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FIRST PASSAGE TIMES OF PH/PH/1/K AND PH/PH/1 QUEUES

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Abstract First passage times for PH/PH/1/K and PH/PH/1 queues have been studied. The Laplace transform for the busy period probability density function (p.d.f.), inter-overflow time p.d.f. and transition probability for system states between two arbitrary time points are represented by some recurrence formulas. Dimensions of the inverse matrices included in each recurrence formula do not exceed the number either of arrival phases or of service phases. Hence, if either the number of arrival phases or number of service phases is small, the Laplace transforms of the above-mentioned p.d.f.s and transition probability can easily be computed. The moment characteristics of busy period and inter-overflow time are presented using some numerical examples.

1. Introduction

A PH/PH/1/K queue with finite capacity K and a PH/PH/1 queue with infinite capacity where the interarrival and service time distributions are both of phase type are very important as approximations for GI/G/1/K and GI/G/1 queues. This paper examines the first passage times of PH/PH/1/K and PH/PH/1 queues. A Laplace transform is provided for the first passage time probability density function (p.d.f.) from customers n to n+1 (or n-1) in the system. Inter-overflow times and busy period are represented by such first passage times. The Laplace transform for the stansition probability of system states between two arbitrary time points is also represented using the Laplace transform of these first passage time p.d.f.s. The first passage time p.d.f. for the PH/PH/1 queue is given as a special case of the PH/PH/1/K queue. All formulae are represented as simple recurrences. In particular, the fact that the dimensions of the inverse matrices included in each recurrence do not exceed either the number of arrival phases or service phases is worthy of attention.

Results for inter-overflow times are very important for telephone network dimensioning, because telephone networks employ alternate routing. These results make it possible to extend discussions previously presented for the M/PH/1/K [4] and PH/M/1/K [1,5] queues. The effect of interarrival time and service time variations for inter-overflow time variations will be discussed using numerical examples.

Results for busy periods are important in analyzing the priority queueing models [2]. For example, models in which the arrival process of non-priority customers is a Poisson process can easily be analyzed. The results presented here are an extension of the results about the M/G/1/K queue [8,9]. The effect of interarrival time and service time variations on the busy period is discussed here using numerical examples.

The results of transition probability are important in analyzing models with two independent inputs. For example, a $PH^{(X)}$+PH/PH/1/K queue with mixed PH-renewal group arrivals and PH-renewal with simple arrivals can be analyzed by applying our results to the theory of piecewise Markov Processes [3].

2. Model and Notations

Consider a finite queueing system, PH/PH/1/K, and an infinite queueing system, PH/PH/1, in which the interarrival time p.d.f., $f(t)$, and the service time p.d.f., $g(t)$, are p.d.f.s of phase types. Let $A = [1,2,\dots,m_1]$ and $B = [1,2,\dots,m_2]$ denote the phase state spaces of $f(t)$ and $g(t)$, respectively. Consider the Cartesian product set $A \times B = [(1,1),(1,2),\dots,(1,m_2),(2,1),\dots,(2,m_2),\dots,(m_1,1),\dots,(m_1,m_2)]$ and assign a number, $(i-1)m_2+j$, for each (i,j), i.e., $A \times B = [1,2,\dots,m_1m_2]$.

Under the above notations, let $Z(t)$, $W(t)$ and $X(t)$ denote respectively the arrival and service phase state, the service phase state and the arrival phase state at time t. Consider the bivariate stochastic process,

$$(2.1) \qquad [Y(t),\ Z(t);\ t \geq 0],$$

where $Y(t)$ denotes the number of customers in the system at time t and $Z(t)$ denotes the phase state at time t. When $Y(t) > 0$, $Z(t)$ takes a value in the arrival and service phase state space $A \times B$. When $Y(t) = 0$, $Z(t)$ takes a value in arrival phase state space A.

Let t_ℓ $(\ell = 1,2,\dots)$ denote the arrival epoch. The phase state at the arrival epoch is free from the arrival phase states and $W(t_\ell) = j$ can be written for $Z(t_\ell - dt) = (i-1)m_2 + j$. The process

(2.2) $[Y(t_\ell),\ W(t_\ell);\ 0 \leqq t_1 < t_2 < \dots]$

is Markovian, where $Y(t_\ell)$ denotes the number of customers in the system immediately before customer arrival epoch t_ℓ, and $W(t_\ell)$ denotes the service phase state for the same epoch. When $Y(t_\ell) > 0$, $W(t_\ell)$ takes a value in service phase state space B. When $Y(t_\ell) = 0$, t_ℓ is the renewal point and $W(t_\ell)$ is free from the phase states.

Let $s_\ell(\ell = 1,2,\dots)$ denote the service completion epoch. The phase state at the service completion epoch is free from the service phase states and $X(s_\ell) = i$ can be written for $Z(s_\ell - dt) = (i-1)m_2 + j$. The process

(2.3) $[Y(s_\ell),\ X(s_\ell);\ 0 < s_1 < s_2 < \dots]$

is Markovian, where $Y(s_\ell)$ denotes the number of customers in the system immediately before service completion epoch s_ℓ, and $X(s_\ell)$ denotes the arrival phase state for the same epoch. When $Y(s_\ell) > 0$, $X(s_\ell)$ takes a value in arrival phase state space A.

3. First Passage Time P.D.F.s and Their Application

First passage times, such as the inter-overflow time, busy period and so on, are discussed in this section. The Laplace transforms of these life-time p.d.f.s are represented by recurrence formulas.

3.1 First Passage Time P.D.F.s for PH/PH/1/K Queue

Let us consider the first passage time from the arbitrary time point τ_0. Let x_n^+ denote the first passage time from $Y(\tau_0) = n$ to $Y(\tau_0 + t) = n+1$ (ω when $n = K$), where ω denotes the occurrence of an overflow. That is,

$$x_n^+ = \inf_t [Y(\tau_0 + t) = n+1 \ /\ Y(\tau_0) = n],\ n = 0,\ 1,\ \dots,\ K-1,$$

and

$$x_K^+ = \inf_t [Y(\tau_0 + t) = \omega \ /\ Y(\tau_0) = K].$$

This first passage is realized by the customer arrival, and the phase states at this first passage time point are free from the arrival phase states. The d.f. of this first passage time is defined as

$$g_{i.}^+(t;\ 0)dt :=P[t < x_0^+ < t+dt \ /\ Z(\tau_0) = i]\ ,$$

(3.1) $i = 1,\ 2,\ \dots,\ m_1.$

$$g^{+}_{ij}(t;\ n)dt:\ = P[t < x^{+}_{n} < t+dt,\ W(\tau_0 + x^{+}_{n}) = j\ /\ Z(\tau_0) = i]\ ,$$
$$n = 1,\ 2,\ \dots,\ K,\ i = 1,\ 2,\ \dots,\ m_1m_2,\ j = 1,\ 2,\ \dots,\ m_2.$$

Let x^{-}_{n} denote the first passage time from $Y(\tau_0) = n$ to $Y(\tau_0 + t) = n-1$ without overflow occurrence. That is,

$$x^{-}_{n} = \inf_{t}[Y(\tau_0 + t) = n-1,\ \text{and no overflow occurs in } (\tau_0,\ \tau_0+t)\ /\ Y(\tau_0) = n]\ .$$

Then, the d.f. of this first passage time can be defined as

$$(3.2)\quad g^{-}_{ij}(t;\ n)dt\ :\ = P[t < x^{-}_{n} < t+dt,\ X(\tau_0 + x^{-}_{n}) = j\ /\ Z(\tau_0) = i]\ ,$$
$$n = 1,\ 2,\ \dots,\ K,\ i = 1,\ 2,\ \dots,\ m_1m_2,\ j = 1,\ 2,\ \dots,\ m_1\ .$$

Let $x^{\pm}_{n}$ denote the first passage time from $Y(\tau_0) = \mathrm{n}$ to $Y(\tau_0+t) = \mathrm{n}-1$. That is,

$$x^{\pm}_{n} = \inf_{t}[Y(\tau_0+t) = n-1\ /\ Y(\tau_0) = n]\ .$$

Then, the d.f. of this first passage time can be defined as

$$(3.3)\quad g^{\pm}_{ij}(t;\ n)dt:\ = P[t < X^{\pm}_{n} < t+dt,\ X(\tau_0+X^{\pm}_{n}) = j\ /\ Z(\tau_0) = i]\ ,$$
$$n = 1,\ 2,\ \dots,\ K,\ i = 1,\ 2,\ \dots,\ m_1m_2,\ j = 1,\ 2,\ \dots,\ m_1\ .$$

The main object of this section is to represent (3.1) through (3.3) using transition probabilities $f^{+}_{ij}(t;\ n)$, $f^{-}_{ij}(t;\ n)$, $f^{\pm}_{ij}(t;\ n)$, $h^{+}_{\cdot j}(0)$, $h^{+}_{ij}(n)$ and $h^{-}_{ij}(n)$ defined by (3.4) through (3.9). The Laplace transforms of $f^{+}_{ij}(t;\ n)$, $f^{-}_{ij}(t;\ n)$ and $f^{\pm}_{ij}(t;\ K)$ will be concretely given by (3.32) through (3.34). $h^{+}_{\cdot}j$, $h^{+}_{ij}(n)$ and $h^{-}_{ij}(n)$ will be given by (3.35) through (3.37).

For the first passage time from $Y(\tau_0) = n$ to $Y(\tau_0+t) = n+1$ (or ω) without service completitions in $[\tau_0,\ \tau_0+t]$, that is for

$$y^{+}_{n} = \inf_{t}[Y(\tau_0+t) = n+1\ (\text{or } \omega),\ Y(s) = n \text{ for all } s\varepsilon[\tau_0,\ \tau_0+t)\ /\ Y(\tau_0) = n]\ ,$$

let

$$(3.4)\quad \begin{aligned} &f^{+}_{i.}(t;\ 0)dt:\ = P[t < y^{+}_{0} < t+dt\ /\ Z(\tau_0) = i]\ , \\ &\qquad i = 1,\ 2,\ \dots,\ m_1, \\ &f^{+}_{ij}(t;\ n)dt:\ = P[t < y^{+}_{n} < t+dt,\ W(\tau_0 + y^{+}_{n}) = j\ /\ Z(\tau_0) = i]\ , \\ &\qquad n = 1,\ 2,\ \dots,\ K,\ i = 1,\ 2,\ \dots,\ m_1m_2,\ j = 1,\ 2,\ \dots,\ m_2. \end{aligned}$$

For the first passage time from $Y(\tau_0) = n$ to $Y(\tau_0 + t) = n-1$ without customer arrivals in $[\tau_0, \tau_0+t]$, that is for

$$y_n^- = \inf_t [Y(\tau_0+t) = n-1,\ Y(s) = n \text{ for all } s\varepsilon[\tau_0, \tau_0+t) \text{ and no customer arrives in } [\tau_0, \tau_0+t)\ /\ Y(\tau_0) = n] ,$$

let

$$(3.5) \qquad f_{ij}^-(t;\ n)dt: = P[t < y_n^- < t+dt,\ X(\tau_0 + y_n^-) = j\ /\ Z(\tau_0) = i] ,$$
$$n = 1, 2, \ldots, K,\ i = 1, 2, \ldots, m_1m_2,\ j = 1, 2, \ldots, m_1 .$$

Further, for the first passage time from $Y(\tau_0) = K$ to $Y(\tau_0 + t) = K-1$, that is for

$$y_K^{\pm} = \inf_t [Y(\tau_0+t) = K-1,\ Y(s) = K \text{ for all } s\varepsilon[\tau_0, \tau_0+t)\ /\ Y(\tau_0) = K] ,$$

let

$$(3.6) \qquad f_{ij}^{\pm}(t;\ K)dt: = P[t < y_K^{\pm} < t+dt,\ X(\tau_0 + y_K^{\pm}) = j\ /\ Z(\tau_0) = i] ,$$
$$i = 1, 2, \ldots, m_1m_2,\ j = 1, 2, \ldots, m_1 .$$

The state transition probabilities between pre- and post-arrival epochs are

$$(3.7) \qquad h_{\cdot j}^+(0): = P[Y(t_\ell+dt) = 1,\ Z(t_\ell+dt) = j\ /\ Y(t_\ell-dt) = 0] ,$$
$$j = 1, 2, \ldots, m_1m_2 ,$$

and

$$(3.8) \qquad h_{ij}^+(n): = P[Y(t_\ell+dt) = n+1 \text{ (when } n < K),\ \ \omega \text{ (when } n = K) ,$$
$$Z(t_\ell+dt) = j\ /\ Y(t_\ell-dt) = n,\ W(t_\ell) = i] ,\ n = 1, 2, \ldots, K,$$
$$i = 1, 2, \ldots, m_2,\ j = 1, 2, \ldots, m_1m_2 .$$

The state transition probabilities between pre- and post-service completion epochs are

$$h_{ij}^-(1): = P[Y(s_\ell+dt) = 0,\ X(s_\ell+dt) = j$$
$$/\ Y(s_\ell-dt) = 1,\ X(s_\ell) = i] ,\ i,j = 1, 2, \ldots, m_1,$$

and

$$h_{ij}^-(n): = P[Y(s_\ell+dt) = n-1,\ Z(s_\ell+dt) = j$$
$$(3.9) \qquad /\ Y(s_\ell-dt) = n,\ X(s_\ell) = i] ,\ n = 2, 3, \ldots, K,$$
$$i = 1, 2, \ldots, m_1,\ j = 1, 2, \ldots, m_1m_2 .$$

Since $f^+_{ij}(t;\ n)$ $(n = 1, 2, \ldots, K)$, $f^-_{ij}(t;\ n)$ $(n = 1, 2, \ldots, K)$, $h^+_{ij}(n)$ $(n = 1, 2, \ldots, K)$ and $h^-_{ij}(n)$ $(n = 2, 3, \ldots, K)$ do not depend on n, then $f^+_{ij}(t)$, $f^-_{ij}(t)$, h^+_{ij} and h^-_{ij} are respectively written instead of these.

Now, let us define the following matrices for each definition from (3.1) through (3.9).

$$G^+_0(t):=(g^+_{i.}(t;\ 0))_{i\,=\,1,\,2,\,\ldots,\,m_1}\quad (m_1\times 1\ \text{matrix}),$$

(3.10) $$G^+_n(t):=(g^+_{ij}(t;\ n))_{i\,=\,1,\,2,\,\ldots,\,m_1m_2,\ j\,=\,1,\,2,\,\ldots,\,m_2}$$
$$(m_1m_2\times m_2\ \text{matrix}),\ n=1,\ 2,\ \ldots,\ K.$$

(3.11) $$G^-_n(t):=(g^-_{ij}(t;\ n))_{i\,=\,1,\,2,\,\ldots,\,m_1m_2,\ j\,=\,1,\,2,\,\ldots,\,m_1}$$
$$(m_1m_2\times m_1\ \text{matrix}).$$

(3.12) $$G^\pm_n(t):=(g^\pm_{ij}(t;\ n))_{i\,=\,1,\,2,\,\ldots,\,m_1m_2,\ j\,=\,1,\,2,\,\ldots,\,m_1}$$
$$(m_1m_2\times m_1\ \text{matrix}).$$

$$F^+_0(t):=(f^+_{i.}(t;\ 0))_{i\,=\,1,\,2,\,\ldots,\,m_1}\quad (m_1\times 1\ \text{matrix}),$$

(3.13) $$F^+(t):=(f^+_{ij}(t))_{i\,=\,1,\,2,\,\ldots,\,m_1m_2,\ j\,=\,1,\,2,\,\ldots,\,m_2}$$
$$(m_1m_2\times m_2\ \text{matrix}).$$

(3.14) $$F^-(t):=(f^-_{ij}(t))_{i\,=\,1,\,2,\,\ldots,\,m_1m_2,\ j\,=\,1,\,2,\,\ldots,\,m_1}$$
$$(m_1m_2\times m_1\ \text{matrix}).$$

(3.15) $$F^\pm(t):=(f^\pm_{ij}(t;\ K))_{i\,=\,1,\,2,\,\ldots,\,m_1m_2,\ j\,=\,1,\,2,\,\ldots,\,m_1}$$
$$(m_1m_2\times m_1\ \text{matrix}).$$

(3.16) $$H^+_0:=(h^+_{.j}(0))_{j\,=\,1,\,2,\,\ldots,\,m_1m_2}\quad (1\times m_1m_2).$$

(3.17) $$H^+:=(h^+_{ij})_{i\,=\,1,\,2,\,\ldots,\,m_2,\ j\,=\,1,\,2,\,\ldots,\,m_1m_2},$$
$$(m_2\times m_1m_2\ \text{matrix}).$$

$$H_1^- := (h_{ij}^-(1))_{i=1,2,\ldots,m_1,\ j=1,2,\ldots,m_1} \quad (m_1 \times m_1 \text{ matrix}).$$

(3.18)

$$H^- := (h_{ij}^-)_{i=1,2,\ldots,m_1,\ j=1,2,\ldots,m_1m_2} \quad (m_1 \times m_1m_2 \text{ matrix}).$$

The foregoing makes it clear that H_1^- is equivalent to an $m_1 \times m_1$ identity matrix. Let us next consider $G_n^+(t)$. Suppose that ℓ upward transitions from $Y(t_m) = n-1$ to $Y(t_m+dt) = n$ ($m = 1, 2, \ldots, \ell$) occur in $[\tau_0, \tau_0+t]$, and divide $[\tau_0, \tau_0+t]$ into its subintervals $[\tau_0, t_1]$, $(t_1, t_2]$, $\ldots$, $(t_{\ell-1}, t_\ell]$ and $(t_\ell, \tau_0+t]$. The transition probabilities of the stochastic process $[Y(t), Z(t);\ t \geqq 0]$ between t_m+dt and t_{m+1} (τ_0 and t_1) are given by $F^-(t)H_1^- * G_0^+(t)H_0^+$ for $n = 1$ and $F^-(t)H^- * G_{n-1}^+(t)H^+$ for $n = 2, 3, \ldots, K$. Since the transition probabilities between $t_\ell+dt$ and τ_0+t are given by $F^+(t)$, the following recurrence formula (3.19) can be derived from a similar discussion by Machihara [5, 6] or Keilson et al. [11].

$$G_0^+(t) = F_0^+(t) ,$$

$$(3.19) \qquad G_1^+(t) = \sum_{\ell=0}^{\infty} X_0^{+\ell *}(t) * F^+(t) ,$$

$$G_n^+(t) = \sum_{\ell=0}^{\infty} X_{n-1}^{+\ell *}(t) * F^+(t),\ n = 2, 3, \ldots, K$$

where

$$X_0^+(t) = F^-(t)H_1^- * G_0^+(t)H_0^+$$

and

$$X_{n-1}^+(t) = F^-(t)H^- * G_{n-1}^+(t)H^+ .$$

Let us consider $G_n^-(t)$ and $G_n^{\pm}(t)$. Suppose that ℓ downward transitions from $Y(s_m) = n+1$ to $Y(s_m+dt) = n$ occur in $[\tau_0, \tau_0+t]$ and divide this into its subintervals $[\tau_0, s_1]$, $(s_1, s_2]$, $\ldots$, $(s_{\ell-1}, s]$ and $(s_\ell, \tau_0+t]$. By considering the transition probabilities of the stochastic process $[Y(t), Z(t);\ t \geqq 0]$ between s_m+dt and s_{m+1} (τ_0 and s_1), it follows that

$$G_K^-(t) = F^-(t) ,$$

(3.20)

$$G_n^-(t) = \sum_{\ell=0}^{\infty} X_{n+1}^{-\ *}(t) * F^-(t) ,$$

$$n = K-1, K-2, \ldots, 1 ,$$

where

$$X^{-}_{n+1}(t) = F^{+}(t)H^{+} * G^{-}_{n+1}(t)H^{-} ,$$

and

$$G^{\pm}_{K}(t) = F^{\pm}_{K}(t) = \sum_{m=0}^{\infty} (F^{+}(t)H^{+})^{m*} * F^{-}(t) ,$$

(3.21) $$G^{\pm}_{n}(t) = \sum_{\ell=0}^{\infty} X^{\pm\ell*}_{n+1}(t) * F^{-}(t) ,$$

$$n = K-1,\ K-2,\ \ldots,\ 1 ,$$

where

$$X^{\pm}_{n+1}(t) = F^{+}(t)H^{+} * G^{\pm}_{n+1}(t)H^{-} .$$

Let us consider the Laplace transform matrices of (3.19) through (3.21), where $\tilde{X}(s)$ denotes the Laplace transform of matrix $X(t)$. Then, equations (3.19), (3.20) and (3.21) give

$$\tilde{G}^{+}_{0}(s) = \tilde{F}^{+}_{0}(s) ,$$

(3.22) $$\tilde{G}^{+}_{1}(s) = \sum_{\ell=0}^{\infty} (\tilde{X}^{+}_{0}(s))^{\ell}\ \tilde{F}^{+}(s) ,$$

$$\tilde{G}^{+}_{n}(s) = \sum_{\ell=0}^{\infty} (\tilde{X}^{+}_{n-1}(s))^{\ell}\ \tilde{F}^{+}(s) ,$$

$$n = 2,\ 3,\ \ldots,\ K .$$

Here, $\tilde{X}^{+}_{i}(s)$ is the Laplace transform of $\tilde{X}^{+}_{i}(t)$ defined in (3.19).

$$\tilde{G}^{-}_{K}(s) = \tilde{F}^{-}(s) ,$$

(3.23) $$\tilde{G}^{-}_{n}(s) = \sum_{\ell=0}^{\infty} (\tilde{X}^{-}_{n+1}(s))^{\ell}\ \tilde{F}^{-}(s) ,$$

$$n = K-1,\ K-2,\ \ldots,\ 1 .$$

Here, $\tilde{X}^{-}_{i}(s)$ is the Laplace transform of $\tilde{X}^{-}_{i}(t)$ defined in (3.20).

$$\tilde{G}^{\pm}_{K}(s) = \tilde{F}^{\pm}_{K}(s) = \sum_{m=0}^{\infty} (\tilde{F}^{+}(s)H^{+})^{m}\ \tilde{F}^{-}(s) ,$$

(3.24) $$\tilde{G}^{\pm}_{n}(s) = \sum_{\ell=0}^{\infty} (\tilde{X}^{\pm}_{n+1}(s))^{\ell}\ \tilde{F}^{-}(s) ,$$

$$n = K-1,\ K-2,\ \ldots,\ 1 .$$

Here $\tilde{X}^{\pm}_{i}(s)$ is the Laplace transform of $\tilde{X}^{\pm}_{i}(t)$ defined in (3.21).

Matrices $\tilde{G}^{+}_{n}(s)$ ($n = 0, 1, \ldots, K$) of (3.22) are represented by the recurrence formula

$$\tilde{G}_0^+(s) = \tilde{F}_0^+(s) ,$$

$$\tilde{G}_1^+(s) = (I - \tilde{X}_0^+(s))^{-1} \tilde{F}^+(s), \tag{3.25}$$

and

$$\tilde{G}_n^+(s) = (I - \tilde{X}_{n-1}^+(s))^{-1} \tilde{F}^+(s) .$$

Let $e^C(n)$ denote the n-column vector with all its components equal to one. Since $\tilde{F}^-(0)e^C(m_1) < e^C(m_1m_2)$, $H^-e^C(m_1m_2) = e^C(m_1)$, $\tilde{G}_0^+(0) = e^C(m_1)$ and $H_0^+e^C(m_1m_2) = 1$, the inverse of $I - \tilde{X}_0^+(s)$ exists if the real part of s is non-negative. Similarly, the inverse of $I - \tilde{X}_{n-1}^+(s)$ exists.

The dimension of each inverse matrix of (3.25) is equal to m_1m_2. Computation of the inverse matrix is usually cumbersome when the dimension is large. In order to reduce this dimension, let us consider other expressions for $(I - \tilde{X}_{n-1}^+(s))^{-1}$ of (3.25). From the definitions of $\tilde{X}_0^+(s) = \tilde{F}^-(s)H_1^- \tilde{G}_0^+(s)H_0^+$ and $\tilde{X}_{n-1}^+(s) = \tilde{F}^-(s)H^- * \tilde{G}_{n-1}^+(s)H^+$ $(n \geq 2)$, we have

$$\begin{aligned}(I - \tilde{X}_{n-1}^+(s))^{-1} &= I + \tilde{F}^-(s)\textstyle\sum_{j=0}^{\infty}(\tilde{Y}_{n-1}^+(s))^j H^-\tilde{G}_{n-1}^+(s)H^+ \\ &= I + \tilde{F}^-(s)(I - \tilde{Y}_{n-1}^+(s))^{-1}H^-\tilde{G}_n^+(s)H^+ ,\end{aligned} \tag{3.26a}$$

or

$$\begin{aligned}(I- \tilde{X}_{n-1}^+(s))^{-1} &= I + \tilde{F}^-(s)H^-\tilde{G}_{n-1}^+(s)\textstyle\sum_{j=0}^{\infty}(\tilde{Z}_{n-1}^+(s))^j H^+ \\ &= I + \tilde{F}^-(s)H^-\tilde{G}_n^+(s)(I - \tilde{Z}_{n-1}^+(s))^{-1}H^+ ,\end{aligned} \tag{3.26b}$$

$$n = 1, 2, \ldots, K,$$

where

$$\tilde{Y}_0^+(s) = H_1^-\tilde{G}_0^+(s)H_0^+\tilde{F}^-(s) \qquad (m_1 \times m_1 \text{ matrix}),$$

$$\tilde{Y}_{n-1}^+(s) = H^-\tilde{G}_{n-1}^+(s)H^+\tilde{F}^-(s) \qquad (m_1 \times m_1 \text{ matrix}),$$

$$\tilde{Z}_0^+(s) = H_0^+\tilde{F}^-(s)H_1^-\tilde{G}_0^+(s) \qquad (\text{scalar}),$$

and

$$\tilde{Z}_{n-1}^+(s) = H^+\tilde{F}^-(s)H^-\tilde{G}_{n-1}^+(s) \qquad (m_2 \times m_2 \text{ matrix}).$$

Equation (3.26a) should be used of $m_1 < m_2$, and (3.26b) should be used if $m_1 > m_2$. The dimensions of the inverse matrices can be reduced from m_1m_2 to $\min(m_1, m_2)$, and hence the computation may be greatly simplified.

Equation (3.23) gives

$$\tilde{G}_K^{-}(s) = \tilde{F}^{-}(s),$$

(3.27) and

$$\tilde{G}_n^{-}(s) = (I - \tilde{X}_{n+1}^{-}(s))^{-1}\ \tilde{F}^{-}(s),\ n = K-1,\ K-2,\ \ldots,\ 1\ .$$

The inverse of $I - \tilde{X}_{n+1}^{-}(s)$ can be transformed as

$$(3.28a)\qquad (I - \tilde{X}_{n+1}^{-}(s))^{-1} = I + \tilde{F}^{+}(s)H^{+}\tilde{G}_{n+1}^{-}(s)(I - \tilde{Y}_{n+1}^{-}(s))^{-1}H^{-}$$

or

$$(3.28b)\qquad (I - \tilde{X}_{n+1}^{-}(s))^{-1} = I + \tilde{F}^{+}(s)(I - \tilde{Z}_{n+1}^{-}(s))^{-1}H^{+}\tilde{G}_{n+1}^{-}(s)H^{-}\ ,$$

where

$$\tilde{Y}_{n+1}^{-}(s) = H^{-}\tilde{F}^{+}(s)H^{+}\tilde{G}_{n+1}^{-}(s)\quad (m_1 \times m_1 \text{ matrix})$$

and

$$\tilde{Z}_{n+1}^{-}(s) = H^{+}\tilde{G}_{n+1}^{-}(s)H^{-}\tilde{F}^{+}(s)\quad (m_2 \times m_2 \text{ matrix})\ .$$

Equation (3.24) gives

$$\tilde{G}_K^{\pm}(s) = \tilde{F}_K^{\pm}(s) = \tilde{F}^{-}(s) + \tilde{F}^{+}(s)(I - H^{+}\tilde{F}^{+}(s))^{-1}\ H^{+}\tilde{F}^{-}(s)\ ,$$

(3.29) and

$$\tilde{G}_n^{\pm}(s) = (I - \tilde{X}_{n+1}^{\pm}(s))^{-1}\ \tilde{F}^{-}(s),\ n = K-1,\ K-2,\ \ldots,\ 1\ .$$

The inverse $(I - \tilde{X}_{n+1}(s))^{-1}$ can be transformed as

$$(3.30a)\qquad (I - \tilde{X}_{n+1}^{\pm}(s))^{-1} = I + \tilde{F}^{+}(s)H^{+}\tilde{G}_{n+1}^{\pm}(s)(I - \tilde{Y}_{n+1}^{\pm}(s))^{-1}H^{-}$$

or

$$(3.30b)\qquad (I - \tilde{X}_{n+1}^{\pm}(s))^{-1} = I + \tilde{F}^{+}(s)(I - \tilde{Z}_{n+1}^{\pm}(s))^{-1}H^{+}G_{n+1}^{\pm}(s)H^{-}\ ,$$

where

$$\tilde{Y}_{n+1}^{\pm}(s) = H^{-}\tilde{F}^{+}(s)H^{+}\tilde{G}_{n+1}^{\pm}(s)\quad (m_1 \times m_1 \text{ matrix})$$

and

$$\tilde{Z}_{n+1}^{\pm}(s) = H^{+}\tilde{G}_{n+1}^{\pm}(s)H^{-}\tilde{F}^{+}(s)\quad (m_2 \times m_2 \text{ matrix}).$$

The representation of $\tilde{G}_n^{+}(s)$, $\tilde{G}_n^{-}(s)$ and $\tilde{G}_n^{\pm}(s)$ have been obtained using $\tilde{F}_0^{+}(s)$, $\tilde{F}^{+}(s)$, $\tilde{F}^{-}(s)$, H_0^{+}, H^{+}, H_1^{-}(identity matrix) and H^{-}.

Suppose that phase-type p.d.f.s $f(t)$ and $g(t)$ have Neuts, representations [7], (α, T) and (β, S), such that

$$f(t) = \alpha\exp(Tt)T_0$$

(3.31) and

$$g(t) = \beta\exp(St)S_0 \ .$$

It follows from the definition of phase states that

$$\tilde{F}_0^+(s) = (sI(m_1) - T)^{-1}\ T_0\ , \tag{3.32}$$

$$\tilde{F}^+(s) = (sI(m_1m_2) - T \otimes I(m_2) - I(m_1) \otimes S)^{-1}\ (T_0 \otimes I(m_2))\ , \tag{3.33}$$

$$\tilde{F}^-(s) = (sI(m_1m_2) - T \otimes I(m_2) - I(m_1) \otimes S)^{-1}\ (I(m_1) \otimes S_0)\ , \tag{3.34}$$

$$H_0^+ = \alpha \otimes \beta\ , \tag{3.35}$$

$$H^+ = \alpha \otimes I(m_2)\ , \tag{3.36}$$

and

$$H^- = I(m_1) \otimes \beta\ . \tag{3.37}$$

Here, $I(n)$ denotes an $n \times n$ identity matrix and $\otimes$ denotes Kronecker's matrix product [7, p.53]. See Appendix A for the derivation of (3.32)-(3.37).

3.1.1. Inter-overflow Time Structure

Let $t_\ell^{(\omega)}$ $(\ell = 0, 1, 2, \dots)$ denote overflow occurrence time epochs. The p.d.f. of the inter-overflow times has the following matrix structure.

$$Q(t) = (q_{ij}(t))_{i=1,\ 2,\ \dots,\ m_2,\ j=1,\ 2,\ \dots,\ m_2}, \tag{3.38}$$

where

$$q_{ij}(t)dt := P[t < t_{\ell+1}^{(\omega)} - t_\ell^{(\omega)} < t+dt,\ W(t_{\ell+1}^{(\omega)}) = j\ /\ W(t_\ell^{(\omega)}) = i]\ .$$

It is easily seen from this definition that $Q(t)$ can be represented by

$$Q(t) = H^+G_K^+(t)\ . \tag{3.39}$$

Hence, the Laplace transform $\tilde{Q}(s)$ of $Q(t)$ is

$$\tilde{Q}(s) = H^+\tilde{G}_K^+(s) = H^+(I - \tilde{X}_{K-1}^+(s))^{-1}\tilde{F}^+(s) \quad (m_2 \times m_2 \text{ matrix}). \tag{3.40}$$

The mean, m_ω, and variance, v_ω, of the inter-overflow time are respectively given by

$$m_\omega = -q d\tilde{Q}(s)/ds\Big|_{s=0}\ e^C(m_2)$$

and

$$v_\omega = q d^2\tilde{Q}(s)/ds^2\Big|_{s=0}\ e^C(m_2) - m_\omega{}^2 ,$$

where vector q is the invariant probability vector satisfying the linear equation

$$q\tilde{Q}(0) = q , \qquad q\ e^C(m_2) = 1 .$$

The first and second moments of (3.25) can be obtained as follows:

$$d\tilde{G}^+_n(s)/ds = (d(I - \tilde{X}^+_{n-1}(s))^{-1}/ds)\ \tilde{F}^+(s) + (I - \tilde{X}^+_{n-1}(s))^{-1}\ d\tilde{F}^+(s)/ds$$

and

$$d^2\tilde{G}^+_n(s)/ds^2 = (d^2(I - \tilde{X}^+_{n-1}(s))^{-1}/ds^2)\ \tilde{F}^+(s)$$
$$+ 2(d(I - \tilde{X}^+_{n-1}(s))^{-1}/ds)\ \tilde{F}^+(s)/ds$$
$$+ (I - \tilde{X}^+_{n-1}(s))^{-1}\ d^2\tilde{F}^+(s)/ds^2 ,$$

where

$$d(I - \tilde{X}^+_{n-1}(s))^{-1}/ds = (I - \tilde{X}^+_{n-1}(s))^{-1}\ (d\tilde{X}^+_{n-1}(s)/ds)(I - \tilde{X}^+_{n-1}(s))^{-1}$$

and

$$d^2(I - \tilde{X}^+_{n-1}(s))^{-1}/ds^2$$
$$= (d(I - \tilde{X}^+_{n-1}(s))^{-1}/ds)\ (d\tilde{X}^+_{n-1}(s)/ds)\ (I - \tilde{X}^+_{n-1}(s))^{-1}$$
$$+ (I - \tilde{X}^+_{n-1}(s))^{-1}\ (d^2\tilde{X}^+_{n-1}(s)/ds^2)\ (I - \tilde{X}^+_{n-1}(s))^{-1}$$
$$+ (I - \tilde{X}^+_{n-1}(s))^{-1}\ (d\tilde{X}^+_{n-1}(s)/ds)\ (d(I - \tilde{X}^+_{n-1}(s))^{-1}/ds) .$$

From $d\tilde{Q}(s)/ds = H^+ d\tilde{G}^+_K(s)/ds$ and $d^2\tilde{Q}(s)/ds^2 = H^+ d^2\tilde{G}^+_K(s)/ds^2$, m_ω and v_ω can be computed.

Consider the convariance of inter-overflow times, $cov(t^{(\omega)}_{k+1} - t^{(\omega)}_k, t^{(\omega)}_{\ell+1} - t^{(\omega)}_\ell)$ $(k < \ell)$. This covariance is given by

$$cov(t^{(\omega)}_{k+1} - t^{(\omega)}_k, t^{(\omega)}_{\ell+1} - t^{(\omega)}_\ell)$$
$$= q\ (d\tilde{Q}(s)/ds\Big|_{s=0})\ \tilde{Q}^{\ell-k-1}(0)\ (d\tilde{Q}(s)/ds\Big|_{s=0})\ e^C(m_1 m_2) - m_\omega{}^2 .$$

The covariance is not always equal to 0, thus the overflow process from the PH/PH/1/K cannot be always thought to be renewal. In particular, for the case of erponential services, since $\tilde{Q}(s)$ is scalar and $\tilde{Q}(0) = q = 1$, the covariance is equal to 0.

Now, let us consider the moment characteristics of inter-overflow time. The service time and interarrival time p.d.f.s agreeing with mean m and variance v are given by

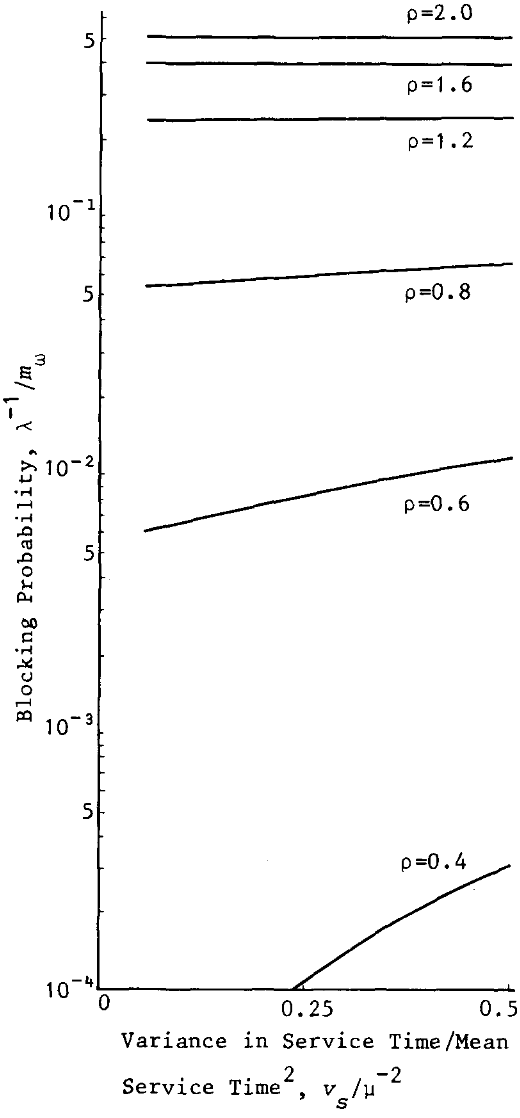
k-phase Erlangian E_k for $v/m^2 < 1$,

and

Morse type hyperexponential H_2[cf. 10] for $v/m^2 > 1$.

The effects of service time variations on the overflow processes are illustrated in Figs. 1 and 2. λ^{-1}, v_a, μ^{-1}, v_s and ρ denote the mean interarrival time, variance in interarrival time, mean service time, variance in service time, and offered load, λ/μ. Figure 1 depicts that blocking probabilities increase monotonically as variations in the service times increase. When the offered load, ρ, is small, the effect of service time variations is very significant. Figure 2 depicts that interoverflow time variations increase monotonically as the service time variations increase. It should be noted that when the coefficient of variation of service time is fixed, the coefficient of variance of interoverflow time is not monotomic as ρ increases.

The effects of interarrival time variations on overflow processes are illustrated in Figs. 3 and 4. Blocking probabilities are very sensitive to variations in interarrival times; this is especially so, when ρ is smaller than one. The effect on the coefficient of interoverflow time is also worthy of notice. When the coefficient of variation for interarrival times is greater than three, the coefficient of variation for interoverflow times increases monotinically as ρ increases.



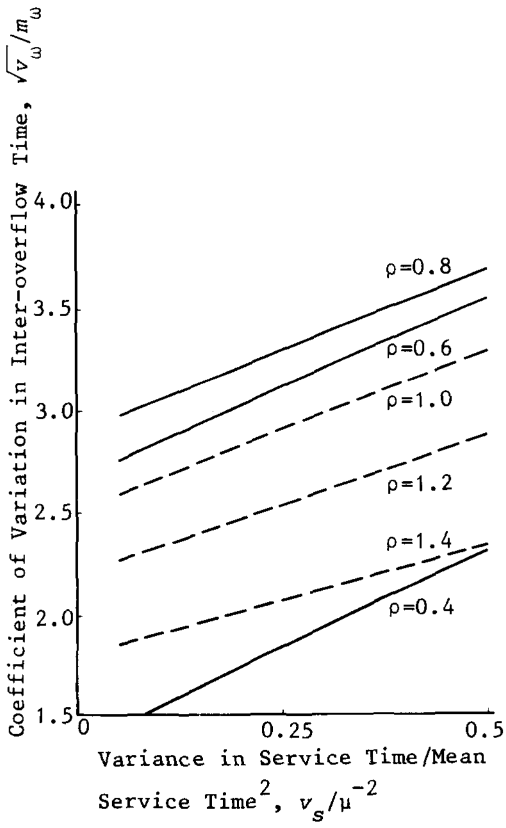


Figure 2. Variation in Service Time vs. Variation in Inter-overflow Time ($\sqrt{v_a}/\lambda^{-1}$=2) (K=11, $\sqrt{v_a}/\lambda^{-1}$=2)

Figure 1. Variation in Service Time vs. Blocking Probability (K=11, $\sqrt{v_a}/\ ^{-1}$=2)

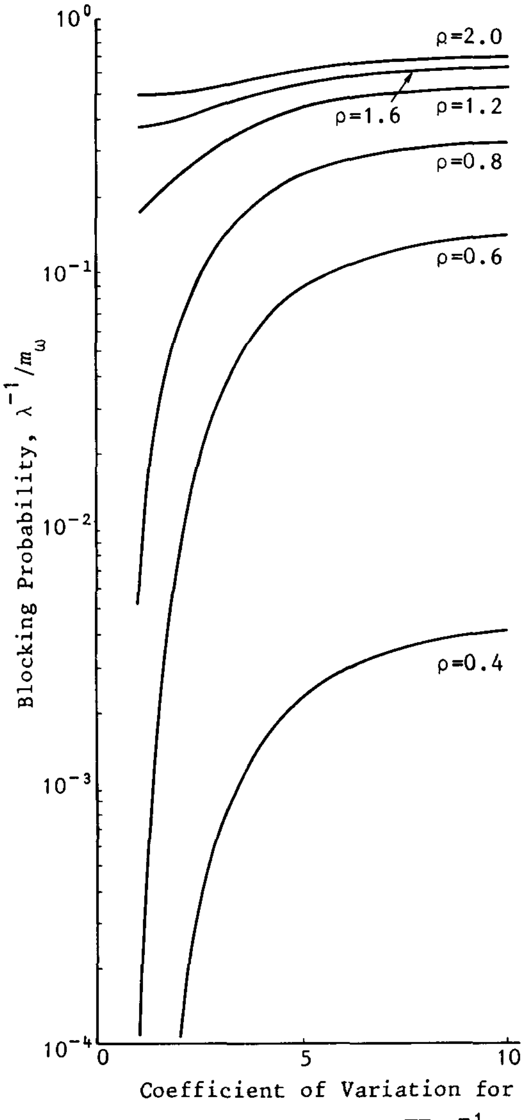


Figure 3. Variation in Interarrival Time VS. Blocking Probability (K=11, v_s/μ^{-2}=0.25)

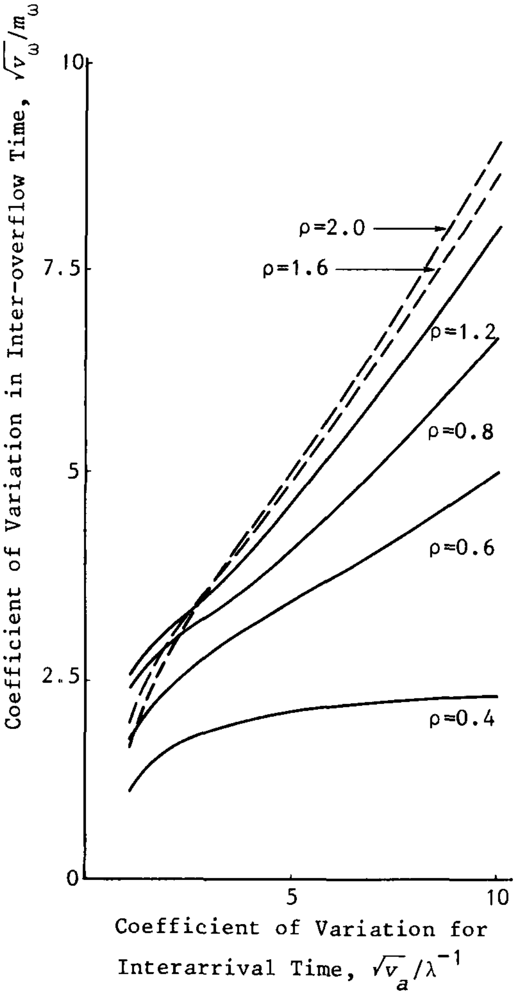


Figure 4. Variation in Interarrival Time VS. Variation in Interoverflow Time (K=11, v_s/μ^{-2}=0.25)

3.1.2. Busy Period Structure

Since the busy period is defined as the first passage time length between the arrival of a customer to an empty queue and the first epoch thereafter that the queue becomes empty again, its p.d.f. has the vector structure

$$B(t) = H_0^+ G_1^{\pm}(t) \ . \tag{3.41}$$

The Laplace transform, $\tilde{B}(s)$, of $B(t)$ is given by the equality,

$$\tilde{B}(s) = H_0^+ \tilde{G}_1^{\pm}(s) \ . \tag{3.42}$$

$\tilde{G}_1^{\pm}(s)$, $d\tilde{G}_1^{\pm}(s)/ds$ and $d^2\tilde{G}_1^{\pm}(s)/ds^2$ can be computed similarly as $\tilde{G}_K^{+}(s)$, $d\tilde{G}_K^{+}(s)/ds$ and $d^2\tilde{G}_K^{+}(s)/ds^2$, respectively.

It should be noted that idle time p.d.f. is given by $\tilde{B}(0)\exp(Tt)T_0$, because $\tilde{B}(0)$ is the stationary arrival phase state probability vector immediately before the idle period starting point. Mean, m_b, and variance, v_b, for the busy period are respectively

$$m_b = -d\tilde{B}(s)/ds\Big|_{s=0} \, e^C(m_1) = -H_0^+ d\tilde{G}_1^{\pm}(s)/ds\Big|_{s=0} \, e^C(m_1),$$

and

$$v_b = d^2\tilde{B}(s)/ds^2\Big|_{s=0} \, e^C(m_1) - m_b^2 = H_0^+ d^2\tilde{G}_1^{\pm}(s)/ds^2\Big|_{s=0} \, e^C(m_1) - m_b^2 \ .$$

The effects of service time variations for the busy period are illustrated in Figs. 5 and 6. Figure 5 depicts that the mean busy period decreases monotonically as the service time variation increases. This is undoubtedly because the blocking probability increases and the number of customers completing services in one busy period decreases. However, this effect is not so noticeable when ρ is small. When ρ is smaller than 0.2, the effect of service time variation is negligible. Figure 6 indicates that the busy period variation increases monotonically as the service time variation increases.

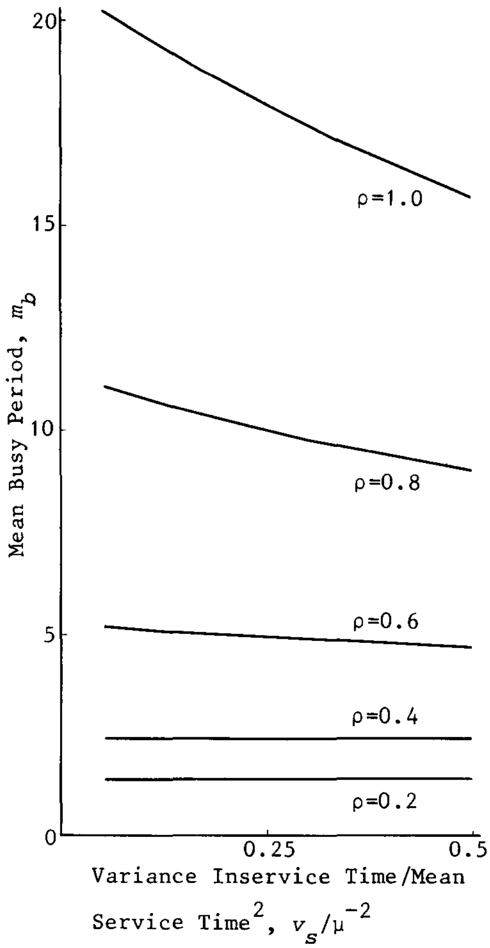


Figure 5. Variation in Service Time VS. Mean Busy Period (K=11, $\sqrt{v_a}/\lambda^{-1}$=2)

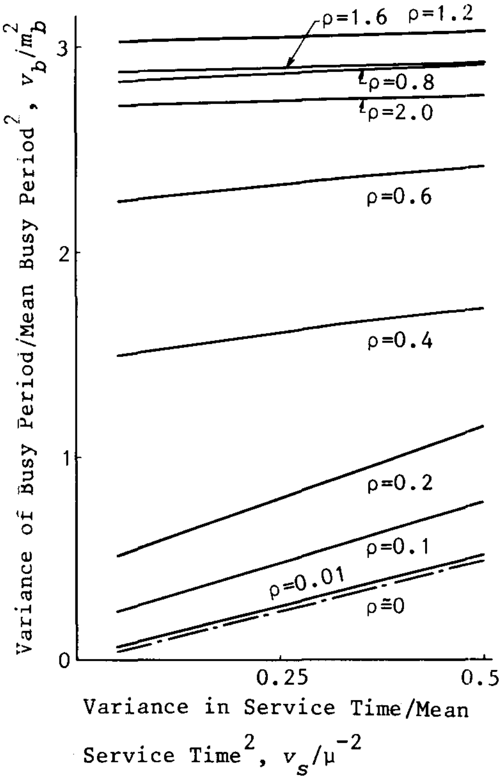


Figure 6. Variation in Service Time VS. Variation in Busy Period (K=11, $\sqrt{v_a}/\lambda^{-1}$=2)

Let us consider a more detailed structure for the busy period. Let $[0,T]$ denote the busy period and let $O(T)$ denote the number of overflows during this period. Consider the following joint d.f. matrix

$$(3.43)\qquad B(t,\ j):=(b_{.\ell}(t,\ j))_{\ell=1,\ 2,\ \ldots,\ m_1}$$

where

$$b_{.\ell}(t,\ j)dt:=P[t<T<t+dt,\ O(T)=j,\ X(T)=\ell\ /\ Y(0)=1],\ j=0,\ 1,\ \ldots,\ \ell=1,\ 2,\ \ldots,\ m_1\ .$$

Let z_n^+ denote the first passage time from $Y(\cdot)=n$ to $Y(\cdot)=n+1$ without intermediate passage to $Y(\cdot)=0$. That is,

$$z_n^+:=\inf_t[Y(\tau_0+t)=n+1,\ Y(s)\neq 0 \text{ for all } s\in[\tau_0,\ \tau_0+t]/Y(\tau_0)=n]$$
$$n=1,\ 2,\ \ldots,\ K-1\ ,$$

and

$$z_K^+\ :=\inf_t[Y(\tau_0+t)=\omega,\ Y(s)\neq 0 \text{ for all } s\in[\tau_0,\ \tau_0+t]/Y(\tau_0)=K].$$

Then, we define

$$(3.44)\qquad \begin{aligned} &g_{ij}^+(t;\ n;\ Y>0)dt:=P[t<z_n^+<t+dt,\ W(\tau_0+z_n^+)=j/Z(\tau_0)=i]\ ,\\ &n=1,\ 2,\ \ldots,\ K,\ i=1,\ 2,\ \ldots,\ m_1m_2,\ j=1,\ 2,\ \ldots,\ m_2\ .\end{aligned}$$

If a matrix with the Laplace transforms of (3.44) as elements, that is,

$$(3.45)\qquad \tilde{G}_n(s;\ Y>0):=(\tilde{g}_{ij}(s;,n,\ Y>0))_{i=1,\ 2,\ \ldots,\ m_1m_2,j=1,\ 2,\ \ldots m_2}.$$

is defined, the following equations can be obtained in a way similar to (3.25).

$$(3.46)\qquad \begin{aligned} &\tilde{G}_1^+(s;\ Y>0)=\tilde{F}^+(s)\\ &\tilde{G}_n^+(s;\ Y>0)=(I-\tilde{X}_{n-1}^+(s;\ Y>0))^{-1}\tilde{F}^+(s),\end{aligned}$$

where

$$\tilde{X}_{n-1}^+(s;\ Y>0)=\tilde{F}^-(s)\ H^-\tilde{G}_{n-1}^+(s;\ Y>0)\ H^+\ .$$

Inverse matrix $(I-\tilde{X}_{n-1}^+(s;\ Y>0))^{-1}$ may be transformed like this:

$$(I - \tilde{X}^{+}_{n-1}(s;\ Y>0))^{-1} = I + \tilde{F}^{-}(s)\ (I - \tilde{Y}^{+}_{n-1}(s;\ Y>0))^{-1}$$

$$\cdot H^{-}\tilde{G}^{+}_{n-1}(s;\ Y>0)\ H^{+}\tilde{F}^{+}(s)$$

or

$$(I - \tilde{X}^{+}_{n-1}(s;\ Y>0))^{-1} = I + \tilde{F}^{-}(s)\ H^{-}\tilde{G}^{+}_{n-1}(s;\ Y>0)$$

$$\cdot (I - \tilde{Z}^{+}_{n-1}(s;\ Y>0))^{-1}\ H^{+}\tilde{F}^{+}(s)\ ,$$

where

$$\tilde{Y}^{+}_{n-1}(s;\ Y>0) = H^{-}\tilde{G}^{+}_{n-1}(s;\ Y>0)\ H^{+}\tilde{F}^{-}(s)$$

and

$$\tilde{Z}^{+}_{n-1}(s;\ Y>0) = H^{+}\tilde{F}^{-}(s)\ H^{-}\tilde{G}^{+}_{n-1}(s;\ Y>0)\ .$$

The Laplace transform, $\tilde{B}(s,\ j)$, of $B(t,\ j)$ can be written from the definitions of $\tilde{G}^{+}_{n}(s;\ Y>0)$, $\tilde{G}^{-}_{n}(s)$, H^{+}_{0}, H^{+} and H^{-} as

(3.47)
$$\tilde{B}(s;\ j) = H^{+}_{0}\tilde{G}^{+}_{1}(s;\ Y>0)[\pi^{k-1}_{i=2}\ (H^{+}\tilde{G}^{+}(s;\ Y>0))][(H^{+}\tilde{G}^{+}(s;\ Y>0)]^{j}$$
$$\cdot H^{+}[\pi^{K-2}_{i=0}\ \tilde{G}^{-}_{K-i}(s)H^{-}]\ \tilde{G}^{-}_{1}(s),\ j = 1,\ 2,\ \ldots,$$

where the product operator, $\pi^{\ell}_{i=k}$, is defined as the identity operator for $\ell < k$.

It is clear that

(3.48) $$\tilde{B}(s,\ 0) = H^{+}_{0}\tilde{G}^{-}_{1}(s).$$

Defining the generating function

$$\tilde{B}^{*}(s,\ z) := \textstyle\sum^{\infty}_{j=0}\ z^{j}\ \tilde{B}(s,\ j),$$

from (3.47) and (3.48), it follows that

$$\tilde{B}^{*}(s,\ z) = H^{+}_{0}\tilde{G}^{-}_{1}(s) + H^{+}_{0}\tilde{G}^{+}_{1}(s;\ Y>0)[\pi^{k-1}_{i=2}\ (H^{+}\tilde{G}^{+}(s;\ Y>0))]$$

(3.49)
$$\cdot [zH^{+}\tilde{G}^{+}_{K}(s;\ Y>0)][I - zH^{+}\tilde{G}^{+}_{K}(s;\ Y>0)]^{-1}\ H^{+}$$

$$\cdot [\pi^{K-2}_{i=0}\ \tilde{G}^{-}_{K-i}(s)H^{-}]\tilde{G}^{-}_{1}(s).$$

3.2. First Passage Time P.D.E. for PH/PH/1 Queue

Since the first passage time p.d.f.s from $Y(\cdot) = n$ to $Y(\cdot) = n+1$ defined in (3.1) have no relationship to queueing capacity, the Laplace transforms of the p.d.f.s for the PH/PH/1 queue can also be given by (3.25). The first passage time p.d.f.s from $Y(\cdot) = n+1$ to $Y(\cdot) = n$, are satisfied by

$$G_n^-(t) = G_{n+1}^-(t) \text{ for all } n = 1, 2, \ldots\ldots\ldots . \tag{3.50}$$

With writing of $\tilde{G}^-(s)$ instead of $\tilde{G}_n^-(s)$ $(n = 1, 2, \ldots.)$, it follows from (3.20) that

$$\tilde{G}^-(s) = \sum_{\ell=0}^{\infty} (\tilde{F}^+(s)H^+\tilde{G}^-(s)H^-)^{\ell}\, F^-(s). \tag{3.51}$$

That is,

$$\tilde{G}^-(s) = \tilde{F}^-(s) + \tilde{F}^+(s)H^+\tilde{G}^-(s)H^-\tilde{G}^-(s). \tag{3.52}$$

$\tilde{G}^-(s)$ can be obtained by successive substitutions starting with the zero matrix (See Neuts [7. pp.95-101]).

4. Transition Probability Between Two Arbitrary Points In Time

This section provides a look at the Laplace transforms for transition probabilities of the stochastic process $[Y(t), Z(t), t \geq 0]$ for the PH/PH/1/K and PH/PH/1 queues. $\tilde{G}_n^+(s)$, $\tilde{G}_n^-(s)$, $\tilde{G}_n^{\pm}(s)$, $\tilde{F}_0^+(s)$, $\tilde{F}^+(s)$, $\tilde{F}^-(s)$, H_0^+, H^+ and H^-, derived in Section 3, will be used.

4.1. Transition Probability for PH/PH/1/K Queue Between Two Arbitrary Points In Time

The transition probabilities for the PH/PH/1/K queue are defined by

$$\begin{aligned} p_{ij}(t;\, k,\, \ell) &:= P[Y(\tau_0+t) = j,\ Z(\tau_0+t) = \ell \;/\; Y(\tau_0) = i,\ Z(\tau_0) = k], \\ & i, j = 0, 1, \ldots, K, \quad k, \ell = 1, 2, \ldots, m_1m_2 . \end{aligned} \tag{4.1}$$

First, let us consider the case where $i = j$. In order to analyze $p_{jj}(t;\, k,\, \ell)$, let us introduce the transition probability

$$\begin{aligned} p_{jj}(t;\, k,\, \ell;\, n) &:= P[Y(\tau_0+t) = j,\ Z(\tau_0+t) = \ell, \text{ the number of} \\ & \text{upward transitions from } Y(\cdot) = j-1 \text{ to } Y(\cdot) = j \text{ is exactly} \\ & n \text{ in } [\tau_0,\ \tau_0+t] \;/\; Y(\tau_0) = j,\ Z(\tau_0) = k] \end{aligned} \tag{4.2}$$

and

$$q_{jj}(t;\, k,\, \ell) := P[Y(u) = j \text{ for all } u \in [\tau_0,\, \tau_0+t],\ Z(\tau_0+t) = \ell \,/\, Y(\tau_0) = j,\, Z(\tau_0) = k] . \tag{4.3}$$

From this definition, it is clear that

$$p_{jj}(t;\, k,\, \ell) = \sum_{n=0}^{\infty} p_{jj}\ (t;\, k,\, \ell;\, n) . \tag{4.4}$$

Now, let $P_{jj}(t;\, n)$ and $Q_{jj}(t)$ denote the matrices which contain elements $p_{jj}(t;\, k,\, \ell;\, n)$ and $q_{jj}(t;\, k,\, \ell)$ $(k,\, \ell = 1,\, 2,\, \ldots,\, m_1m_2)$. That is,

$$P_{jj}(t;\, n) := (p_{jj}(t;\, k,\, \ell;\, n))_{k=1,\, 2,\, \ldots,\, m_1m_2,\ \ell=1,\, 2,\, \ldots,\, m_1m_2}. \tag{4.5}$$

Then, from Appendix A, it is clear that the Laplace transform, $\tilde{Q}_{jj}(s)$, of $Q_{jj}(t)$ can be represented by

$$\tilde{Q}_{00}(s) = (sI(m_1) - T)^{-1} ,$$

$$\tilde{Q}_{jj}(s) = (sI(m_1m_2) - T \otimes I(m_2) - I(m_1) \otimes S)^{-1},$$

$$j = 1,\, 2,\, \ldots,\, K-1 ,$$

$$\tilde{Q}_{KK}(s) = (sI(m_1m_2) - T \otimes I(m_2) - I(m_1) \otimes S)^{-1}$$

$$+ \tilde{F}^+(s)\ (I(m_2) - H^+\tilde{F}^+(s))^{-1}\ H^+(sI(m_1m_2) -$$

$$- T \otimes I(m_2) - I(m_1) \otimes S)^{-1} .$$

(i) $n = 0$ case

Noting the epochs of the downward transitions from $Y(\cdot) = j+1$ to $Y(\cdot) = j$ $(j = 0,\, 1,\, \ldots,\, K-1)$, it follows that

$$P_{00}(t;\, 0) = \sum_{i=0}^{\infty} (F_0^+(t)H_0^+ * G_1^{\pm}(t)H_1^-)^{i*} * Q_{00}(t), \tag{4.7}$$

and

$$P_{jj}(t;\, 0) = \sum_{i=0}^{\infty} (F^+(t)H^+ * G_{j+1}^{\pm}(t)H^-)^{i*} * Q_{jj}(t), \quad j = 1,\, 2,\, \ldots,\, K-1 . \tag{4.8}$$

It is clear that

$$P_{KK}(t;\, 0) = Q_{KK}(t) . \tag{4.9}$$

(ii) $n = 1, 2, \ldots.$ case.

It is clear that

$$P_{00}(t;\ n) = 0\ . \tag{4.10}$$

Noting the epochs of the upward transitions from $Y(\cdot) = j-1$ to $Y(\cdot) = j$ ($j = 1, 2, \ldots, K$), it follows from (4.7) that

$$P_{jj}(t;\ n) = \begin{cases} (G_1^{\pm}(t)H_1^{-} * F_0^{+}(t)H_0^{+})^{n*} * P_{11}(t;\ 0),\ j = 1. \\ (G_j^{\pm}(t)H^{-} * G_{j-1}^{+}(t)H^{+})^{n*} * P_{jj}(t;\ 0), \\ j = 2, 3, \ldots, K. \end{cases} \tag{4.11}$$

Equations (4.7) and (4.10) give the expression

$$P_{00}(t) = \textstyle\sum_{i=0}^{\infty} (F_0^{+}(t)H_0^{+} * G_1^{\pm}(t)H_1^{-})^{i*} * Q_{00}(t), \tag{4.12}$$

and (4.11) gives

$$P_{jj}(t) = \begin{cases} \sum_{i=0}^{\infty} (G_1^{\pm}(t)H_1^{-} * F_0^{+}(t)H_0^{+})^{i*} * P_{11}(t;\ 0), \\ j = 1. \\ \sum_{i=0}^{\infty} (G_j^{\pm}(t)H^{-} * G_{j-1}^{+}(t)H^{+})^{i*} * P_{jj}(t;\ 0), \\ j = 2, 3, \ldots, K. \end{cases} \tag{4.13}$$

Taking the Laplace transforms of (4.12) and (4.13),

$$\begin{aligned} \tilde{P}_{00}(s) &= (I - \tilde{F}_0^{+}(s)H_0^{+}\tilde{G}_1^{\pm}(s)H_1^{-})^{-1}\ \tilde{Q}_{00}(s), \\ &= (I + \tilde{F}_0^{+}(s)\ (1 - H_0^{+}\tilde{G}_1^{\pm}(s)\ H_1^{-}\tilde{F}_0^{+}(s))^{-1}\ H_0^{+}\tilde{G}_1^{\pm}(s)H_1^{-})\ \tilde{Q}_{00}(s), \end{aligned} \tag{4.14}$$

$$\begin{aligned} \tilde{P}_{11}(s) &= (I - \tilde{G}_1^{\pm}(s)H^{-}\tilde{F}_0^{+}(s)H_0^{+})^{-1}\ \tilde{P}_{11}(s;\ 0) \\ &= (I + \tilde{G}_1^{\pm}(s)H^{-}\tilde{F}_0^{+}(s)(1 - H_0^{+}\tilde{G}_1^{\pm}(s)H_1^{-}\tilde{F}_0^{+}(s))^{-1}H_0^{+})\ \tilde{P}_{11}(s;\ 0). \\ \tilde{P}_{jj}(s) &= (I - \tilde{G}_j^{\pm}(s)\tilde{H}^{-}\tilde{G}_{j-1}^{+}(s)H^{+})^{-1}\ \tilde{P}_{jj}(s;\ 0), \\ &\quad j = 2, 3, \ldots, K, \end{aligned} \tag{4.15}$$

where

$$\begin{aligned} &(I - \tilde{G}_j^{\pm}(s)H^{-}\tilde{G}_{j-1}^{+}(s)H^{+})^{-1} \\ &= I + \tilde{G}_j^{\pm}(s)\ (I - H^{-}\tilde{G}_{j-1}^{+}(s)H^{+}\tilde{G}_j^{\pm}(s))^{-1}H^{-}\tilde{G}_{j-1}^{+}(s)H^{+} \end{aligned}$$

or

$$(I - \tilde{G}_j^{\pm}(s)H^-\tilde{G}_{j-1}^{+}(s)H^+)^{-1}$$

$$= I + \tilde{G}_j^{\pm}(s)H^-\tilde{G}_{j-1}^{+}(s) \ (I - H^+\tilde{G}_j^{\pm}(s)H^-\tilde{G}_{j-1}^{+}(s))^{-1}H^+ .$$

Here, from (4.8) and (4.9), the $P_{jj}(s;\ 0)$ of (4.15) can be represented by

$$\tilde{P}_{jj}(s;\ 0) = (I - \tilde{F}^+(s)H^+\tilde{G}_{j+1}^{\pm}(s)H^-)^{-1}\tilde{Q}_{jj}(s) , \tag{4.16}$$

$$j = 1,\ 2,\ \ldots,\ K-1 ,$$

and

$$\tilde{P}_{KK}(s;\ 0) = \tilde{Q}_{KK}(s) ,$$

where

$$(I - \tilde{F}^+(s)H^+\tilde{G}_{j+1}^{\pm}(s)H^-)^{-1}$$

$$= I + \tilde{F}^+(s)H^+\tilde{G}_{j+1}^{\pm}(s) \ (I - H^-\tilde{F}^+(s)H^+\tilde{G}_{j+1}^{\pm}(s))^{-1}H^- ,$$

or

$$(I - \tilde{F}^+(s)H^+\tilde{G}_{j+1}^{\pm}(s)H^-)^{-1}$$

$$= I + \tilde{F}^+(s) \ (I - H^+\tilde{G}_{j+1}^{\pm}(s)H^-\tilde{F}^+(s))^{-1}H^+\tilde{G}_{j+1}^{\pm}(s)H^- .$$

Now $\tilde{P}_{jj}(s)$ has been obtained for all $j = 0,\ 1,\ \ldots,\ K$.

Next, for $i \neq j$, consider the transition probability defined by

$$P_{ij}(t) := (p_{ij}(t;\ k,\ \ell))_{k=1,\ 2,\ \ldots,\ m_1m_2,\ \ell=1,\ 2,\ \ldots,\ m_1m_2} . \tag{4.17}$$

Since $P_{ij}(t)$ is represented as the convolution of the first passage p.d.f. from $Y(\cdot) = i$ to $Y(\cdot) = j$ and $P_{jj}(t)$, the Laplace transform $\tilde{P}_{ij}(s)$ of $P_{ij}(t)$ can be obtained by

$$\tilde{P}_{ij}(s) = \begin{cases} \tilde{G}_0^+(s)H_0^+[\pi_{\ell=1}^{j-1}\ (\tilde{G}_\ell^+(s)H^+)]\tilde{P}_{jj}(s),\ i = 0,\ j > 0 . \\ [\pi_{\ell=i}^{j-1}\ (\tilde{G}_\ell^+(s)H^+)]\tilde{P}_{jj}(s),\ 0 < i < j . \\ [\pi_{\ell=0}^{i-j-1}\ (\tilde{G}_{i-\ell}^{\pm}(s)H^-)]\tilde{P}_{jj}(s),\ 0 < j < i . \\ [\pi_{\ell=0}^{i-2}\ (\tilde{G}_{i-\ell}^{\pm}(s)H^-)]\tilde{G}_1^{\pm}(s)H_0^-\tilde{P}_{00}(s),\ i > 0,\ j = 0 . \end{cases} \tag{4.18}$$

4.2. Transition Probability for PH/PH/1 Queue Between Two Arbitrary Points In Time

By subsituting $\tilde{G}^{-}(s)$ obtained in (3.52), into $\tilde{G}_j^{\pm}(s)$ in (4.14) through (4.16), the transition probabilities for the PH/PH/1 queue can also be derived.

5. Conclusion

First passage times for the PH/PH/1/K queue have been studied. A Laplace transform for the inter-overflow time probability density function (p.d.f.) was obtained. The Laplace and z double transform of the joint distribution for the busy period and the number of overflows in this busy period was also obtained. In addition, the transition probability of system states between two arbitrary time points was obtained using the Laplace transform. All results could be represented by simple recurrences. Moreover, the Laplace transform of the busy period p.d.f. and transition probability of the system states for the PH/PH/1 queue have been analyzed as a special case of the PH/PH/1/K queue.

Appendix A Proof of (3.32)-(3.37)

(3.32) is clear.

Consider $Q_{jj}(t)$ defined (4.6) for $j \geqq 1$. If the arrival phase state process with generator T defined in space A is considered a process in space A × B, the arrival phase state transitions form a Markov process with generator $T \otimes I(m_2)$. Similarly, the service phase state process can be considered a Markov process with generator $I(m_1) \otimes S$ in space A × B. Since these two processes are independent of each other, $Q_{jj}(t)$ is governed by the Markov process with generator $T \otimes I(m_2) + I(m_1) \otimes S$. Thus, $\tilde{Q}_{jj}(s) = (sI(m_1m_2) - T \times I(m_2) - I(m_1) \times S)^{-1}$ is proved.

Next, consider arrival rate vector T_0, which is a mapping from space A to one state space {1}, as mapping X from space A × B to space B. Since a customer arrival never influences the service phase states, X is represented by $T_0 \otimes I(m_2)$. Thus, (3.33) is proved. (3.34) can be similarly obtained.

When service begins simultaneously with customer arrival, the phase state at the time point immediately after the service begins is $k = (i-1)m_2 + j$ with probability $\alpha_i\beta_j$. Thus, (3.35) is proved. When the service phase state shows no change due to the customer arrival, (3.36) can be easily derived.

Since the arrival phase state shows no change due to customer service completion, (3.37) can be derived.

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