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AN ASSET SELLING PROBLEM WITH RECALL

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Abstract This paper considers an asset selling problem with recall where the states of the economy follow a Markov chain and the cost of holding the asset depends not only on the state but on the period Especially, analytic properties of an optimal sell-timing strategy in the model are investigated. Additionally we explore the impact of uncertainty about the states of the economy and the price offer distribution.

1. Introduction

This paper considers the problem of selling an asset in the open market where the states of the economy follow a Markov chain and the cost of holding the asset depends not only on the state but on the period.

Rosenfield et al [3] investigate conditions under which an optimal policy is characterized by a reservation offer in the asset selling problem, where any previously received offer may be accepted (recall allowed) when the distribution of offers is known and the seller's prior distribution of offers undergoes Baysian updating as successive offers are received. But their paper assumes that the cost incurred for each unit of time the asset remains unsold is constant and that there is only a single state of the economy. Also Sawaki and Nishida [5] study a similar model with emphasis on a job search but without recall. This paper investigates conditions under which an optimal selling policy of the asset selling problem with recall is characterized by a reservation offer, when the states of the economy follow a Markov chain and the distribution of the offer at some period is determined by the state in the previous period to that one, and furthermore the cost incurred for each unit of time (period) the asset remains unsold depends not only on the state but on the period. It also explores whether increasing the riskness of search environments is beneficial or not to the asset holder.

2. An Asset Selling Problem with Recall

Suppose that an individual has an asset, a piece of real estate, and wants to sell it during the finite planning horizon N. At the beginning of each period he receives an offer and then has to make a decision of whether he accepts the offer and stops or waits for the next offer. When he decides to wait, he is incurred the cost of holding the asset that depends not only on the state but the number of periods remaining. On the other hand the state of the economy changes according to a Markov chain with known transitoin probabilities $\{P_{ij}\}$. We assume that the offer at any period is a random variable which depends only on the state in the previous period and that the offers' distributions are known. The objective is to maximize the expected net return. Let subscript t of all variables represent the number of periods remaining, $t=0,1,\ldots,N$. We use the following notations:

S_t: a random variable denoting the state of the economy at period $(N-t)$, which takes one of integers, $1,2,\ldots,s$, where S_t changes according to a Markov chain with the known transition probability

$$P_{ij} =\{Pr\{S_{t-1}=j|S_t=i\},\ i,j=1,\ldots,s,$$

X_t^i: the nonnegative continuous random variable denoting the offer at period $(n-t)$ with the known distribution $F_i(.)$ when $S_{t+1}=i$,

$C_t(j)$: the cost incurred at period $(N-t)$ when the asset holder does not sell his asset and $S_t=j$.

Let $X_t=(x_N,x_{N-1},\ldots,x_t)$ be the vector of observed offers up to period $(N-t)$, where x_n is the offer observed at period $(N-n)$, $n=N,N-1,\ldots,t$.
We define

$$z(X_t) = \max(x_N,x_{N-1},\ldots,x_t).$$

Let $\bar{V}_t(i,z(X_t))$ be the maximal expected net return given X_t and that $S_t=i$, $S_t=i_t,\ldots,S_N=i_N$. From the principle of optimality, we have

$$\text{(1)}\qquad \begin{aligned} \bar{V}_t(i,z(X_t)) &= \max \{z(X_t)-\textstyle\sum_{n=t+1}^{N}C_n(i_n),\sum_{j=1}^{s}P_{ij}\int_0^\infty\bar{V}_{t-1}(j,z(X_t\rho)dF_i(\rho)\} \\ \bar{V}_0(i,z(X_0)) &= \max \{z(X_0)-\textstyle\sum_{n=1}^{N}C_n(i_n),\ -\sum_{n=0}^{N}C_n(i_n)\}, t=0,1,\ldots,N \end{aligned}$$

For convenience we let

$$\text{(2)}\qquad V_t(i,z(X_t)) = \bar{V}_t(i,z(X_t)) + \textstyle\sum_{n=t+1}^{N}C_n(i_n).$$

Then the recursive equation (1) can be rewritten as follows:

$$(3)\quad \begin{aligned} V_t(\mathrm{i},Z(X_t)) &= \max\,\{Z(X_t),\ -C_t(\mathrm{i}) + \textstyle\sum_{j=1}^{s} P_{ij}\int_0^\infty V_{t-1}(\mathrm{j},Z(X_t\rho))dF_\mathrm{i}(\rho)\} \\ V_0(\mathrm{i},Z(X_0)) &= \max\,\{Z(X_0),\ -C_0(\mathrm{i})\} \end{aligned}$$

We present the following assumptions:

(A) $F_1(x) \geqq F_2(x) \geqq \ldots \geqq F_s(x)$ for all $x \geqq 0$,

(B) $\sum_{j=k}^{s} P_{ij}$ is nondecreasing in i for each k,

(C) $C_t(\mathrm{i})$ is nonincreasing in i and nonincreasing in t.

3. Optimal Reservation Policies

In this section we investigate conditions under which optimal selling policies are characterized by a reservation offer. We also present a numerical example.

For the sake of the subsequent analysis, we let

$$(4)\quad H_t(\mathrm{i},Z(X_t)) = V_t(\mathrm{i},Z(X_t)) - Z(X_t).$$

Then the equivalent equation to (3), i.e.

$$(5)\quad \begin{aligned} H_t(\mathrm{i},Z(X_t)) &= \max\,\{0,\ -C_t(\mathrm{i}) + \textstyle\sum_{j=1}^{s} P_{ij}\int_0^\infty H_{t-1}(\mathrm{j},Z(X_t\rho))dF_\mathrm{i}(\rho) \\ &\qquad + \textstyle\int_{Z(X_t)}^\infty (\rho - Z(X_t))dF_\mathrm{i}(\rho)\} \\ &= \max\,\{0,\ J_t(\mathrm{i},Z(X_t))\},\quad t \geqq 1, \\ H_0(\mathrm{i},Z(X_0)) &= \max\,\{0,\ -C_0(\mathrm{i}) - Z(X_0)\} \equiv \max\,\{0,\ J_0(\mathrm{i},Z(X_0))\} = 0 \end{aligned}$$

is obtained.

Let $G(Z(X_t),\mathrm{i}) = \int_{Z(X_t)}^\infty (\rho - Z(X_t))dF_\mathrm{i}(\rho)$.

We obtain the next lemma.

Lemma 1. Suppose that $E[X_t^\mathrm{i}] < \infty$ for all i and t. Then $G(Z(X_t),\mathrm{i})$ is nonincreasing in $Z(X_t)$ for each i and t, and is nondecreasing in i for each $Z(X_t)$ and t.

Proof: Let $Z(X_t)=z$, $z \geqq 0$. Since we assume that X_t^i is a nonnegative continuous random variable for each i and t, we have

$$\begin{aligned} G(z,\mathrm{i}) &= \textstyle\int_z^\infty (\rho - z)dF_\mathrm{i}(\rho) \\ &= \textstyle\int_z^\infty \rho\, dF_\mathrm{i}(\rho) - z\int_z^\infty dF_\mathrm{i}(\rho) \end{aligned}$$

$$= \int_0^\infty \rho dF_i(\rho) - \int_0^z \rho dF_i(\rho) - z[1 - F_i(z)]$$
$$= \int_0^\infty \rho dF_i(\rho) + \int_0^z F_i(\rho)d\rho - z,$$

where the last equality follows by partially integrating second term. By the assumption, $\int_0^\infty \rho dF_i(\rho) < \infty$. Hence

$$\frac{\partial G}{\partial z} = F_i(z) - 1 \leqq 0.$$

In the following we shall show that G is nondecreasing in i for each $Z(X_t)$ and t. If $i \leqq j$, then we have $F_i(\rho) \geqq F_j(\rho)$ for all ρ(by assumption (A)). Therefore

$$G(z,i) = \int_0^\infty \rho dF_i(\rho) + \int_0^z F_i(\rho)d\rho - z$$
$$= \int_0^\infty [1 - F_i(\rho)]d\rho + \int_0^z [F_i(\rho) - 1]d\rho$$
$$= \int_z^\infty [1 - F_i(\rho)]d\rho$$
$$\leqq \int_z^\infty [1 - F_j(\rho)]d\rho = G(z,j)$$

(Q.E.D.)

Next we derive analytic properties of functions $J_t(i,Z(X_t))$, $H_t(i,Z(X_t))$ and $V_t(i,Z(X_t))$.

Theorem 1. If $E[X_t^i] < \infty$ for all i and t, the next statements hold.
(i) $J_t(i,Z(X_t)$ and $H_t(i,Z(X_t))$ are nonincreasing in $Z(X_t)$ for each i and t.
(ii) $V_t(i,Z(X_t))$ is a convex nondecreasing function on $Z(X_t)$ for each i and t.
(iii) $J_t(i,Z(X_t))$, $H_t(i,Z(X_t))$ and $V_t(i,Z(X_t))$ are nondecreasing in i for each and $Z(X_t)$.
(iv) $J_t(i,Z(X_t))$ and $H_t(i,Z(X_t))$ are nondecreasing in t for each i and $Z(X_t)$.

Proof: The proofs of all statements are by induction on t. At first we show statement (i). Let Z_t be the range of $Z(X_t)$ for $X_t \geqq 0$. For $t = 0$, by (5) we have

$$J_0(i,Z(X_0)) = -C_0(i) - Z(X_0)$$

and statement (i) holds. Take any two elements, $Z^1(X_{t-1})$ and $Z^2(X_{t-1}) \in Z_{t-1}$ such that $Z^1(X_{t-1}) \leqq Z^2(X_{t-1})$. Assume that for all $Z^1(X_t\rho)$, $Z^2(X_t\rho) \in Z_{t-1}$ such that $Z^1(X_t\rho) \leqq Z^2(X_t\rho)$,

$$J_{t-1}(i,Z^1(X_t\rho)) \geqq J_{t-1}(i,Z^2(X_t\rho)) \quad \text{and}$$
$$H_{t-1}(i,Z^1(X_t\rho)) \geqq H_{t-1}(i,Z^2(X_t\rho)) \text{ for each i.}$$

Then by (4) we have

$$J_t(i,z^1(X_t)) = -C_t(i) + \sum_{j=1}^{s} P_{ij}\int_0^\infty H_{t-1}(j,z^1(X_t\rho))dF_i(\rho) + G(z^1(X_t),i)$$
$$\geq -C_t(i) + \sum_{j=1}^{s} P_{ij}\int_0^\infty H_{t-1}(j,z^2(X_t\rho))dF_i(\rho) + G(z^2(X_t),i)$$
$$= J_t(i,z^2(X_t)),$$

where the inequality follows from the induction hypothesis and Lemma 1. Hence, for $H_t(i,Z(X_t))$, its nonincreasing property is obvious by (5).

For statement (ii), first nondecreasing property is shown. By (3), $V_0(i,Z(X_0))$ is nondecreasing in $Z(X_0)$ for each i. Now assume that $V_{t-1}(i,Z(X_{t-1}))$ is nondecreasing in $Z(X_{t-1})$ for each i. For $Z^1(X_t) \leq Z^2(X_t)$, we have

$$Z^1(X_t\rho) = \max\{Z^1(X_t), \rho\} \leq \max\{Z^2(X_t),\rho\} = Z^2(X_t\rho).$$

From the fact above and the induction hypothesis,

$$V_t(i,Z^1(X_t)) = \max\{Z^1(X_t), -C_t(i) + \sum_{j=1}^{s} P_{ij}\int_0^\infty V_{t-1}(j,Z^1(X_t\rho)dF_i(\rho)\}$$
$$\leq \max\{Z^2(X_t), -C_t(i) + \sum_{j=1}^{s} P_{ij}\int_0^\infty V_{t-1}(j,Z^2(X_t\rho)dF_i(\rho)\}$$
$$= V_t(i,Z^2(X_t\rho)) \qquad \text{for each i.}$$

Secondly convexity is proved. For $V_0(i,Z(X_0))$ it is obvious by (3). Assume that $V_{t-1}(i,Z(X_{t-1}))$ is convex on $Z(X_{t-1})$ for each i. To simplify the expression, let $Z(X_t) = z$. Then $Z(X_t\rho) = Z(z\rho)$. For $z^1 \neq z^2$, $0 \leq \alpha \leq 1$, by induction hypothesis we have

$$\alpha V_{t-1}(i,Z(z^1\rho)) + (1-\alpha)V_{t-1}(i,Z(z^2\rho))$$
$$\geq V_{t-1}(i,\alpha Z(z^1\rho) + (1-\alpha)Z(z^2\rho)) \qquad \text{for each i.}$$

Since the maximum of positive linear function of z and convex nondecreasing function is convex on z, it is sufficient to show that

$$\int_0^\infty V_{t-1}(j,Z(z\rho))dF_i(\rho) \text{ is convex on z for each j.}$$

By the induction hypothesis, we have

$$\alpha\int_0^\infty V_{t-1}(j,Z(z^1\rho))dF_i(\rho) + (1-\alpha)\int_0^\infty V_{t-1}(j,Z(z^2\rho))dF_i(\rho)$$
$$\geq \int_0^\infty V_{t-1}(j,Z(z^1\rho) + (1-\alpha)Z(z^2\rho))dF_i(\rho)$$
$$\geq \int_0^\infty V_{t-1}(j,Z((\alpha z^1 + (1-\alpha)z^2)\rho)dF_i(\rho),$$

where the last inequality follows from the fact that $Z(z\rho) = \max\{z,\rho\}$ is convex on z and that $V_{t-1}(j,Z(X_{t-1}))$ is nondecreasing in $Z(X_{t-1})$ for each j.

Now statement (iii) is shown. By (4) we have,

$$J_0(i,Z(X_0)) = -C_0(i) - Z(X_0),$$

which is nondecreasing in i for each $Z(X_0)$ by assumption (C). Hence by (5), $H_0(i,Z(X_0))$ is also nondecreasing in i for each $Z(X_0)$. Assume that $J_{t-1}(i,Z(X_{t-1})) = J_{t-1}(i,Z(X_t\rho))$ is nondecreasing in i for each $Z(X_t\rho)$. Then by (5) $H_{t-1}(i,Z(X_t\rho))$ is nondecreasing in i for each $Z(X_t\rho)$. Also we have

$$\begin{aligned} J_t(i,Z(X_t\rho)) &= -C_t(i) + \textstyle\sum_{j=1}^{s} P_{ij}\int_0^\infty H_{t-1}(j,Z(X_t\rho))dF_i(\rho) + G(Z(X_t),i) \\ &= -C_t(i) + \textstyle\int_0^\infty \sum_{k=1}^{s}[H_{t-1}(k,Z(X_t\rho)) - H_{t-1}(k-1,Z(X_t\rho))]\cdot \\ &\qquad \textstyle\sum_{j=k}^{s} P_{ij}dF_i(\rho) + G(Z(X_t),i), \end{aligned}$$

where $H_{t-1}(0,Z(X_t\rho)) \equiv 0$ for all t and $Z(X_t\rho)$.

From assumptions (A),(B),(C) and Lemma 1, it follows that $J_t(i,Z(X_t))$ and $H_t(i,Z(X_t))$ are nondecreasing in i for each $Z(X_t)$. By (4) it is obvious that $V_t(i,Z(X_t))$ is nondecreasing in i for each $Z(X_t)$.

Lastly we present the proof of statement (iv). By for each i,

$$\begin{aligned} J_1(i,Z(X_1)) &= -C_1(i) + G(Z(X_1),i) \\ &\geqq -C_0(i) - Z(X_0) \\ &= -C_0(i) - Z(X_1\rho) = J_0(i,Z(X_1\rho)) \quad \text{for each } Z(X_1\rho). \end{aligned}$$

Therefore for each i we have

$$H_1(i,Z(X_1)) \geqq H_0(i,Z(X_1\rho)) \qquad \text{for each } Z(X_1\rho).$$

Assume that for each i,

$$\begin{aligned} &J_{t-1}(i,Z(X_{t-1})) \geqq J_{t-2}(i,Z(X_{t-1}\rho)) \quad \text{and} \\ &H_{t-1}(i,Z(X_{t-1})) \geqq H_{t-2}(i,Z(X_{t-1}\rho)) \quad \text{for each } Z(X_{t-1}\rho). \end{aligned}$$

Then we have

$$\begin{aligned} J_t(i,Z(X_t)) &= -C_t(i) + \textstyle\sum_{j=1}^{s} P_{ij}\int_0^\infty H_{t-1}(j,Z(X_t\rho))dF_i(\rho) + G(Z(X_t),i) \\ &\geqq -C_{t-1}(i) + \textstyle\sum_{j=1}^{s} P_{ij}\int_0^\infty H_{t-2}(j,Z(X_{t-1}\rho))dF_i(\rho) \\ &= J_{t-1}(i,Z(X_t\rho)) \quad \text{for each } Z(X_t\rho), \end{aligned}$$

where the inequality follows from assumption (C), statement (i) of Theorem 1 and Lemma 1, since $z(X_t) \leqq z(X_t\rho) = \max\{z(X_t),\rho\}$ for all $z(X_t)$ and ρ. This completes the proof. (Q.E.D.)

Now we show the existence of an optimal policy characterized by a single value. For this purpose we define $R_t(i)$ (we call this *reservation value*) as follows:

For each i and t, define the set

$$A_t^i \equiv \{z(X_t) : J_t(i,z(X_t)) > 0\}, \text{ and we let}$$

$$(6) \qquad R_t(i) = \begin{cases} \sup_{z(X_t)} A_t^i, & A_t^i \neq \phi \\ 0 \quad , & A_t^i = \phi. \end{cases}$$

We are ready to show one of the main results.

Theorem 2. Suppose that $S_t = i$ and X_t are given, $t=0,1,\ldots,N$. Then there exists an optimal reservation policy as follows:

Stop and sell the asset at that period, if $z(X_t) > R_t(i)$

wait one more period for the next offer, otherwise.

Furthermore $R_t(i)$ has the following properties:

(I) for each $i=1,\ldots,s$, $R_0(i) \leqq R_1(i) \leqq \ldots \leqq R_N(i)$

(II) for $t=0,1,\ldots,N$, $R_t(1) \leqq R_t(2) \leqq \ldots \leqq R_t(s)$.

Proof: The existence of an optimal policy characterized by reservation value $R_t(i)$ follows from (i) of Theorem 1 and the definition of $R_t(i)$. The properties (I) and (II) of $R_t(i)$ immediately follow from (iii) and (iv) of Theorem 1, respectively. (Q.E.D.)

Remarks: Theorem 2 states that there exists an optimal reservation policy characterized by reservation values $R_t(i)$'s. It also asserts that the better state of the economy at period (N-t) is the higher the reservation value is and that the larger number of periods remains, the higher the reservation value is. These results are consistent with our economic intuition.

A Numerical Example

Here we present a numerical example in which Theorem 2 holds. Set N = 2 and s = 2, and let

$P_{ij}=1/2$, for $i,j=1,2$, $c_0(1) = c_0(2) = 50$

$c_1(1) = 4/5$, $c_1(2) = 2/5$,

$c_2(1) = 3/5$, $c_2(2) = 1/5$.

Also assume that offers' distributions are

$x_t^1 \sim U(4,14)$,

$x_t^2 \sim U(4,24)$, $= 0,1$,

where U(a,b) represents a uniform distribution with parameters a and b(a < b).

These data satisfy assumptions (A),(B) and (C). For convenience we denote $z(X_0)$, $z(X_1)$ and $z(X_2)$ by z_0, z_1, z_2, respectively.

For $t = 0$, we have

$J_0(1,z_0) = J_0(2,z_0) = -50 - z_0 < 0$, and from the definition of A_0^i,

$A_0^i = \phi$, $i = 1,2$.

Therefore by (6)

$R_0(i) = 0$, $i = 1,2$.

For $t = 1$, BY (5) we have

$$J_1(1,z_1) = \begin{cases} 9 - 7z_1/5 + z_1^2/20, & 4 \leqq z_1 \leqq 14, \\ -4/5 & , 14 < z_1 \end{cases}$$

and

$$J_1(2,z_1) = \begin{cases} 14 - 6z_1/5 + z_1^2/40, & 4 \leqq z_1 \leqq 24, \\ -2/5 & , 24 < z_1, \end{cases}$$ and from (6)

$R_1(1) = 10$, $R_1(2) = 20$.

For $t = 2$, we obtain

$$J_2(1,z_2) = \begin{cases} 3449/300 - 22z_2/25 - 3z_2^2/100 + z_2^3/400, & 4 \leqq z_2 < 10, \\ 3439/300 - 22z_2/25 + 3z_2^2/200 + z_2^3/1200, & 10 \leqq z_2 < 14, \\ -3/5 & , 14 < z_2, \end{cases}$$

and

$$J_2(2,z_2) = \begin{cases} 929/60 - 47z_2/50 - 3z_2^2/200 + z_2^3/800, & 4 \leqq z_2 < 10, \\ 232/15 - 27z_2/25 + 3z_2^2/400 + z_2^3/2400, & 10 \leqq z_2 < 20, \\ 71/5 - 6z_2/5 + z_2^2/40 & , 20 \leqq z_2 \leqq 24, \end{cases}$$

$-1/5 \qquad\qquad\qquad\qquad , \ 24 < z_2.$

By (6), we have $R_2(1) = 14$, $R_2(2) = 24 - 2\sqrt{2}$.

Hence, $\quad R_0(1) = 0 < R_1(1) = 10 < R_2(1) = 14$

$\qquad\qquad R_0(2) = 0 < R_1(2) = 20 < R_2(2) = 24 - 2\sqrt{2}.$

These results satisfy the properties (I) and (II) of Theorem 2. See Figures 1,2 and 3.

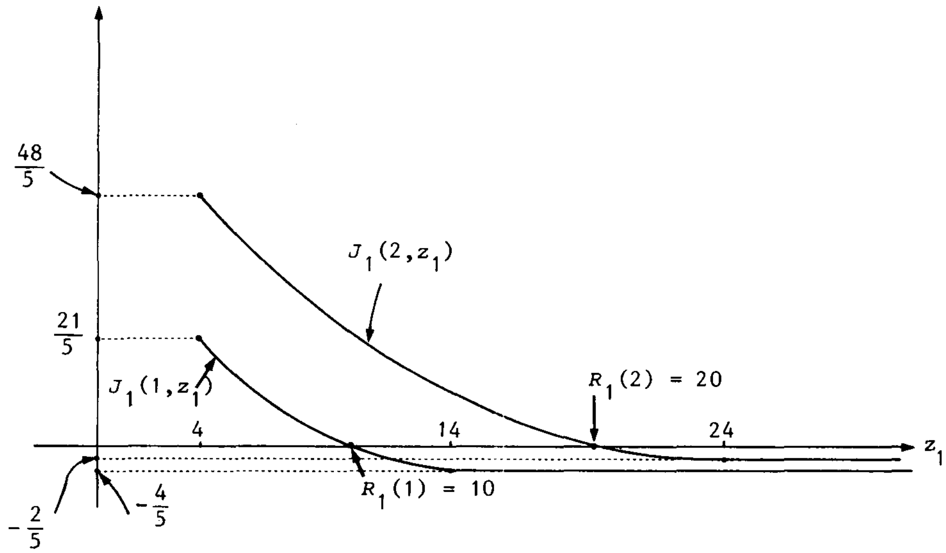

Fig. 1

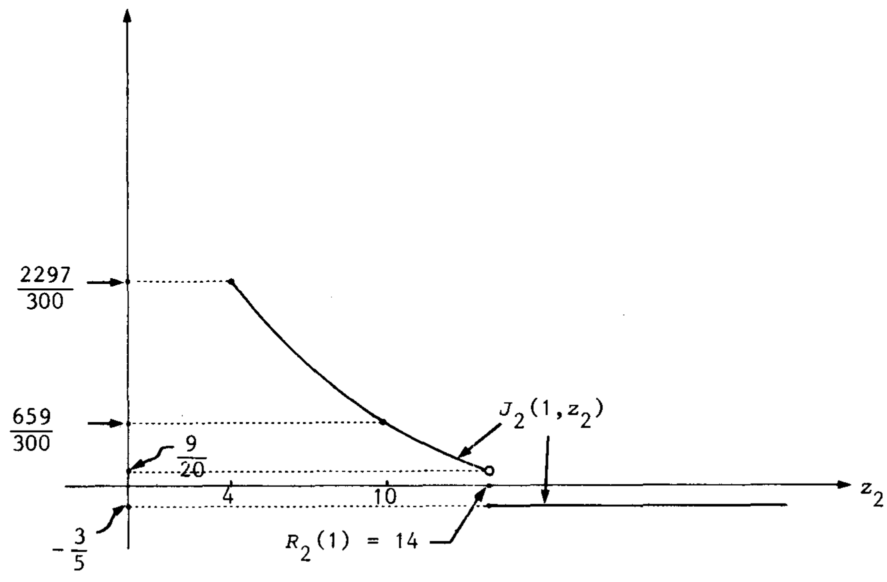

Fig. 2

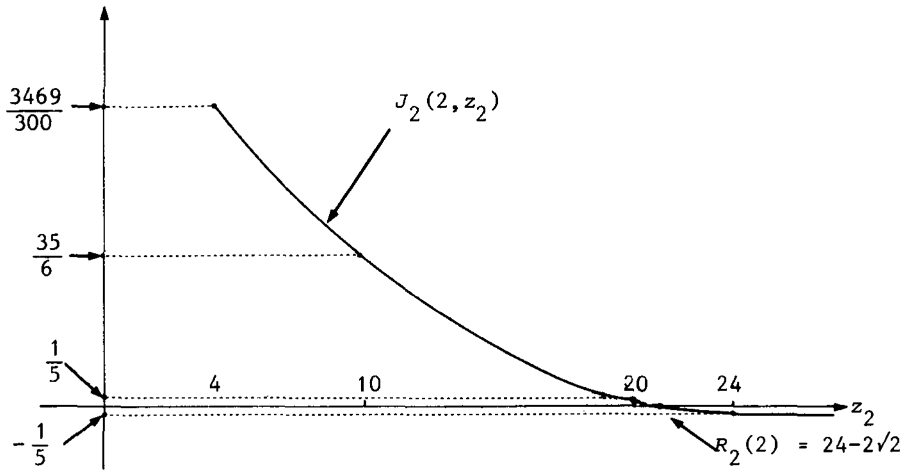

Fig. 3

4. The Impact of Uncertainty

In this section we investigate the impact on optimal value $V_t(i,Z(X_t))$ when uncertainty about porobability distributions P_{ij} and $F_i(.)$ increases. At first, to investigate the effect of uncertainty about transition probabilities, we write

$$P^1 > P^2 \quad \text{whenever} \quad \sum_{j=k}^{s} P^1_{ij} \geq \sum_{j=k}^{s} P^2_{ij} \quad \text{for all k and define}$$

$$V_t(i,Z(X_t),P^1) = \max\{Z(X_t),\ -C_t(i) + \sum_{j=1}^{s} P_{ij}\int_0^\infty V_{t-1}(j,Z(X_t\rho),P)dF_i(\rho)\}.$$

Theorem 3. If $P^1 > P^2$, then for each i, $Z(X_t)$ and t, we have

$$V_t(i,Z(X_t),P^1) \geq V_t(i,Z(X_t),P^2).$$

Proof: The proof is again by induction on t. For $t = 0$, the assertion holds with equality. Assume that for $t-1$ that $V_{t-1}(.,.,P^1) \geq V_{t-1}(.,.,P^2)$ for $P^1 > P^2$. Then by (3) we have

$$V_t(i,Z(X_t),P^1) = \max\{Z(X_t),\ -C_t(i) + \sum_{j=1}^{s} P^1_{ij}\int_0^\infty V_{t-1}(j,Z(X_t\rho),P^1)dF_i(\rho)\}$$

$$\geq \max\{Z(X_t),\ -C_t(i) + \sum_{j=1}^{s} P^1_{ij}\int_0^\infty V_{t-1}(j,Z(X_t\rho),P^2)dF_i(\rho)\}$$

$$= \max\{z(X_t),\ -c_t(i) + \int_0^\infty \sum_{k=1}^s [V_{t-1}(k,z(X_t\rho),P^2) - V_{t-1}(k-1,z(X_t\rho),P^2)]dF_i(\rho)\cdot\sum_{j=k}^s P_{ij}^1\}$$

$$\geqq \max\{z(X_t),\ -c_t(i) + \int_0^\infty \sum_{k=1}^s [V_{t-1}(k,z(X_t\rho),P^2) - V_{t-1}(k-1,z(X_t\rho),P^2)]dF_i(\rho)\cdot\sum_{j=k}^s P_{ij}^2\}$$

$$= V_t(i,z(X_t),P^2)$$

Corollary. If P^1 and P^2 is the lower and upper bound of P, respectively, that is, $P^1 < P < P^2$, then we have

$V_t(i,z(X_t),P^1) \leqq V_t(i,z(X_t),P) \leqq V_t(i,z(X_t),P^2)$ for each i, t and $z(X_t)$. (Proof is omitted.)

The result asserts, that the higher probability that the state of the economy goes to any subset of the states set $\{k,k+1,\ldots,s\}$ brings the asset holder the higher expected return.

Next we explore the effect of increasing uncertainty about offer distribution $F_i(.)$. To show explicitly the dependence of F_i on V_t, we write

$$V_t(i,z(X_t),F) = \max\{z(X_t),\ -c_t(i) + \sum_{j=1}^s P_{ij}\int_0^\infty V_{t-1}(j,z(X_t\rho),F)dF_i(\rho)\}.$$

Theorem 4. For two distinct offer distributions F^1, F^2 such that $E[X^1] = E[X^2]$ and

$\int_0^z F_i^1(\rho)d\rho \leqq \int_0^z F_i^2(\rho)d\rho$ for all z, we have

$V_t(i,z(X_t),F^1) \leqq V_t(i,z(X_t),F^2)$ for each i, $z(X_t)$ and t.

Proof: It is known that, if $E[X^1] = E[X^2]$ and $\int_0^z F_i^1(\rho)d\rho \leqq \int_0^z F_i^2(\rho)d\rho$ for all z, then for any convex function $h(x)$, we have

$E[h(x^1)] \leqq E[h(x^2)]$. (See Whitmore and Findlay [6].)

Hence by (3), it is sufficient to show that $V_{t-1}(i,z(X_t\rho),F)$ is convex on ρ for each i and $z(X_t)$. From (ii) of Theorem 1, it follows that $V_{t-1}(i,z(X_t\rho),F)$ is convex nondecreasing function of $z(X_t\rho)$ for each i. Therefore letting $z(X_t) = z$, for ρ^1, ρ^2 such that $\rho^1 \neq \rho^2$ and $\alpha\varepsilon[0,1]$ we have

$$\alpha V_{t-1}(i,Z(z\rho^1),F) + (1-\alpha)V_{t-1}(i,Z(z\rho^2),F)$$
$$\geqq V_{t-1}(i,\alpha Z(z\rho^1) + (1-\alpha)Z(z\rho^2),F)$$
$$\geqq V_{t-1}(i,Z(z,\alpha\rho^1 + (1-\alpha)\rho^2),F),$$

where the last inequality follows from the fact that $Z(z\rho) = \max\{z,\rho\}$ is convex on ρ and that $V_{t-1}(.,Z(z\rho),.)$ is nondecreasing in $Z(z\rho)$. Hence we have

$$\int_0^\infty V_{t-1}(j,Z(X_t\rho),F^1)dF_i^1(\rho) \leqq \int_0^\infty V_{t-1}(j,Z(X_t\rho),F^2)dF_i^2(\rho),$$

and the result follows. (Q.E.D.)

Remarks. Note that the conditions $E[X^1] = E[X^2]$ and $\int_0^z F^1(\rho)d\rho \leqq \int_0^z F^2(\rho)d\rho$ for all z, together imply that $\mathrm{Var}(X^1) \leqq \mathrm{Var}(X^2)$. If the variance can be interpreted as the measure of risk, Theorem 4 states that increasing the risk of the offer distribution is beneficial to the asset holder.

5. Conclusion

In this paper we considered an asset selling problem *with recall* where the states of the economy follow a Markov chain and the cost of holding the asset depends not only on the state but on the period. Under some conditions, we showed that (1) there exists an optimal reservation policy characterized by critical numbers (*reservation values*), (2) they depend on the state of the economy at each period and on the number of periods remaining, (3) the better state of the economy at each period is or the larger number of periods remain, the higher values they are, (4) maximal return increases as the economy improves, (5) increasing the risk of offer distribution is beneficial to the asset holder. Additionally we present a numerical example of the results (1) and (2).

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