

ON THE $M^X/G/1$ QUEUE WITH AND WITHOUT VACATION TIME UNDER NONPREEMPTIVE LAST-COME FIRST-SERVED DISCIPLINE

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Abstract We consider the $M^X/G/1$ queue with and without vacation time under nonpreemptive last-come first-served (LCFS). The Laplace Stieltjes transform of the distribution function of the steady-state waiting time is found. The first two moments of the waiting time are obtained. We find the relationship between the second moments of the waiting time in the $M^X/G/1$ queue with and without vacation time under first-come first-served (FCFS) and LCFS.

Key words: $M^X/G/1$ queue, vacation time, last-come first-served, waiting time

1 Introduction

In this paper, we study the batch arrival $M^X/G/1$ queue with and without vacation time under nonpreemptive last-come first-served (LCFS).

In some queueing systems, the server takes a vacation of random length each time the system becomes empty. The vacation time is utilized for some additional work. If the server returns from a vacation and he finds one or more customers waiting, he works until the system empties, then takes another vacation. If the server returns from a vacation and finds no customers waiting, he takes another vacation immediately.

The $M/G/1$ queue with vacation time has been studied by a number of authors, e. g. Cooper [5], Levy and Yechiali [9], Scholl and Kleinrock [10]. In particular [10] has studied the waiting time of the $M/G/1$ queue with vacation time under three queueing disciplines, which are independent of service time: first-come first-served (FCFS), random order of service (ROS) and nonpreemptive LCFS.

Several authors have studied the batch arrival queueing system $M^X/G/1$ under FCFS (see Chaudhry [3], Chaudhry and Templeton [4], Cooper [6], Gross and Harris [7] and Kleinrock [8]). Recently Baba [1] has studied the $M^X/G/1$ queue with vacation time under FCFS. However, there has been no work for the $M^X/G/1$ queue with vacation time under other service discipline.

Nonpreemptive LCFS queueing discipline is applicable to many practical situations such as push-down stack and inventory systems, etc. The batch arrival queueing model appears in many situations such as computer communication systems. So it is important to analyze the model that will be studied in this paper. In the batch arrival queueing systems under nonpreemptive LCFS, the following two queueing disciplines are considered.

(1) Queueing discipline 1

The customers which are included in the same batch as the customer in service are served previously than the customers which arrived at the system after the customer in service. (This discipline is considered as the case that arrived batch is treated as a super customer)

(2) Queueing discipline 2

The order of service is nonpreemptive LCFS with respect to batches and the order of service in a batch is random. (This discipline is considered as the nonpreemptive LCFS discipline with respect to all customers in the system)

In this paper, we study the queueing system under queueing discipline 2. This paper studies Laplace Stieltjes transform (LST) of the distribution function (d. f.) of the steady-state waiting time for the $M^X/G/1$ queue with and without vacation time under nonpreemptive LCFS. The first two moments of the waiting time are obtained.

The relationship that we find in this paper between the second moments of the waiting time for FCFS and LCFS in the $M^X/G/1$ queue with and without vacation time is an

extension of the result which has been found in [10] in the $M/G/1$ queue with and without vacation time.

2 Assumptions and Notations

We consider the $M^X/G/1$ queue with and without vacation time. We study this queueing system under queueing discipline 2 stated in Section 1. For the $M^X/G/1$ queue, it is assumed that customers arrive in batches according to a time homogeneous Poisson process with rate λ . The batch size X is a random variable and

$$(1) \quad P(X = n) = g_n, \quad n = 1, 2, \dots$$

with probability generating function (p. g. f.)

$$(2) \quad G(z) = \sum_{k=1}^{\infty} g_k z^k \quad |z| \leq 1.$$

We assume that n -th ($n = 1, 2, 3$) factorial moments of X are finite and defined by

$$(3) \quad g = E(X) = G^{(1)}(1),$$

$$(4) \quad g^{(2)} = E(X(X-1)) = G^{(2)}(1),$$

$$(5) \quad g^{(3)} = E(X(X-1)(X-2)) = G^{(3)}(1).$$

Let $S(x)$ be the d. f. of the service time S of a typical customer and $S^*(\theta)$ be the LST of S . Denote by \hat{S} the remaining service time for the customer in service. The d. f. $\hat{S}(x)$ and the LST $\hat{S}^*(\theta)$ of S are given by (see [8])

$$(6) \quad \hat{S}(x) = \frac{1}{E(S)} \int_0^x [1 - S(t)] dt,$$

$$(7) \quad \hat{S}^*(\theta) = [1 - S^*(\theta)] / \theta E(S).$$

In order to analyze the queue with vacation time, we denote by $V(x)$ the d. f. of a typical vacation time V and $V^*(\theta)$ the LST of V . Further denote by \hat{V} the remaining vacation time for the server on vacation. The d. f. $\hat{V}(x)$ and the LST $\hat{V}^*(\theta)$ of \hat{V} are

$$(8) \quad \hat{V}(x) = \frac{1}{E(V)} \int_0^x [1 - V(t)] dt,$$

$$(9) \quad \hat{V}^*(\theta) = [1 - V^*(\theta)] / \theta E(V).$$

Let B be the random variable of busy period that starts with one customer. The busy period starts with one customer is called 1-busy period. Denote by $B(x)$ the d. f. of the random variable B and denoted by $B^*(\theta)$ the LST of B . Suppose that a 1-busy period starts by one customer and its service time $S_1 = x$. Let N_1 be the number of customer arrivals during the interval S_1 . By conditioning on S_1 and N_1 , we have the following conditional transform

$$(10) \quad E(e^{-\theta B} \mid S_1 = x, N_1 = n) = e^{-\theta x} [B^*(\theta)]^n.$$

Unconditioning on N_1 , we have

$$(11) \quad \begin{aligned} E(e^{-\theta B} \mid S_1 = x) &= \sum_{n=0}^{\infty} E(e^{-\theta B} \mid S_1 = x, N_1 = n) P(N_1 = n) \\ &= \sum_{n=0}^{\infty} e^{-\theta x} [B^*(\theta)]^n \sum_{k=0}^n \frac{(\lambda x)^k}{k!} g_n^{*k} e^{-\lambda x} \\ &= \exp[-\{\theta + \lambda - \lambda G(B^*(\theta))\}x], \end{aligned}$$

where g_n^{*k} is the k -fold convolution of g_n with itself, with $g_i^{*0} = \delta_{i0}$. Finally we have

$$(12) \quad \begin{aligned} B^*(\theta) &= E(e^{-\theta B}) \\ &= \int_0^{\infty} \exp[-\{\theta + \lambda - \lambda G(B^*(\theta))\}x] dS(x) \\ &= S^*(\theta + \lambda - \lambda G(B^*(\theta))). \end{aligned}$$

In the case of the traffic intensity $\rho = \lambda g E(S) < 1$, the n -th ($n = 1, 2, 3$) moments of $B(x)$ are given by

$$(13) \quad E(B) = E(S)/(1 - \rho),$$

$$(14) \quad E(B^2) = \frac{E(S^2) + \lambda g^{(2)} [E(S)]^2}{(1 - \rho)^3},$$

$$(15) \quad \begin{aligned} E(B^3) &= \frac{E(S^3) + \lambda g^{(3)} [E(S)]^3}{(1 - \rho)^4} \\ &+ \frac{3\lambda g [E(S^2)]^2 + 6\lambda g^{(2)} [E(S)]^2 E(S^2) + 3\lambda [g^{(2)}]^2 [E(S)]^5}{(1 - \rho)^5} \end{aligned}$$

3 Waiting Time Distribution

3.1 The $M^X/G/1$ Queue without Vacation Time under LCFS

Let r_n ($n = 1, 2, \dots$) be the probability of an arbitrary customer being in the n -th position of an arrived batch. Burke [2] showed, using a result in the renewal theory, that it is given by

$$(16) \quad r_n = \frac{1}{g} \sum_{k=n}^{\infty} g_k.$$

Define the p. g. f.

$$(17) \quad R(z) = \sum_{n=1}^{\infty} r_n z^n = \frac{z[1 - G(z)]}{g(1 - z)}.$$

At the time of an arbitrary test customer's arrival, there will be a number of customers arriving in his batch who will be served before him. This number is $(n-1)$ with probability r_n .

Suppose that an arbitrary test customer named C_0 arrives when the server is idle. the $(n-1)$ customers arriving in C_0 's batch who will be served before him are named C_1, \dots, C_{n-1} , respectively. Observe that the effect of C_i ($i = 1, \dots, n-1$) on the waiting time of C_0 is the 1-busy period B_i generated by C_i . Since B_1, \dots, B_{n-1} are mutually independent and the proportion of time that the server is idle in an $M^X/G/1$ queue is $1 - \rho$ (see e. g. [3]), the steady-state waiting time of C_0 is distributed as $B_1 + \dots + B_{n-1}$ with probability $(1 - \rho)r_n$. Next suppose that C_0 arrives when the server is busy. Let the remaining service time of the customer in service at C_0 's arrival epoch be \hat{S} . Suppose that m (a random variable) customers named C_n, \dots, C_{m+n-1} arrive during \hat{S} . Denote by B_i ($i = 1, \dots, m+n-1$) the 1-busy period generated by C_i . Similar to the case that the server is idle, the steady-state waiting time of C_0 is distributed as $B_1 + \dots + B_{n-1} + B_n + \dots + B_{m+n-1}$ with probability ρr_n . Let W_L and $W_L^*(\theta)$ be the steady-state waiting time of an arbitrary customer and its LST, respectively. Since B_1, \dots, B_{n-1} are independent of \hat{S} , we have

$$(18) \quad \begin{aligned} W_L^*(\theta) &= \sum_{n=1}^{\infty} (1 - \rho)r_n [B^*(\theta)]^{n-1} \\ &+ \sum_{n=1}^{\infty} \rho r_n \sum_{m=0}^{\infty} \sum_{k=0}^m [B^*(\theta)]^{m+n-1} \int_0^{\infty} \frac{(\lambda t)^k}{k!} e^{-(\lambda+\theta)t} g_m^k d\hat{S}(t) \\ &= \frac{(1 - \rho)R(B^*(\theta))}{B^*(\theta)} + \sum_{n=1}^{\infty} \rho r_n [B^*(\theta)]^{n-1} \hat{S}^*(\theta + \lambda - \lambda G(B^*(\theta))) \\ &= \frac{(1 - \rho)[1 - G(B^*(\theta))]}{g[1 - B^*(\theta)]} + \frac{\lambda[1 - G(B^*(\theta))]}{\theta + \lambda - \lambda G(B^*(\theta))} \end{aligned}$$

By taking the first and second derivatives of (18) at $\theta = 0$, we obtain the following expressions:

$$(19) \quad E(W_L) = \frac{(1 - \rho)g^{(2)}E(B)}{2g} + \frac{\lambda g^{(2)}[E(B)]^2 + \lambda gE(B^2)}{2[1 + \lambda gE(B)]^2}$$

$$= \frac{\lambda gE(S^2)}{2(1 - \rho)} + \frac{g^{(2)}E(S)}{2g(1 - \rho)},$$

$$(20) \quad E(W_L^2) = \frac{(1 - \rho)}{6g} [3g^{(2)}E(B^2) + 2g^{(3)}\{E(B)\}^2]$$

$$+ \frac{\lambda}{6[1 + \lambda gE(B)]^3} [2[1 + \lambda gE(B)][g^{(3)}\{E(B)\}^3$$

$$+ 3g^{(2)}E(B)E(B^2) + g(B^3)]$$

$$- 3\lambda[g^{(2)}\{E(B)\}^2 + gE(B^2)]^2]$$

$$= \frac{\lambda gE(S^3)}{3(1 - \rho)^2} + \frac{\lambda^2 g^2\{E(S^2)\}^2}{2(1 - \rho)^3} + \frac{g^{(3)}\{E(S)\}^2}{3g(1 - \rho)^2}$$

$$+ \frac{\lambda\{g^{(2)}\}^2\{E(S)\}^3 + (1 + \rho)g^{(2)}E(S^2)}{2g(1 - \rho)^3}.$$

Remark 1. Let W_F be the steady-state waiting time without vacation time when the service discipline is FCFS. Its LST, $W_F^*(\theta)$, is given by (see Cooper [6])

$$(21) \quad W_F^*(\theta) = \frac{(1 - \rho)\theta[1 - G(S^*(\theta))]}{g[\theta - \lambda + \lambda G(S^*(\theta))][1 - S^*(\theta)]}.$$

By taking the first and second derivatives of (21) at $\theta = 0$, we have

$$(22) \quad E(W_F) = \frac{\lambda gE(S^2)}{2(1 - \rho)} + \frac{g^{(2)}E(S)}{2g(1 - \rho)},$$

$$(23) \quad E(W_F^2) = \frac{\lambda gE(S^3)}{3(1 - \rho)} + \frac{\lambda^2 g^2\{E(S^2)\}^2}{2(1 - \rho)^2} + \frac{g^{(3)}\{E(S)\}^2}{3g(1 - \rho)}$$

$$+ \frac{\lambda\{g^{(2)}\}^2\{E(S)\}^3 + (1 + \rho)g^{(2)}E(S^2)}{2g(1 - \rho)^2}.$$

Since the order of service is independent of service time, we see immediately that the mean queue size and the mean waiting time for LCFS must be same as for FCFS. Thus it is clear that $E(W_L) = E(W_F)$ seen from (19) and (22). However, the second moment $E(W_L^2)$ is larger than $E(W_F^2)$. Comparing (20) to (23), we have

$$(24) \quad E(W_L^2) = E(W_F^2)/(1 - \rho).$$

It is surprising to find that this result holds for the $M^X/G/1$ queue as well as for the $M/G/1$ queue.

3.2 The $M^X/G/1$ Queue with Vacation Time under LCFS

As is the case with 3.1, since the proportion of time that the server is vacationing in the $M^X/G/1$ queue with vacation time is $1 - \rho$ (see Baba [1]), the steady-state waiting time of an arbitrary test customer is distributed as $\hat{V} + B_1 + \dots + B_{m+s-1}$ with probability $(1 - \rho)r_s$, where m is the number of customers arrived during the residual vacation time $hat{V}$ and as $\hat{S} + B_1 + \dots + B_{m+s-1}$ with probability ρr_s , where m is the number of customers arrived during the residual service time \hat{S} . Let W_{VL} and $W_{VL}^*(\theta)$ be the steady-state waiting time of an arbitrary customer and its LST, respectively. By using (7) and (17), we have

$$\begin{aligned}
 (25) \quad W_{VL}^*(\theta) &= \sum_{s=1}^{\infty} (1 - \rho)r_s \sum_{m=0}^{\infty} \sum_{k=0}^m [B^*(\theta)]^{m+s-1} \int_0^{\infty} \frac{(\lambda t)^k}{k!} e^{-(\lambda+\theta)t} g_{m,k}^* d\hat{V}(t) \\
 &+ \sum_{s=1}^{\infty} \rho r_s \sum_{m=0}^{\infty} \sum_{k=0}^m [B^*(\theta)]^{m+s-1} \int_0^{\infty} \frac{(\lambda t)^k}{k!} e^{-(\lambda+\theta)t} g_{m,k}^* d\hat{S}(t) \\
 &= \sum_{s=1}^{\infty} (1 - \rho)r_s [B^*(\theta)]^{s-1} \hat{V}(\theta + \lambda - \lambda G(B^*(\theta))) \\
 &+ \sum_{s=1}^{\infty} \rho r_s [B^*(\theta)]^{s-1} \hat{S}(\theta + \lambda - \lambda G(B^*(\theta))) \\
 &= \frac{(1 - \rho)[1 - V^*(\theta + \lambda - \lambda G(B^*(\theta)))]R(B^*(\theta))}{B^*(\theta)[\theta + \lambda - \lambda G(B^*(\theta))]E(V)} \\
 &+ \frac{\rho[1 - S^*(\theta + \lambda - \lambda G(B^*(\theta)))]R(B^*(\theta))}{B^*(\theta)[\theta + \lambda - \lambda G(B^*(\theta))]E(S)} \\
 &= \frac{(1 - \rho)[1 - V^*(\theta + \lambda - \lambda G(B^*(\theta)))] [1 - G(B^*(\theta))]}{g[1 - B^*(\theta)][\theta + \lambda - \lambda G(B^*(\theta))]E(V)} \\
 &+ \frac{\lambda[1 - G(B^*(\theta))]}{\theta + \lambda - \lambda G(B^*(\theta))}
 \end{aligned}$$

By taking the first and second derivatives of (25) at $\theta = 0$, we have

$$(26) \quad E(W_{VL}) = \frac{E(V^2)}{2E(V)} + \frac{\lambda g E(S^2)}{2(1 - \rho)} + \frac{g^{(2)}E(S)}{2g(1 - \rho)},$$

$$\begin{aligned}
 (27) \quad E(W_{VL}^2) &= \frac{g^{(2)}E(S)E(V^2)}{2g(1 - \rho)^2E(V)} + \frac{E(V^3)}{3(1 - \rho)E(V)} + \frac{\lambda g E(S^2)E(V^2)}{2(1 - \rho)^2E(V)} \\
 &+ \frac{\lambda g E(S^3)}{3(1 - \rho)^2} + \frac{\lambda^2 g^2 [E(S^2)]^2}{2(1 - \rho)^3} + \frac{g^{(3)}[E(S)]^2}{3g(1 - \rho)^2} \\
 &+ \frac{\lambda [g^{(2)}]^2 [E(S)]^3 + (1 + \rho)g^{(2)}E(S^2)}{2g(1 - \rho)^3}
 \end{aligned}$$

Remark 2. Let W_{VF} be the waiting time with vacation time when the service discipline is FCFS. Its LST, $W_{VF}^*(\theta)$, is given by (see [1])

$$(28) \quad W_{VF}^*(\theta) = \frac{(1 - \rho)[1 - G(S^*(\theta))][1 - V^*(\theta)]}{g[\theta - \lambda + \lambda G(S^*(\theta))][1 - S^*(\theta)]E(V)}.$$

By taking the first and second derivatives of (28) at $\theta = 0$, we have

$$(29) \quad E(W_{VF}) = \frac{E(V^2)}{2E(V)} + \frac{\lambda g E(S^2)}{2(1 - \rho)} + \frac{g^{(2)} E(S)}{2g(1 - \rho)},$$

$$(30) \quad E(W_{VL}^2) = \frac{g^{(2)} E(S) E(V^2)}{2g(1 - \rho) E(V)} + \frac{E(V^3)}{3E(V)} + \frac{\lambda g E(S^2) E(V^2)}{2(1 - \rho) E(V)}$$

$$+ \frac{\lambda g E(S^3)}{3(1 - \rho)} + \frac{\lambda^2 g^2 [E(S^2)]^2}{2(1 - \rho)^2} + \frac{g^{(3)} [E(S)]^2}{3g(1 - \rho)}$$

$$+ \frac{\lambda [g^{(2)}]^2 [E(S)]^3 + (1 + \rho) g^{(3)} E(S^2)}{2g(1 - \rho)^2}.$$

As is the case with 3.1, it holds that $E(W_{VL}) = E(W_{VF})$ and $E(W_{VL}^2) = E(W_{VF}^2)/(1 - \rho)$.

4 Conclusion

We have studied the waiting time of the $M^X/G/1$ queue without and with vacation time under nonpreemptive LCFS. Comparing to FCFS, we have shown that

$$(31) \quad E(W_L) = E(W_F),$$

$$(32) \quad E(W_L^2) = E(W_F^2)/(1 - \rho),$$

$$(33) \quad E(W_{VL}) = E(W_{VF})$$

$$(34) \quad E(W_{VL}^2) = E(W_{VF}^2)/(1 - \rho).$$

For the $M/G/1$ queue, equation (31)–(34) hold (see [10]). The results obtained in this paper is the extension to the batch arrival $M^X/G/1$ queue.

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