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AN INFINITELY-MANY SERVER QUEUE HAVING MARKOV RENEWAL ARRIVALS AND HYPEREXPONENTIAL SERVICE TIMES

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Abstract This paper studies an infinitely-many server queue which has Markov renewal inputs and hyper-exponential service times. The stationary distributions of the number of busy servers both immediately before an arrival epoch and at an arbitrary time are obtained. These results can be applied to engineering telephone networks.

1. Introduction

An infinite server queue with Markov renewal arrivals has potential applicability for telephone network planning in which the equivalent random method is used [7, 12]. Since Wilkinson and Bretschneider first proposed this method, it has been assumed that the service times of all customers are independent and identically distributed according to an exponential distribution. However, it is too restrictive to assume that the distribution is exponential. Burke has studied the equivalent random method in the case where the service time distribution is a unit distribution [1]. The results of an analysis of field data indicate that the service time distribution is hyperexponential when its coefficient of variation is greater than one [8, 9]. Therefore, it seems that Burke's results cannot be directly applied to network planning because the unit service time distribution differs from the real service time distribution. When the equivalent random method is applied under the assumption that the service time distribution is hyperexponential, an $MR/H_m/\infty$ queue having Markov renewal inputs and hyperexponential service times has an essential role. This is because the overflow process from the $M/H_m/S/S$ loss system is Markov renewal process [10] and the overflow traffic cannot be characterized without analyzing the $MR/H_m/\infty$ queue. In this paper, the r-th binomial moment of the number of customers in a system in equilibrium immediately before a customer arrival epoch is analyzed using the embedded Markov chain method. Then, the relationship between the stationary state probabilities at an arbitrary time and the stationary state probabilities immediately before a customer arrival epoch is derived and the r-th binomial moment of the number of customers in the system at an arbitrary time is explicitly provided. These results are an extension of Franken's results [3]. Smith [5] has also extended Franken's results, but to a model with a mean service time which varies according to the type of customer.

2. Stationary State Distribution Immediately Before a Customer Arrival Epoch

Consider the following model:

i) The customer arrival process is a Markov renewal process with a semi-Markov kernel [2]:

$$(2.1) \qquad F(x) = (F_{\alpha\beta}(x)) \qquad 1 \leq \alpha,\beta \leq n.$$

The Laplace-Stieltjes transform of $F(x)$ is written as

$$(2.2) \qquad \Phi(s) = (\psi_{\alpha\beta}(s)) \qquad 1 \leq \alpha,\beta \leq n,$$

where

$$\psi_{\alpha\beta}(s) = \int_0^\infty e^{-sx} dF_{\alpha\beta}(x)$$

ii) There are an infinite number of servers and each customer is served as soon as he arrives. The service times of all customers are independent and identically distributed according to the distribution

$$(2.3) \qquad H(x) = 1 - \sum_{i=1}^{m} k_i e^{-\mu_i x}$$

The Laplace-Stieltjes transform of $H(x)$ is written as

$$(2.4) \qquad \eta(s) = \int_0^\infty e^{-sx} dH(x).$$

The stage whose mean sojourn time is μ_i^{-1} is hereafter referred to as "service stage i". Let (t_0, z_0), (t_1, z_1), ..., (t_n, z_n), ... $(t_0<t_1< \dots <t_n< \dots)$ denote the pairing of the customer arrival epoch and an element in the kernel of the Markov renewal arrival process. Let $Y(t_k-0) = (n_1, n_2, \dots, n_m)$ denote that the number of customers in service stage i is n_i immediately before t_k, where $\sum_{i=1}^{m} n_i$ is the total number of customers in the system. It is clear from the assumptions that the sequence $(Y(t_k-0),\ Z_k)$ $(k=0,1,2,\ \dots)$ forms a homogeneous Markov chain which contains the transition probability of successive arrival points

$$
\begin{aligned}
&\Pi^{(\alpha\beta)}_{i_1,\ i_2,\ \dots,\ i_m;\ j_1,\ j_2,\ \dots,\ j_m} \\
(2.5)\qquad &:= P\{Y(t_{n+1}-0) = (j_1,\ j_2,\ \dots,\ j_m),\ Z_{n+1} = \beta \\
&\qquad / \ Y(t_n-0) = (i_1,\ i_2,\ \dots,\ i_m),\ Z_n = \alpha\}.
\end{aligned}
$$

The probability that a customer arriving at t_k will enter service stage ℓ immediately after t_k is k_ℓ. Thus, (2.5) can be expressed as

$$
\begin{aligned}
&\Pi^{(\alpha\beta)}_{i_1,\ i_2,\ \dots,\ i_m;\ j_1,\ j_2,\ \dots,\ j_m} \\
(2.6)\qquad &= \int_0^\infty \Big[\sum_{\ell=1}^{m} k_\ell \binom{i_1}{j_1} \dots \binom{i_\ell+1}{j_\ell} \dots \binom{i_m}{j_m} e^{-j_1\mu_1 x}(1-e^{-\mu_1 x})^{i_1-j_1} \\
&\quad \dots e^{-j_\ell\mu_\ell x}(1-e^{-\mu_\ell x})^{i_\ell+1-j_\ell} \dots e^{-j_m\mu_m x}(1-e^{-\mu_m x})^{i_m-j_m} \\
&\quad \cdot dF_{\alpha\beta}(x)\ .
\end{aligned}
$$

The stationary probability distribution can now be defined as follows:

$$
(2.7)\qquad P^{(\beta)}_{j_1,\ j_2,\ \dots,\ j_m} := \lim_{n\to\infty} P\{Y(t_n-0)=(j_1,\ j_2,\ \dots,\ j_m),\ Z_n=\beta\}.
$$

The $(r_1,\ r_2,\ \dots,\ r_m)$th binomial moment of the distribution $\{P^{(\beta)}_{j_1,\ j_2,\ \dots,\ j_m}\}$ is denoted by:

$$
\begin{aligned}
(2.8)\qquad b^{(\beta)}_{r_1,\ r_2,\ \dots,\ r_m} &:= \sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\cdots\sum_{j_m=0}^{\infty}\binom{j_1}{r_1}\binom{j_2}{r_2} \\
&\quad \cdots \binom{j_m}{r_m} P^{(\beta)}_{j_1,\ j_2,\ \dots,\ j_m}\ ,
\end{aligned}
$$

where we define $\binom{i}{j} = 0$ for $i<j$.

From the definition of the transition probability (2.5), it is possible to obtain

$$
\begin{aligned}
(2.9)\qquad P^{(\beta)}_{j_1,j_2,\dots,\ j_m} &= \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\cdots\sum_{i_m=0}^{\infty}\sum_{\alpha=1}^{n} P^{(\alpha)}_{i_1,i_2,\dots,i_m} \\
&\quad \cdot\ \Pi^{(\alpha\beta)}_{i_1,i_2,\dots,i_m;j_1,j_2,\dots,j_m}\ .
\end{aligned}
$$

The binomial moment (2.8) is given by the following theorem:

Theorem 2.1. The $(r_1,r_2,\ldots,r_m)$th binomial moment vector

$$B_{r_1,r_2,\ldots,r_m} := (b^{(1)}_{r_1,r_2,\ldots,r_m},\ b^{(2)}_{r_1,r_2,\ldots,r_m},\ldots,\ b^{(n)}_{r_1,r_2,\ldots,r_m})$$

of the distribution $\{P^{(\beta)}_{j_1,j_2,\ldots,j_m}\}(\beta=1,2,\ldots,n)$ is given by

$$B_{0,0,\ldots,0} = q \tag{2.10}$$

and

$$B_{r_1,r_2,\ldots,r_m} = \Big(\sum_{\ell=1}^{m} k_\ell B_{r_1,\ldots,r_\ell-1,\ldots,r_m}\Big)\Phi\Big(\sum_{i=1}^{m} r_i\mu_i\Big)\Big(I-\Phi\Big(\sum_{i=1}^{m} r_i\mu_i\Big)\Big)^{-1}, \tag{2.11}$$

where q is an invariant probability vector of $\Phi(0)$ which satisfies linear equations

$$q\Phi(0) = q \qquad \text{and} \qquad qe^{c} = 1 \ (e=(1,1,\ldots,1)).$$

Proof: Multiplying (2.9) by $\binom{j_1}{r_1}\binom{j_2}{r_2}\cdots\binom{j_m}{r_m}$ and adding the results obtained by this multiplication for every $j_1,j_2,\ldots,j_m$, it follows from (2.6) and (2.8) that

$$\begin{aligned} b^{(\beta)}_{r_1,r_2,\ldots r_m} &= \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\cdots\sum_{i_m=0}^{\infty}\sum_{\alpha=1}^{n}\int_0^\infty P^{(\alpha)}_{i_1,i_2,\ldots,i_m} \\ &\quad\cdot\sum_{\ell=1}^{m} k_\ell\Big\{\sum_{j_1=0}^{\infty}\binom{j_1}{r_1}\binom{i_1}{j_1}e^{-j_1\mu_1x}(1-e^{-\mu_1x})^{i_1-j_1}\Big\}\cdots \\ &\quad\cdots\Big\{\sum_{j_\ell=0}^{\infty}\binom{j_\ell}{r_\ell}\binom{i_\ell+1}{j_\ell}e^{-j_\ell\mu_\ell x}(1-e^{-\mu_\ell x})^{i_\ell+1-j_\ell}\Big\}\cdots \\ &\quad\cdots\Big\{\sum_{j_m=0}^{\infty}\binom{j_m}{r_m}\binom{i_m}{j_m}e^{-j_m\mu_m x}(1-e^{-\mu_m x})^{i_m-j_m}\Big\}dF_{\alpha\beta}(x) \\ &= \sum_{\alpha=1}^{n}\sum_{\ell=1}^{m} k_\ell\,(b^{(\alpha)}_{r_1,\ldots,r_\ell,\ldots r_m} + b^{(\alpha)}_{r_1,\ldots,r_\ell-1,\ldots r_m})\psi_{\alpha\beta}\Big(\sum_{i=1}^{m} r_i\mu_i\Big), \end{aligned} \tag{2.12}$$

Here, for the transformation from the second term to the third term of (2.12) we use the relationships [6]

$$\sum_{k=0}^{\infty} \binom{k}{r}\binom{n}{k} p^k (1-p)^{n-k} = \binom{n}{r} p^r$$

and

$$\binom{i+1}{r} = \binom{i}{r} + \binom{i}{r-1} .$$

Equation (2.12) gives

$$\begin{aligned} B_{r_1, r_2, \ldots, r_m} &= \sum_{\ell=1}^{m} k_\ell (B_{r_1, \ldots, r_\ell, \ldots, r_m} + B_{r_1, \ldots, r_\ell - 1, \ldots, r_m}) \\ &\quad \cdot \Phi(\sum_{i=1}^{m} r_i \mu_i) \\ &= (B_{r_1, \ldots, r_\ell, \ldots, r_m} + \sum_{\ell=1}^{m} k_\ell \, B_{r_1, \ldots, r_\ell - 1, \ldots, r_m}) \Phi(\sum_{i=1}^{m} r_i \mu_i) \end{aligned} \tag{2.13}$$

and serves as a proof for (2.11).
Equation (2.10) is self-evident.

Equation (2.11) gives

$$B_{N\delta_{1\ell},\, N\delta_{2\ell}, \ldots, N\delta_{m\ell}} = k_\ell^N q \prod_{i=1}^{N} \Phi(i\mu_\ell)(I - \Phi(i\mu_\ell))^{-1}, \tag{2.14}$$

$$\ell = 1, 2, \ldots, m, \qquad N = 1, 2, \ldots, \quad \delta_{ij} = \begin{cases} 1 & (i=j) \\ 0 & (i \neq j) \end{cases}.$$

The proof is as follows:
Assume that

$$B_{(N-1)\delta_{1\ell},\, (N-1)\delta_{2\ell}, \ldots, (N-1)\delta_{m\ell}} = k_\ell^{N-1} q \prod_{i=1}^{N-1} \Phi(i\mu_\ell)(I - \Phi(i\mu_\ell))^{-1}, \tag{2.15}$$

$$\ell = 1, 2, \ldots, m.$$

From (2.11) and (2.15), it is possible to obtain

$$\begin{aligned} & B_{N\delta_{1\ell},\, N\delta_{2\ell},\, \ldots,\, N\delta_{m\ell}} \\ &= k_\ell \, B_{(N-1)\delta_{1\ell},\, (N-1)\delta_{2\ell},\, \ldots,\, (N-1)\delta_{m\ell}} \Phi(N\mu_\ell)(I - \Phi(N\mu_\ell))^{-1} \\ &= k_\ell^N \, q \prod_{i=1}^{N} \Phi(i\mu_\ell)(I - \Phi(i\mu_\ell))^{-1} . \end{aligned}$$

Next, define

$$(2.16)\qquad b_{r_1,r_2,\ldots,r_m} := B_{r_1,r_2,\ldots,r_m} e^C .$$

It is given from (2.10) and (2.11) that

$$b_{0,0,\ldots,0} = 1,$$

$$(2.17)\qquad b_{N\delta_{1\ell},\, N\delta_{2\ell},\ldots,\, N\delta_{m\ell}} = k_\ell^N q \prod_{i=1}^{N} \Phi(i\mu_\ell)(I-\Phi(i\mu_\ell))^{-1} e^C,$$

$$\ell = 1,\ 2,\ \ldots,\ m\ .$$

It should be noted that $b_{N\delta_{1\ell},\, N\delta_{2\ell},\ldots,\, N\delta_{m\ell}}$ is equal to the N-th binomial moment of the stationary distribution for the number of customers in service stage ℓ immediately before an arrival epoch. This can also be obtained from the theory of the MR/M/∞ queue with markov renewal arrivals and exponential service times. Let us consider only the arrival instants in service stage ℓ. These arrival instants are governed by Markov renewal process in which the Laplace-Stieltjes transform of the semi-Markov kernel is

$$k_\ell\Phi(s)(I-(1-k_\ell)\Phi(s))^{-1} \ (= \sum_{n=0}^{\infty} \{(1-k_\ell)\Phi(s)\}^n\, k_\ell\Phi(s)).$$

This satisfies the following relation:

$$k_\ell\Phi(s)(I-(1-k_\ell)\Phi(s))^{-1}[I-k_\ell\Phi(s)(I-(1-k_\ell)\Phi(s))^{-1}]^{-1}$$

$$= k_\ell\Phi(s)(I-\Phi(s))^{-1}.$$

Since the service time distribution of customers in stage ℓ is an exponential one with mean μ_ℓ^{-1}, (2.17) can also be obtained by substituting $k_\ell\Phi(i\mu_\ell)(I-\Phi(i\mu_\ell))^{-1}$ into S_i of Franken's equation (15) of Ref. [3].

3. Stationary State Distribution at an Arbitrary Time

The stationary state distribution at an arbitrary time is defined by:

$$(3.1)\qquad P^*_{j_1,j_2,\ldots,j_m} := \lim_{t\to\infty} P\{Y(t) = (j_1,\ j_2,\ \ldots,\ j_m)\}.$$

The $(r_1,\ r_2,\ \ldots,\ r_m)$th binomial moment of the distribution $\{P^*_{j_1,j_2,\ldots,j_m}\}$ is denoted by

$$(3.2)\qquad b^*_{r_1,r_2,\ldots,r_m} = \sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\cdots\sum_{j_m=0}^{\infty}\binom{j_1}{r_1}\binom{j_2}{r_2}\cdots\binom{j_m}{r_m}p^*_{j_1,j_2,\ldots,j_m},$$

$$r_\ell = 0,\ 1,\ \ldots\ .$$

Theorem 3.1. The following relationship exists between $b^*_{r_1,r_2,\ldots,r_m}$ defined in (3.2) and $b_{r_1,r_2,\ldots,r_m} := B_{r_1,r_2,\ldots,r_m}e^C$ derived in the previous section.

$$(3.3)\qquad b^*_{r_1,r_2,\ldots,r_m} = \nu\sum_{\ell=1}^{m} k_\ell\, b_{r_1,\ldots,r_\ell-1,\ldots,r_m} \Big/ \sum_{\ell=1}^{m} r_\ell\mu_\ell,$$

where ν^{-1} is the mean interarrival time, i.e.,

$$\nu^{-1} = -q\frac{d}{ds}\Phi(s)\Big|_{s=0}e^C .$$

Proof: The rate conservation principle [4] provides that

$$\nu p_{0,0,\ldots,0} = \sum_{\ell=1}^{m}\mu_\ell p^*_{\delta_{1\ell},\delta_{2\ell},\ldots,\delta_{m\ell}},$$

$$(3.4)\qquad \nu p_{j_1,j_2,\ldots,j_m} + \sum_{\ell=1}^{m} j_\ell\mu_\ell p^*_{j_1,j_2,\ldots,j_m}$$

$$= \nu\sum_{\ell=1}^{m} k_\ell p_{j_1,\ldots,j_\ell-1,\ldots,j_m} + \sum_{\ell=1}^{m}(j_\ell+1)\mu_\ell p^*_{j_1,\ldots,j_\ell+1,\ldots,j_m},$$

where $p_{j_1,j_2,\ldots,j_m} := \sum_{\beta=1}^{n} p^{(\beta)}_{j_1,j_2,\ldots,j_m}$.

Multiplying every equation of (3.4) of by $\binom{j_1}{r_1}\binom{j_2}{r_2}\ldots\binom{j_m}{r_m}$ and adding the results obtained by this multiplication for every j_1, j_2, ..., j_m, it is possible to obtain

$$\nu b_{r_1,r_2,\ldots,r_m} + \sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\cdots\sum_{j_m=0}^{\infty}\binom{j_1}{r_1}\binom{j_2}{r_2}\cdots\binom{j_m}{r_m}\sum_{\ell=1}^{m} j_\ell\mu_\ell p^*_{j_1,j_2,\ldots,j_m}$$

$$(3.5)\qquad = \nu\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\cdots\sum_{j_m=0}^{\infty}\binom{j_1}{r_1}\binom{j_2}{r_2}\cdots\binom{j_m}{r_m}\sum_{\ell=1}^{m} k_\ell p_{j_1,j_2,\ldots,j_\ell-1,\ldots,j_m}$$

$$+ \sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\cdots\sum_{j_m=0}^{\infty}\binom{j_1}{r_1}\binom{j_2}{r_2}\cdots\binom{j_m}{r_m}\sum_{\ell=1}^{m}(j_\ell+1)\mu_\ell p^*_{j_1,j_2,\ldots,j_\ell+1,\ldots,j_m} .$$

Since

$$\binom{j_\ell}{r_\ell} j_\ell = \binom{j_\ell}{r_\ell+1}(r_\ell+1) + \binom{j_\ell}{r_\ell} r_\ell,$$

$$\binom{j_\ell}{r_\ell} (j_\ell+1) = \binom{j_\ell+1}{r_\ell+1}(r_\ell+1),$$

and

$$\binom{j_\ell}{r_\ell} = \binom{j_\ell-1}{r_\ell} + \binom{j_\ell-1}{r_\ell-1},$$

it follows that

$$\begin{aligned} &\nu b_{r_1,r_2,\ldots,r_m} + \sum_{\ell=1}^{m} \mu_\ell((r_\ell+1)b^*_{r_1,\ldots,r_\ell+1,\ldots,r_m} \\ &\quad + r_\ell b^*_{r_1,\ldots,r_\ell,\ldots,r_m}) \\ (3.6)\qquad &= \nu \sum_{\ell=1}^{m} k_\ell\ (b_{r_1,\ldots,r_\ell\ldots,r_m} + b_{r_1,\ldots,r_\ell-1,\ldots,r_m}) \\ &\quad + \sum_{\ell=1}^{m} (r_\ell+1)\mu_\ell b^*_{r_1,\ldots,r_\ell+1,\ldots,r_m} . \end{aligned}$$

Thus (3.3) is proved.

In particular, the first and second binomial moments of the stationary distribution of the number of customers in the $MR/H_m/\infty$ queue at an arbitrary time, $b^{*(1)}$ and $b^{*(2)}$, are respectively provided by the following corollary:

Corollary 3.2.

$$(3.7)\qquad \begin{aligned} b^{*(1)} &= (:= \sum_{j_1=0}^{\infty} \cdots \sum_{j_m=0}^{\infty} (\sum_{\ell=1}^{m} j_\ell) p^*_{j_1,\ldots,j_m}) \\ &= \sum_{\ell=1}^{m} b^*_{\delta_{1\ell},\ldots,\delta_{m\ell}} \end{aligned}$$

and

$$(3.8)\qquad \begin{aligned} b^{*(2)} &= (:= \frac{1}{2} \sum_{j_1=0}^{\infty} \cdots \sum_{j_m=0}^{\infty} (\sum_{\ell=1}^{m} j\ )(\sum_{\ell=1}^{m} j_\ell-1)\ p^*_{j_1,\ldots,j_m}) \\ &= \sum_{\ell=1}^{m} (1 + \frac{2\mu_\ell}{k_\ell} \sum_{\substack{i=1 \\ i\neq\ell}} \frac{k_i}{\mu_\ell+\mu_i})\ b^*_{2\delta_{1\ell},\ldots,2\delta_{m\ell}} , \end{aligned}$$

where

$$(3.9)\qquad b^*_{\delta_{1\ell},\ldots,\delta_{m\ell}} = \nu k_\ell/\mu_\ell$$

and

$$(3.10)\qquad b^*_{2\delta_{1\ell},\ldots,2\delta_{m\ell}} = \nu k_\ell^2 q\Phi(\mu_\ell)(1-\Phi(\mu_\ell))^{-1}e^c/2\mu_\ell.$$

Proof: Equation (3.3) gives

$$b^*_{\delta_{1\ell},\delta_{2\ell},\ldots,\delta_{m\ell}} = \nu k_\ell\, b_{0,0,\ldots,0}/\mu_\ell.$$

Thus (3.7) and (3.9) are derived.
$b^{*(2)}$ can be represented as follows:

$$(3.11)\qquad \begin{aligned} b^{*(2)} &= \frac{1}{2}\sum_{j_1=0}^{\infty}\cdots\sum_{j_m=0}^{\infty}\Big(\sum_{\ell=1}^{m} j_\ell(j_\ell-1) + 2\sum_{k\neq\ell} j_k j_\ell\Big)\, p^*_{j_1,\ldots,j_m} \\ &= \sum_{\ell=1}^{m} b^*_{2\delta_{1\ell},\ldots,2\delta_{m\ell}} + \sum_{\ell=1}^{m-1}\sum_{n=\ell+1}^{m} b^*_{\delta_{1\ell}+\delta_{1n},\ldots,\delta_{m\ell}+\delta_{mn}}. \end{aligned}$$

From (3.3), it is possible to obtain

$$(3.12)\qquad \begin{aligned} & b^*_{\delta_{1\ell}+\delta_{1n},\ldots,\delta_{m\ell}+\delta_{mn}} \\ &= \nu(k_\ell\, b_{\delta_{1n},\ldots,\delta_{mn}} + k_n\, b_{\delta_{1\ell},\ldots,\delta_{m\ell}})/(\mu_\ell + \mu_n). \end{aligned}$$

Hence, using (3.3) again,

$$(3.13)\qquad \begin{aligned} & b^*_{\delta_{1\ell}+\delta_{1n},\ldots,\delta_{m\ell}+\delta_{mn}} \\ &= 2\Big(\frac{\mu_n k_\ell}{k_n}\, b^*_{2\delta_{1n},\ldots,2\delta_{mn}} + \frac{\mu_\ell k_n}{k_\ell}\, b^*_{2\delta_{1\ell},\ldots,2\delta_{m\ell}}\Big)/(\mu_\ell+\mu_n). \end{aligned}$$

Substituting (3.13) into (3.11), it is possible to obtain (3.8).

Let us consider an infinite trunk group which is split into a finite first choice group with S servers and an infinite overflow group as shown in Fig. 1. Customers who find all serves busy in the first choice group overflow to the infinite overflow group. It is assumed that the arrival process is Poisson. Determining the stationary distribution of the number of customers in the overflow group at an arbitrary time, in particular the mean α, variance v and peakedness $z(= v/\alpha)$ of the number of customers, is basic to the method of engineering overflow groups in telephone networks. This method is called "the equivalent random method" originally proposed by Wilkinson [7] and

Bretschneider [12]. For the case where the service time distribution is an exponential one, α, v and z have been found by Kosten [11] and have already been used for engineering telephone networks. For the case where the service time distribution is a unit one, α, v and z have been found by Burke [1]. However, an analysis of the field data shows that the service time distribution is a 2-order hyperexponential one whose coefficient of variation is greater than 1 [8, 9]. Now, let us consider the first and second moments of the number of customers in the infinite overflow group for the case of 2-order hyperexponential service times, where the Poisson arrival rate is λ and the service time distribution is defind by (2.3) for $m = 2$.

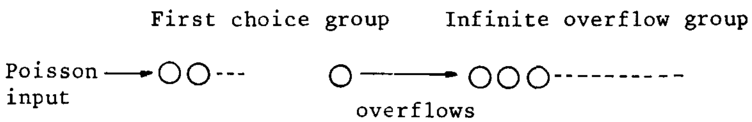

Figure 1. Split Infinite Server Group

Theorem 3.3. [10] The overflow process from the first-choice group, that is, the arrival process for the infinite overflow group is a Markov renewal one and the Laplace transform $\Phi_S(s)$ of the semi-Markov kernel can be represented by

$$\Phi_0(s) = \frac{\lambda}{s+\lambda},$$

$$\Phi_n(s) = (I(n+1) - \lambda^{-1}Q_n(s)A_{n,\,n-1}\Phi_{n-1}(s)A_{n-1,\,n})^{-1}\lambda Q_n(s),$$

$$n = 1, 2, \ldots, S,$$

where $I(n+1)$ is the $(n+1 \times n+1)$ identify matrix,

$$Q_n(s) = \begin{pmatrix} \frac{1}{s+n\mu_1+\lambda} & & & & \\ & \frac{1}{s+(n-1)\mu_1+\mu_2+\lambda} & & & \\ & & \ddots & & \\ & & \frac{1}{s+(n-i)\mu_1 + i\mu_2+\lambda} & & \\ & & & \ddots & \\ & & & & \frac{1}{s+n\mu_2+\lambda} \end{pmatrix} \quad ((n+1 \times n+1) \text{ matrix}),$$

$$A_{n,n-1} = \begin{pmatrix} n\mu_1 & & & & & \\ \mu_2 & (n-1)\mu_1 & & & & \\ & 2\mu_2 & (n-2)\mu_1 & & & \\ & & \ddots & \ddots & & \\ & & & i\mu_2 & (n-i)\mu_1 & \\ & & & & \ddots & \ddots \quad \mu_1 \\ & & & & & \ddots \quad n\mu_2 \end{pmatrix} \qquad ((n+1) \times n \text{ matrix}),$$

and

$$A_{n-1,n} = \lambda \begin{pmatrix} k_1 & k_2 & & & \\ & k_1 & k_2 & & \\ & & \ddots & \ddots & \\ & & & k_1 & k_2 \end{pmatrix} \qquad (n \times (n+1) \text{ matrix}).$$

$b^{(1)*}$ and $b^{(2)*}$ can be derived by substituting $\Phi_S(s)$ into $\Phi(s)$ in Corollary 3.2. It is well known that the stationary distribution of the number of customers in the first-choice group is independent of the service time distribution for a given load $\lambda\mu^{-1}$. Thus, the blocking probability in this group is given by

$$B_S = \frac{(\lambda\mu^{-1})^S/S!}{\sum_{i=0}^{S} (\lambda\mu^{-1})^i/i!}, \quad \text{where} \quad \mu^{-1} = \sum_{\ell=1}^{2} k_\ell \mu_\ell^{-1} .$$

Hence, the input rate to the infinite trunk group, ν, is given by

$$\nu = \lambda B_S$$

and thus

$$\alpha = b^{*(1)} = \nu\mu^{-1}.$$

The variance v is provided by

$$v = 2b^{*(2)} + \alpha - \alpha^2.$$

Figure 2 shows the effect of the coefficient of variation of service time, c_s, on peakedness of overflow streams, z. We fit a hyperexponential distribution with balanced mean [13]. As c_s increases, z decreases monotonically.

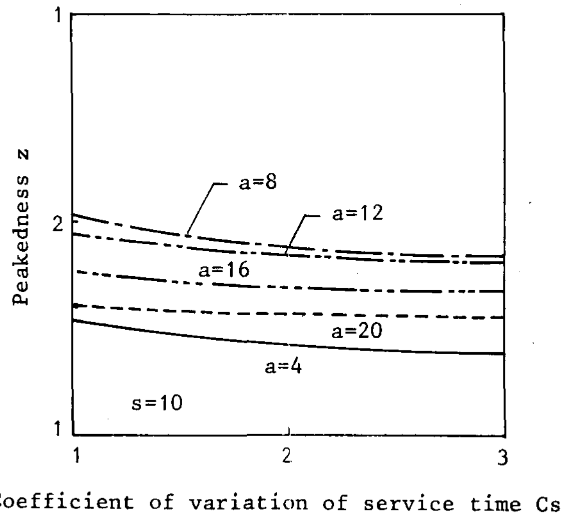

Figure 2. Coefficient of Variation of Service Time VS Peakedness ($a = \lambda\mu-1$)

4. Conclusion

An $MR/H_m/\infty$ queue having Markov renewal inputs and hyperexponential service times is analyzed. Stationary state distributions are derived at both the customer arrival epoch and an arbitrary time point. The most important measures, the mean and the variance of the number of customers in the system at an arbitrary time are explicitly represented. By applying these results to the equivalent random method, it is possible to engineer telephone networks under the assumption that the service time distribution is hyperexponential.

References

[1] P.J. Burke: The Overflow Distribution for Constant Holding Time, *Bell Syst. Tech. J.*, 50, PP. 3195-3210 (1971)

[2] E. Çinlar: *Introduction to Stochastic Processes*, Prentice-Hall Inc., Englewood Cliffs, New Jersey (1975)

[3] V.P. Franken: Erlangsche Formeln für Semimarkowschen Eingang, *Electron. Informationsverarb. Kybernetik*, 4, PP. 197-204 (1968)

[4] A. Kuczura: Piecewise Markov Processes, *SIAM J. Appl. Math.*, 24, PP. 169-181 (1973)

[5] W. Smith: The Infinitely-Many Server Queue with Semi-Markovian Arrivals

and Customer-Dependent Exponential Service Times, *Oper. Res.*, 20, PP. 907-913 (1972)

[6] L. Takács: *Introduction to the Theory of Queues*, Oxford Univ. Press, New York (1962)

[7] R.I. Wilkinson: Theories for Toll Traffic Engineering in U.S.A., *Bell Syst. Tech. J.*, 35, PP. 421-514 (1956)

[8] V.B. Iversen: Analyses of Real Teletraffic Processes Based on Computerized Measurement, *Ericsson Tech.*, 29, PP. 3-64 (1973)

[9] E. Takemori, Y. Usui and J. Matsuda: Field Data Analysis for Traffic Engineering, *Proc. 11th Intl. Teletraffic Cong.*, Session 3.2B, 5, Kyoto, Japan (1985)

[10] F. Machihara, Y. Takahashi and Y. Usui: Individual Overflow Processes from the M_1, M_2/M_1, M_2/S/S Loss System. *Review of Elec. Com. Lab.*, 34, PP. 561-567 (1986)

[11] L. Kosten: Über Sperungswahrscheinlichkeiten bei Staffelschaltungen, *Elek. Nachr. Tech.*, 13, PP. 5-12 (1937)

[12] G. Bretschneider: Die Berechnung von Leitungsgruppen für Überfliessenden Verkehr in Fernsprechwahlanlagen, *Nachrichtentechnische Zeitschrift*, 9, PP. 533-540 (1956)

[13] P.M. Morse: *Queues, Inventories and Maintenance*, John Wiley, New York (1958)

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