } \mathbf{R}^n \text{ with } \dim S = j\} \qquad (j = 1,\ldots,n).$$

Since

$$\begin{aligned}
&\sum_{\lambda=1}^{n}\sum_{\kappa=1}^{n} c(x,c_0)^2 H_{\lambda\kappa}\xi^\kappa\xi^\lambda \\
&\quad = \Big[m\sum_{\kappa=1}^{n} c_\kappa\xi^\kappa - \sum_{i=1}^{m}\frac{c(x,c_0)}{a^i(x)}\sum_{\kappa=1}^{n} a^i_\kappa\xi^\kappa\Big]^2 + \sum_{i=1}^{m}\Big[\sum_{\kappa=1}^{n} c_\kappa\xi^\kappa - \frac{c(x,c_0)}{a^i(x)}\sum_{\kappa=1}^{n} a^i_\kappa\xi^\kappa\Big]^2 \\
&\quad \leq (m^2+m)\Big(\sum_{\kappa=1}^{n} c_\kappa\xi^\kappa\Big)^2 + \Big(\sum_{i=1}^{m}\frac{c(x,c_0)}{a^i(x)}\sum_{\kappa=1}^{n} a^i_\kappa\xi^\kappa\Big)^2 + \sum_{i=1}^{m}\Big(\frac{c(x,c_0)}{a^i(x)}\sum_{\kappa=1}^{n} a^i_\kappa\xi^\kappa\Big)^2,
\end{aligned}$$

taking $S = \{\xi \mid \sum_{\kappa=1}^{n} a^{i(k)}_\kappa\xi^\kappa = 0 \ (k = 1,\ldots,j-1)\}$, we see that there are positive constants α_j, β_j independent of x such that

$$\sigma_j \leq \sigma(S) \leq \alpha_j\tilde{\sigma}_j + \beta_j.$$

Concerning $\det c(x,c_0)^2 H_{\lambda\kappa} = \prod_{j=1}^{n}\sigma_j$, from the Claim in Proposition 5.1 in [5], there are positive constants α_0 and β_0 such that

$$\alpha_0\prod_{j=1}^{n}\tilde{\sigma}_j \leq \prod_{j=1}^{n}\sigma_j \leq \beta_0\prod_{j=1}^{n}\tilde{\sigma}_j.$$

Furthermore, there is a positive constant γ_0 independent of x such that $\tilde{\sigma}_n \geq \gamma_0$.

Combining the discussions above, we obtain the theorem. □

Thus, in this case, all constraints inactive at the unique basic optimum solution, and hence the optimum basis can be found if the current solution is sufficiently close to the optimum. However, if the dual problem is degenerate, this method alone obviously fails to find any optimum basis.

5. Use of Bases

In this section, we present two ways of making use of bases in the algorithm, one in obtaining an optimum basic solution from the current interior point, and the other in solving the system of linear equations (3.1) quickly.

For $x \in \text{Int } X$, arrange m vectors $(a^i_\kappa \mid \kappa = 1, \ldots, n)$ $(i = 1, \ldots, m)$ in nondecreasing order of $\frac{a^i(x)}{\|a^i\|}$, where $\|a^i\| = \sqrt{\sum_{\kappa=1}^n (a^i_\kappa)^2}$, and make a maximum independent set of vectors $(a^i_\kappa \mid \kappa = 1, \ldots, n)$ $(i \in I(x))$ by choosing independent vectors in that order $(|I(x)| = n)$. The basis thus constructed will be identified with the index set $I(x)$.

As $x^{(\nu)}$ converges to an optimum, this basis is expected to be an optimum basis if the dual problem is not degenerate (see the end of this section). If the basis form with respect to $I(x^{(\nu)})$ is at hand, the optimality of the basis can be easily checked, and, updating the basis can be performed by pivots. This strategy for obtaining an optimum basis works efficiently if the total number of pivots during the iterations is small. (It should be noted that this strategy is first used in [13] (see also [9]) in connection with the use of bases in Karmarkar's algorithm in its gradient projection steps.)

As in [13] (also, [9]), the basis $I(x)$ can further be utilized in solving the system of linear equations (3.1) as follows. For $I \in \{1, \ldots, m\}$, define $H_{\lambda\kappa}(x, c_0; I)$ by

$$\begin{aligned} H_{\lambda\kappa}(x, c_0; I) =& [m\frac{c_\lambda}{c(x, c_0)} - \sum_{i \in I} \frac{a^i_\lambda}{a^i(x)}][m\frac{c_\kappa}{c(x, c_0)} - \sum_{i \in I} \frac{a^i_\kappa}{a^i(x)}] \\ &+ \sum_{i \in I} [\frac{c_\lambda}{c(x, c_0)} - \frac{a^i_\lambda}{a^i(x)}][\frac{c_\kappa}{c(x, c_0)} - \frac{a^i_\kappa}{a^i(x)}] \\ &+ (m - |I|)\frac{c_\lambda}{c(x, c_0)}\frac{c_\kappa}{c(x, c_0)}. \end{aligned}$$

When each $a^j(x)$ $(j \notin I(x))$ is sufficiently larger than $a^i(x)$ $(i \in I(x))$, $H_{\lambda\kappa}(x, c_0; I(x))$ can be regarded as a good approximation of $H_{\lambda\kappa}(x, c_0)$. $H_{\lambda\kappa}(x, c_0; I(x))$ is easily seen to be positive definite, and, if the basis form with respect to $I(x)$ is at hand, a system of linear equations $\sum_{\kappa=1}^n H_{\lambda\kappa}(x, c_0; I(x))\xi^\kappa = -\eta_\lambda$ can be solved in $O(n^2)$ time in total, as will be shown in the following.

The problem is reduced to computing the inverse of $G_{\lambda\kappa} = \sum_{i=1}^{n+2} D^i_\lambda D^i_\kappa$ where

$$\begin{aligned} D^i_\kappa &= w_\kappa - \delta^i_\kappa \qquad (i = 1, \ldots, n) \\ &= \sqrt{m - n} \cdot w_\kappa \qquad (i = n + 1) \\ &= mw_\kappa - 1 \qquad (i = n + 2) \end{aligned}$$

We claim that the inverse of $G_{\lambda\kappa}$ can be computed in $O(n^2)$ time. Although we can write down explicitly the inverse of $G_{\lambda\kappa}$, doing so is rather tedious and the inverse itself is too messy to show its validity, so that we shall here write down the inverse $\overline{G}^{\lambda\kappa}$ of $\tilde{G}_{\lambda\kappa} = \sum_{i=1}^{n+1} D^i_\lambda D^i_\kappa$, and show that the inverse $\overline{G}^{\lambda\kappa}$ can be computed in $O(n^2)$ time. Then, the claim can be shown by using the well-known technique of the rank-one modification of a matrix.

Applying the Binet-Chauchy Formula to the principal term of the expression for $\tilde{G}_{\lambda\kappa}$ to get

$$\det \tilde{G}_{\lambda\kappa} = \Big(\sum_{\kappa=1}^n w_\kappa - 1\Big)^2 + (m - n)\sum_{\kappa=1}^n (w_\kappa)^2.$$

In a similar way, we can compute the determinant of every $(n-1)\times(n-1)$ square submatrix of $\tilde{G}_{\lambda\kappa}$, and have the following, where $\sum_{\mu\neq\kappa}$ is the summation over $\mu=1,\ldots,\kappa-1,\kappa+1,\ldots,n$, and $\sum_{\mu\neq\lambda,\kappa}$ is similarly computed.

$$\overline{G}^{\kappa\kappa}=\Big[\Big(\sum_{\mu\neq\kappa}w_\mu-1\Big)^2+(m-n+1)\sum_{\mu\neq\kappa}(w_\mu)^2\Big]\Big/(\det\tilde{G}_{\lambda\kappa})$$

$$\overline{G}^{\lambda\kappa}=\Big[\sum_{\mu\neq\lambda,\kappa}(w_\mu)^2-\Big(\sum_{\mu\neq\lambda}w_\mu-1\Big)w_\lambda-\Big(\sum_{\mu\neq\kappa}w_\mu-1\Big)w_\kappa-(m-n)w_\lambda w_\kappa\Big]\Big/(\det\tilde{G}_{\lambda\kappa})\qquad(\lambda\neq\kappa)$$

from which we see $\overline{G}^{\lambda\kappa}$ can be computed in $O(n^2)$ time.

A direction obtained by replacing $H_{\lambda\kappa}(x^{(\nu)},c_0)$ by $H_{\lambda\kappa}(x^{(\nu)},c_0;I(x^{(\nu)}))$ in (3.1) might be expected to be a good direction (at least, as an initial solution in solving (3.1) by some iterative method such as the conjugate gradient method, as in the revised Karmarkar's algorithm in [9]).

In the case there is the dual degeneracy in (P), that is, $\widehat{X}$ is not a point, $\hat{x}(c_0)$ does not converge to an extreme point in $\widehat{X}$ (see the proof of Lemma 2.3). In fact, it converges to $\hat{x}(\hat{c}_0)$ which is relatively interior in $\widehat{X}$. Hence, it is expected that the technique for obtaining an optimum basis described above does not necessarily work effectively. Also, in this case, the technique in section 4 cannot find any optimum basis as noted at the end of section 4. Thus, in order to cope with the dual degeneracy, another method would be needed (e.g., see [3]). This difficulty would similarly arise in modified algorithms of Karmarkar's which update the lower bounds of the objective function, such as [8], [12].

6. The Multiplicative Penalty Function and the Analytic Centre

The algorithm proposed here has connection with the "analytical centre" method by Sonnevend [11]. Recently, Renegar [10] proposes an algorithm similar to Sonnevend's, and shows that the algorithm solves the linear programming problem in a polynomial time. In this section, we refer to some of the results in [10, 11], and discuss the connection.

For a system of $\tilde{m}$ linear inequalities $\sum_{\kappa=1}^n\tilde{a}^i_\kappa x^\kappa\geq\tilde{a}^i_0$ $(i=1,\ldots,\tilde{m})$ such that it determines a nonempty, bounded polyhedron P in $\mathbf{R}^n$, its *analytical centre* is a unique solution of a system of equations

$$\sum_{i=1}^{\tilde{m}}\frac{\tilde{a}^i_\lambda}{\sum_{\kappa=1}^n\tilde{a}^i_\kappa x^\kappa-\tilde{a}^i_0}=0\qquad(\lambda=1,\ldots,n).$$

The analytical centre is a unique optimum solution of the problem

$$\min\ \Psi(x)=\prod_{i=1}^{\tilde{m}}\frac{1}{\sum_{\kappa=1}^n\tilde{a}^i_\kappa x^\kappa-\tilde{a}^i_0}\quad\text{in }P,$$

where Ψ is strictly convex (Ψ may be regarded as a special case of Iri's multiplicative penalty function).

The following lemma can be shown easily.

Lemma 6.1. $\hat{x}(c_0)$ is the analytical centre of the system of $m+1$ linear inequalities $a^i(x) \geq 0$ $(i = 1, \ldots, m)$ and $-\sum_{\kappa=1}^{n} c_\kappa x^\kappa \geq -\sum_{\kappa=1}^{n} c_\kappa \hat{x}^\kappa(c_0) - \frac{c(\hat{x}(c_0), c_0)}{m+1}$. □

In [11], it is shown that there exist ellipsoids containing P and contained in P, centered at the analytical centre with similarity ratio $\widetilde{m} - 1$. Analogous ellipsoids for the multiplicative penalty function are given in section 4, whose similarity ratio is $\sqrt{2m(m-1)}$.

As an algorithm for the linear programming problem (P), Sonnevend [11] (and also Renegar [10] as noted above) proposes a continuation method for the analytical centres for $a^i(x) \geq 0$ $(i = 1, \ldots, m)$ and $\sum_{\kappa=1}^{n} c_\kappa x^\kappa \leq \tilde{c}_0$ $(\tilde{c}_0 \downarrow \hat{c}_0)$. Relations of Sonnevend's algorithm and the algorithm in this paper would deserve investigation.

7. Estimating the Optimum Objective Function Value

In this section, we show the following theorem by means of which the optimum value $\hat{c}_0$ of the objective function can be estimated in a good way from c_0 and $\hat{x}(c_0)$ when c_0 is sufficiently close to $\hat{c}_0$.

Theorem 7.1. Suppose there is no degeneracy in the given linear programming problem. Then, we have

$$\lim_{c_0 \uparrow \hat{c}_0} \frac{c(\hat{x}(c_0), \hat{c}_0)}{\hat{c}_0 - c_0} = \frac{n}{m+1-n}$$

and so

$$\hat{c}_0 = \lim_{c_0 \uparrow \hat{c}_0} \frac{(m+1-n)\sum_{\kappa=1}^{n} c_\kappa x^\kappa(c_0) + nc_0}{m+1}.$$

Proof: Define $c(x)$ to be $\sum_{\kappa=1}^{n} c_\kappa x^\kappa$. We have only to show that

$$\lim_{c_0 \uparrow \hat{c}_0} \frac{\mathrm{d}c(\hat{x}(c_0))}{\mathrm{d}c_0} = -\frac{n}{m+1-n}.$$

We can compute it directly by using the Hessian of $F(x, c_0)$, but we here adopt an easier approach by making use of another characterization of $\hat{x}(c_0)$ given in section 6. Let $\tilde{x}(\tilde{c}_0)$ be the analytical centre of the system of $m+1$ linear inequalities $a^i(x) \geq 0$ $(i = 1, \ldots, m)$ and $c(x, \tilde{c}_0) \leq 0$ for $\tilde{c}_0 > \hat{c}_0$. From Lemma 6.1,

$$\tilde{x}\Big(\frac{-c_0}{m+1} + \frac{m+2}{m+1} c(\hat{x}(c_0))\Big) = \hat{x}(c_0) \tag{7.1}$$

for $c_0 < \hat{c}_0$. From (7.1), we see

$$\frac{\mathrm{d}c(\hat{x}(c_0))}{\mathrm{d}c_0} = \frac{\mathrm{d}c(\tilde{x}(\tilde{c}_0))}{\mathrm{d}\tilde{c}_0} \Big/ \Big((m+2)\frac{\mathrm{d}c(\tilde{x}(\tilde{c}_0))}{\mathrm{d}\tilde{c}_0} - (m+1)\Big) \tag{7.2}$$

We shall prove the following lemma for $\tilde{x}(\tilde{c}_0)$, which itself would be of theoretical interest. Combining this lemma and (7.2), we obtain the theorem. □

Lemma 7.1. Suppose there is no degeneracy in the given linear programming problem. Then, we have

$$\lim_{\tilde{c}_0 \downarrow \hat{c}_0} \frac{c(\tilde{x}(\tilde{c}_0), \hat{c}_0)}{\tilde{c}_0 - \hat{c}_0} = \lim_{\tilde{c}_0 \downarrow \hat{c}_0} \frac{\mathrm{d}c(\tilde{x}(\tilde{c}_0))}{\mathrm{d}\tilde{c}_0} = \frac{n}{n+1}.$$

Proof: $\tilde{x}(\tilde{c}_0)$ satisfies the following system of equations:

$$-\sum_{i=1}^{m} \frac{a^i_\kappa}{a^i(x)} - \frac{c_\kappa}{c(x, \tilde{c}_0)} = 0 \qquad (\kappa = 1, \ldots, n).$$

Hence, $\mathrm{d}\tilde{x}^\kappa(\tilde{c}_0)/\mathrm{d}\tilde{c}_0$ is expressed by

$$\frac{\mathrm{d}\tilde{x}^\lambda(\tilde{c}_0)}{\mathrm{d}\tilde{c}_0} = -\sum_{\kappa=1}^{n} G^{\lambda\kappa} \cdot \frac{(-c_\kappa)}{c(\tilde{x}(\tilde{c}_0), \tilde{c}_0)^2},$$

where $G^{\lambda\kappa}$ is the inverse of $H_{\lambda\kappa}$ defined by

$$H_{\lambda\kappa} = \sum_{i=1}^{m} \frac{a^i_\lambda a^i_\kappa}{a^i(\tilde{x}(\tilde{c}_0))^2} + \frac{c_\lambda c_\kappa}{c(\tilde{x}(\tilde{c}_0), \tilde{c}_0)^2}.$$

Let $\hat{x}$ be the unique optimum solution. By the invariance of the system, we consider, under the group of affine transformations in $\mathbf{R}^n$, we can assume without loss of generality that $a^i_\kappa = \delta^i_\kappa$ $(i = 1, \ldots, n)$ where $\delta^i_\kappa = 1$ if $i = \kappa$ and $\delta^i_\kappa = 0$ if $i \neq \kappa$, and $a^i_0 = \sum_{\kappa=1}^{n} \delta^i_\kappa \hat{x}^\kappa$ $(i = 1, \ldots, n)$. Then, by discussions similar to those in section 2, we see

$$\lim_{\tilde{c}_0 \downarrow \hat{c}_0} \frac{-c(\tilde{x}(\tilde{c}_0), \tilde{c}_0)}{a^i(\tilde{x}(\tilde{c}_0))} = \begin{cases} c_i & (i = 1, \ldots, n) \\ 0 & (i = n+1, \ldots, m) \end{cases}$$

and

$$\lim_{\tilde{c}_0 \downarrow \hat{c}_0} c(\tilde{x}(\tilde{c}_0), \tilde{c}_0)^2 H_{\lambda\kappa} = \sum_{i=1}^{n} \delta^i_\lambda \delta^i_\kappa (c_i)^2 + c_\lambda c_\kappa.$$

By elementary calculation, we have

$$\lim_{\tilde{c}_0 \downarrow \hat{c}_0} \frac{G^{\lambda\kappa}}{c(\tilde{x}(\tilde{c}_0), \tilde{c}_0)^2} = \sum_{i=1}^{n} \frac{\delta^\lambda_i \delta^\kappa_i}{(c_i)^2} - \frac{1}{n+1} \frac{1}{c_\lambda c_\kappa},$$

and

$$\lim_{\tilde{c}_0 \downarrow \hat{c}_0} \frac{\mathrm{d}c(\tilde{x}(\tilde{c}_0))}{\mathrm{d}\tilde{c}_0} = \sum_{\lambda=1}^{n} \sum_{\kappa=1}^{n} \left(\sum_{i=1}^{n} \frac{\delta^\lambda_i \delta^\kappa_i}{(c_i)^2} - \frac{1}{n+1} \frac{1}{c_\lambda c_\kappa} \right) \cdot c_\kappa c_\lambda$$
$$= n - \frac{n^2}{n+1} = \frac{n}{n+1}.$$

Hence,

$$\lim_{\tilde{c}_0 \downarrow \hat{c}_0} \frac{c(\tilde{x}(\tilde{c}_0), \hat{c}_0)}{\tilde{c}_0 - \hat{c}_0} = \lim_{\tilde{c}_0 \downarrow \hat{c}_0} \frac{\mathrm{d}c(\tilde{x}(\tilde{c}_0))}{\mathrm{d}\tilde{c}_0} = \frac{n}{n+1}. \qquad \square$$

We prove the theorem under an assumption that there is no degeneracy. In the degenerate case, we can prove the following theorem first by showing it in the case where only the dual problem is degenerate with a proof almost similar to the proof above, and then by applying the dual arguments given in section 2.

Theorem 7.2. Suppose that at most one of the primal and dual problems is degenerate. Then, defining l to be the number of constraints $a^i(x) \geq 0$ which are active at all optimum solutions, we have

$$\lim_{c_0 \uparrow \hat{c}_0} \frac{c(\hat{x}(c_0), \hat{c}_0)}{\hat{c}_0 - c_0} = \frac{l}{m+1-l}. \quad \square$$

It is conjectured that the theorem holds valid even when both primal and dual problems are degenerate, which is left open. Also, it may be possible to apply this technique given in this paper to Karmarkar's algorithm for the standard-form linear programming problem (e.g., see [9]).

8. Computational Experiments

In this section, we show results of preliminary computational experiments of the algorithm in section 3. In this algorithm, we employ a technique in section 5 for obtaining an optimum basis.

The test problem is the random linear programming which was used in several computational experiments (e.g., see [1]). The problem can be described as follows:

$$\begin{aligned} \max\ & e^{\mathrm{T}} x, \\ \text{s.t.}\ & Ax \leq 10000, \\ & x \geq 0, \end{aligned} \tag{8.1}$$

where $e = (1, \ldots, 1)^{\mathrm{T}}$ and A is an $M \times N$ matrix with $M \leq N$ the elements of which are integers randomly generated in the range from 1 to 1000. The number n of variables is N and the number m of constraints is $M + N$. The problems would have many redundant constraints, and, there would be no dependence on m row vectors. Furthermore it seems that the problems of this class are not so practical since their A's are dense while the A's in ordinary linear programming problems being solved in practice are sparse. However, the random problems give us some insight into the performance of the algorithm for dense problems, especially into the effect of the problem size on the number of iterations required by the algorithm for the structured dense problems.

In these computational experiments, we employed a technique for obtaining an optimum basis described in section 5 (recall that there would be no degeneracy in the random problems). When obtaining dual solutions by using Theorem 2.4, we used the original canonical form of the random problem (the random problem is seen to be of good form to obtain dual solutions by Theorem 2.4). We did not use the latter half of Theorem 2.4 but used (2.7). The parameter p, which is the number of dual solutions used for updating c_0 in (2.7), was set to $p = 5$. Concerning the size of the matrix A, we set $M = N = n$ for the number n of variables. As an initial feasible solution $x^{(0)}$ and an initial lower bound

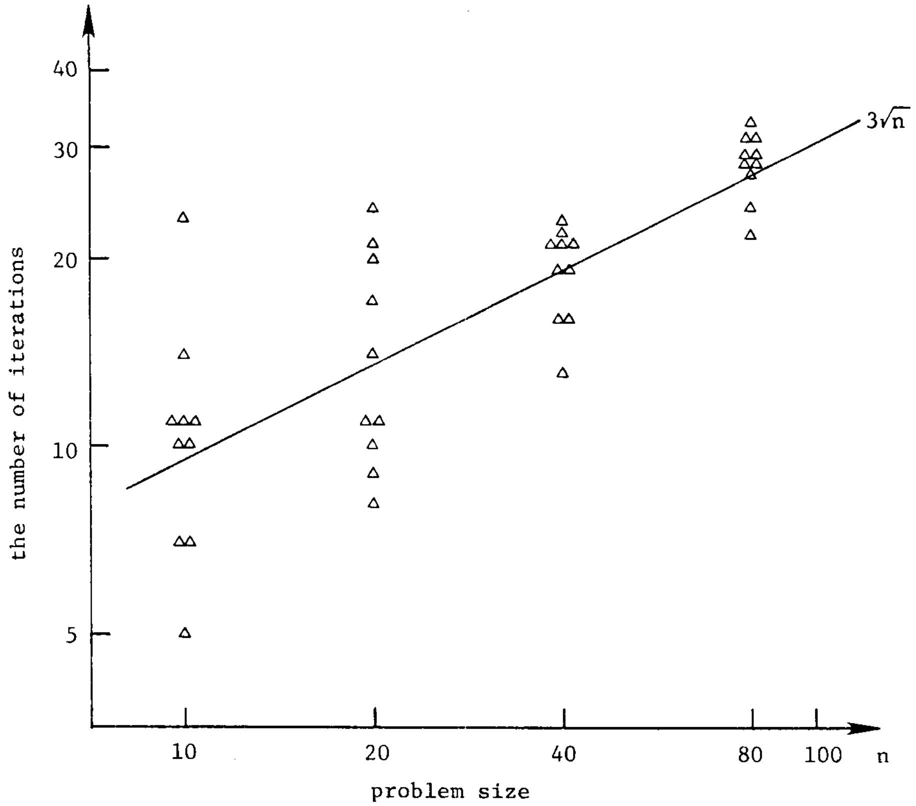

Fig.8.1. Computational results for the random problems: the number of iterations and pivoting required

$c_0^{(0)}$, we set $x^{(0)} = e/n$ and $c_0 = -10000$. The computational experiments were executed for the ten cases with each n = 10, 20, 40, 80. In all cases, the algorithm halts by finding an optimum basis by the technique in section 5. The results are shown in Fig.8.1.

It is seen that the average number of iterations the algorithm requires is proportional to about $3\sqrt{n}$ for n, and, the total number of pivots for maintaining bases throughout (see section 5) was $O(n^{1.5})$. Roughly, the number of iterations required by this algorithm is similar to that by the algorithm in [4] for solving the random problems with combining the primal and dual problems (note that conditions of computational experiments, such as the stopping criteria, are different). The number of pivots, which was observed to be $O(n^{1.5})$, seems large, and so we would need another technique for maintaining bases.

Although the computational experiments done here are rather limited, it is seen that the procedure for obtaining dual feasible solutions and updating the lower bounds of c_0 works quite well so that the algorithm for minimizing the multiplicative penalty function combined with the procedure requires a small number of iterations and runs fast in total.

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