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ON THE REPRESENTATION OF THE RIGID SUB-SYSTEMS OF A PLANE LINK SYSTEM

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Abstract A combinatorial characterization of the rigidity of plane link systems was first established by Laman[9] (he used the term 'skeletal structure' instead of 'link system' of this paper). This paper presents a combinatorial analysis method for the structures of plane link systems. More precisely, the proposed method affords a representation of the family of all the rigid sub-systems of a plane link system.

1. Introduction

A link system is a mechanical object composed of rods and joints(links). A rod is a rigid bar connected with some other rods by joints at its ends and it moves freely around a joint.

A link system was first studied systematically by Laman[9], who established a graph-theoretic characterization of the rigidity of link systems on a 2-dimensional space. But the generalization of the results of Laman to the 3-dimensional case is not yet known. As for the further researches about this problem, we refer to [2][4].

Bolker-Crapo[3] presented a matroid-theoretic approach to the bracing problem on a one-story-building. Lovász[12] reduced the pinning down problem of a plane link system to a matroid parity problem.

This paper presents a method for the representation of the inner structures of a plane link system. The main tool used in this paper is based on matroid theory.

2. Mathematical Preliminaries

In this section we prepare the terminology and the mathematical results needed in this paper. Let S be a nonempty finite set. A relation $<$ on S is called a quasi-order if it satisfies

(1) $a < a$ (for any $a \in S$),

(2) $a < b,\ b < c \Longrightarrow a < c$.

A quasi-order is called a partial order if it satisfies (3) below in addition to (1) and (2).

(3) $a < b,\ b < a \Longrightarrow a = b$.

A quasi-order on S will induce in a natural way a partition of S and a partial order on a class of subsets of the partition. We define a relation $\equiv$ on S by

(2.1) $a \equiv b$ if $a < b$ and $b < a$.

This relation is obviously an equivalence relation on S. Let F be the collection of all the equivalence classes. Then we have a partition

(2.2) $S = \cup A : A \in F$.

A partial order $\leq$ is induced on F from the original quasi-order $<$ as follows:

(2.3) $[\ a\ ] \leq [\ b\ ]$ if $a < b$.

Here $[\ a\ ]$ $(a \in S)$ denotes an equivalence class containing a. It is easily seen that the definition of the partial order (2.3) does not depend on the choice of the representative elements a, b.

A subset $A \subseteq S$ is an ideal with respect to the quasi-order if it satisfies

(2.4) $a < b,\ b \in A \Longrightarrow a \in A$.

Note here that an empty set and the entire set S are necessarily an ideal.

Let H be a family of subsets of S. The relation $\underset{H}{<}$ on S defined by

(2.5) $a \underset{H}{<} b$ if for any $X \in H$, $b \in X$ implies $a \in X$

is a quasi-order. Then the collection R(H) of all the ideals of $\underset{H}{<}$ is a sublattice of 2^S which contains $H \cup \{\emptyset, S\}$ and minimal with respect to this property (where 2^S is the binary lattice of all the subsets of S). In general R(H) is not equal to H, i.e. larger than H. But in case that H is a sublattice of 2^S, R(H) becomes isomorphic to H ([1]).

Let L be a sublattice of 2^S. In other words L is a family of subsets satisfying

(2.8) $\quad X, Y \in L \Longrightarrow X \cap Y \in L,\ X \cup Y \in L.$

The minimal [resp. maximal] element in L is denoted by S^+ [resp. S^M]. And we write as follows:

$$S^- = S - S^M,\ S' = S^M - S^+,\ L' = \{X - S^+ : X \in L\}.$$

The partition (2.2) of S' derived from the quasi-order $\underset{L'}{<}$ is denoted by

$$S' = \cup A : A \in F.$$

Hence we have a partition

(2.9) $\quad (S^+, \{A : A \in F\}, S^-).$

of S and a partial order on F defined by (2.3). Thus a quasi-order is equivalent to a pair of a partition and a partial order. The original lattice L is reconstructed from the quasi-order $\underset{L'}{<}$ by

(2.10) $\quad L = \{S^+ \cup X' : X' \in R(L')\}$

where R(L') is the collection of all the ideals of the quasi-order $\underset{L'}{<}$.

A set-function f on S satisfying

(2.11) $\quad f(X) + f(Y) \geq f(X \cap Y) + f(X \cup Y) \qquad$ for $X, Y \subseteq S$

is called a submodular function. Let us denote the collection of subsets which attain the minimum of a set-function h by

(2.12)
$$\begin{aligned} L &= \left\{X \subseteq S : h(X) = \min\{h(Y) : Y \subseteq S\}\right\} \\ &= \arg\min\{h(X) : X \subseteq S\}. \end{aligned}$$

As is easily seen, if h is submodular then L is a sublattice of 2^S.

Let H be a family of sets. H is called a p-intersecting family (where p is a non-negative integer) if the following holds:

(2.13) $\quad$ if $X, Y \in H$ and $|X \cap Y| \geq p$ then $X \cap Y \in H$ and $X \cup Y \in H$.

A real-valued function g on a p-intersecting family H is called submodular if it satisfies

(2.14) $\quad g(X) + g(Y) \geq g(X \cap Y) + g(X \cup Y)$

$\qquad$ for $X, Y \in H$ with $|X \cap Y| \geq p$.

Furthermore if H is a p-intersecting family and g is a submodular function on H then

(2.15) $\quad D = \{X \in H : g(X) = c\} \qquad$ (where $c = \min\{g(Y) : Y \in H\}$)

is again a p-intersecting family.

A matroid M is defined as a pair (E, r) where E is a nonempty finite set and r is an integer-valued set-function on E which satisfies

$$(2.16)\quad \begin{aligned} &1)\ \ 0 \le r(X) \le |X| \qquad (X \subseteq E),\\ &2)\ \ \text{if } X \subseteq Y \text{ then } r(X) \le r(Y),\\ &3)\ \ r(X) + r(Y) \ge r(X \cap Y) + r(X \cup Y). \end{aligned}$$

r is called the rank function of M. $r(E)$ is called the rank of M and denoted by rank(M). Take any $e \in E$ and let $E' = E - \{e\}$. Here we suppose $r(\{e\}) = 1$. Then

$$(2.17)\quad r'(X') = r(X' \cup \{e\}) - 1 \qquad \text{for } X' \subseteq E'$$

is a rank function of a matroid on E', which we denote by M/e. A subset $A \subseteq E$ is called an independent set of M if $r(A) = |A|$. I(M) denotes the collection of all independent sets of M. Let $G = (V, E)$ be a graph with an edge set E. M(G) denotes the cycle matroid of G. I(M(G)) is equal to the collection of the edge sets which are cycleless.

Let $M_1 = (E, r_1)$ and $M_2 = (E, r_2)$ be a pair of matroids on the same ground set E. Then

$$(2.18)\quad I = \{A \cup B : A \in I(M_1),\ B \in I(M_2)\}$$

can be proved to form the collection of the independent sets of a certain matroid on E. This matroid is denoted by $M_1 \vee M_2$ and called a union matroid of M_1 and M_2. The rank of a union matroid is determined by

$$(2.19)\quad \begin{aligned} &\text{rank } (M_1 \vee M_2)\\ &\qquad = \min \{r_1(X) + r_2(X) + |E - X| : X \subseteq E\}. \end{aligned}$$

For other terms and properties of matroids, we refer to Iri [7][8] and Welsh [20].

Let us denote the partition (2.9) of E associated with the following lattice

$$(2.20)\quad L = \arg\min \{r_1(X) + r_2(X) + |E - X| : X \subseteq E\}$$

by

$$(2.21)\quad (E^+, \{A : A \in F\}, E^-),$$

which we call the partition associated with a union matroid $M_1 \vee M_2$ ([14]). The algorithm to construct a base of a union of two matroids is well-known [5][10][11]. By a slight modification, this algorithm can be easily extended to one which determines the partition (2.21) and the partial order on F [15]

[17].

3. Link Systems, Degree of Freedom and Rigidity

In this paper we consider only plane link systems, i.e. those on a 2-dimensional space. So a link system is defined as a pair of an undirected graph $G = (V, E)$ and a mapping p of V into a 2-dimensional space $\mathbb{R}^2$. Intuitively an edge of G corresponds to a rod and a vertex to a joint. Here p is called a realization of the link system. Naturally $G = (V, E)$ is a simple graph, i.e. containing no self-loops nor parallel edges. For simplicity, G is supposed to be a connected graph throughout this paper. Let $V = \{1, 2, \ldots, n\}$ and $E = \{e_1, e_2, \ldots, e_m\}$.

We shall consider an admissible motion of a link system and define its degree of freedom, its rigidity, etc. following the manner in [13]. Let $p_i(t)$ be a position of $i \in V$ on $\mathbb{R}^2$ at time t. Since a rod is rigid,

$$||p_i(t) - p_j(t)|| = \text{constant} \tag{3.1}$$

holds for $(i, j) \in E$. From this, we have

$$(p_i(0) - p_j(0))\cdot(v_i - v_j) = 0 \tag{3.2}$$

where $v_i = (dp_i/dt)(0)$, $v_j = (dp_j/dt)(0)$.

An assignment $(v_i : i \in V)$ which satisfies (3.2) is an admissible infinitesimal motion of the link system. The collection W of all the admissible infinitesimal motions forms a subspace of $(\mathbb{R}^2)^V$. So the dimension of W is the degree of freedom of the motion of a link system. And the degree of freedom of outer motion is a sum of 2 of parallel movement and 1 of rotation, that is equal to 3. Hence the degree of freedom of the inner motion of a link system is

$$f(G, p) = \dim(W) - 3. \tag{3.3}$$

When $f(G, p) = 0$, a link system (G, p) is said to be rigid.

We use the following notation:

$$p_i = (x_i, y_i), \qquad v_i = (u_i, w_i) \qquad (i \in V),$$

$$x = {}^t[x_1, x_2, \ldots, x_n],\ y = {}^t[y_1, y_2, \ldots, y_n],$$

$$u = {}^t[u_1, u_2, \ldots, u_n],\ w = {}^t[w_1, w_2, \ldots, w_n],$$

For the sake of formulation, we assume here that the edges of G are oriented.

The choice of orientations is arbitrary since the properties of a link system, e.g. rigidity etc., are irrelevant to this choice. So we write

$$(3.4)\qquad a_{ki} = \begin{cases} 1 & \text{edge } e_k \text{ starts from vertex } i, \\ -1 & \text{edge } e_k \text{ ends at vertex } i, \\ 0 & \text{otherwise,} \end{cases}$$

$$a_k = (a_{k1}, a_{k2}, \ldots, a_{kn}) \quad \text{for } k = 1, \ldots, m.$$

Then (3.2) is rewritten as

$$(3.5)\qquad (a_k\, x)(a_k\, u) + (a_k\, y)(a_k\, w) = 0 \qquad \text{for } k = 1, \ldots, m.$$

Combining (3.5), we have

$$(3.6)\qquad H\begin{pmatrix} u \\ w \end{pmatrix} = 0 \qquad \text{where } H = \begin{pmatrix} (a_1\, x)a_1, & (a_1\, y)a_1 \\ \vdots & \vdots \\ (a_m\, x)a_m, & (a_m\, y)a_m \end{pmatrix}.$$

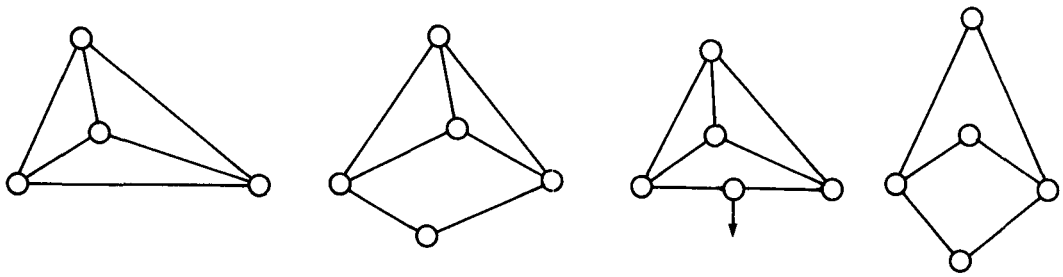
The linear space W is nothing but the kernel space of the linear map H. We denote by $M(G, p)$ the matroid on E derived from the row vectors of H. Then

$$(3.7)\qquad \begin{aligned} f(G, p) &= \dim(W) - 3 = \dim(\ker H) - 3 \\ &= 2n - 3 - \operatorname{rank}(H) = 2n - 3 - \operatorname{rank}(M(G, p)). \end{aligned}$$

If a subset A of E is independent in $M(G, p)$ then it implies that the subsystem spanned by A has no redundant edge with respect to the rigidity, i.e. the deletion of any edge in A will increase the degree of freedom of inner motions. Especially in case that the link system is rigid, any base B of $M(G, p)$ has a property that the system still remains rigid even after the deletion of the edges other than B and B is minimal with respect to this property. A link system (G, p) is called independent if $\operatorname{rank}(M(G, p)) = |E|$. An independent link system contains no redundant edge with respect to its rigidity.

Example 1. Some examples of link systems are shown in Fig. 1. Here (a) and (b) are rigid while (b') and (c) are not rigid. And (b) and (c) are independent while (a) and (b') are not independent.

In the above example even though the underlying graphs of (b) and (b') are the same, (b) is rigid and (b') is not so. The cause of this is that the three vertices on the bottom of (b') lie on a line and hence the infinitesimal motion designated by an arrow in Fig. 1 is admissible. This difficulty can be avoided if it is assumed that the vertices are situated in a 'general' position.



(a) (b) (b') (c)

Fig. 1

Being situated in a 'general' position is guaranteed in a mathematical term if the coordinates x_i, y_i (i = 1, 2,..., n) under a realization p are algebraically independent over the rational field. If this assumption is met, the realization p is called generic. A link system (G, p) is called generic if p is generic.

Let (G, p) be a generic link system and r the rank function of the cycle matroid M(G). Then the rank of the matroid M(G, p) is determined by

$$\text{rank}(M(G, p)) = \min \left\{ \sum_{i=1}^{s} (2r(A_i) - 1) : A_1, A_2, \ldots, A_s \text{ is a partition of } E \text{ into nonempty subsets} \right\} \tag{3.8}$$

(Lovász-Yemini[12]). Furthermore,

Proposition 3.1. The following three conditions are equivalent.

(1) (G, p) is independent, i.e. rank(M(G, p)) = $|E|$,

(2) $2r(X) - 1 \geq |X|$ for any $X \subseteq E$ with $X \neq \emptyset$ (Laman[9]),

(3) $E = E^-$ holds in the partition (2.21) associated with a union matroid of $M(G) \vee M(G)$.

Proof: ((1)$\Longrightarrow$(2)) Take any nonempty set $X \subseteq E$. Consider a partition obtained by letting $A_1 = X$ and deviding $E - X$ into singleton sets. By (3.8),

$$2r(X) - 1 + |E - X| \geq |E|,$$

i.e. $2r(X) - 1 \geq |X|$.

((2)$\Longrightarrow$(1)) Obvious from (3.8) and the following.

$$\sum_{i=1}^{s} (2r(A_i) - 1) \geq \sum_{i=1}^{s} |A_i| = |E|.$$

((2)$\Longleftrightarrow$(3)) Let

$$L = \arg\min \{2r(X) + |E - X| : X \subseteq E\}. \tag{3.9}$$

Then $\quad 2r(X) - 1 \geq |X| \quad$ for $X \neq \emptyset$

$$\Longleftrightarrow \min\{2r(X) + |E - X| : X \neq \emptyset\} \geq |E| + 1$$

$$\Longleftrightarrow L = \{\emptyset\} \Longleftrightarrow E = E^-.$$

Proposition 3.2. Let (G, p) be a generic independent link system and r the rank function of $M(G)$. Then the following four conditions are equivalent.

(1) (G, p) is rigid,

(2) $2r(E) - 1 = |E|$ (Laman[9]),

(3) For $e \in E$, let $G(e)$ denote a graph obtained from G by inserting a new edge e' parallel to e. Then for every $e \in E$, $E^+ = E^- = \emptyset$ holds in the partition (2.21) derived from the union matroid $M(G(e)) \vee M(G(e))$,

(4) For any $e \in E$, there exists a pair of trees T_1 and T_2 in $G(e)$ with $T_1 \cup T_2 = E \cup \{e'\}$ and $T_1 \cap T_2 = \emptyset$.

Proof: ((1)$\Longleftrightarrow$(2)) By the definition of independency, $\mathrm{rank}(M(G, p)) = |E|$. Since G is connected, $r(E) = n - 1$. Hence,

$$f(G, p) = 2n - 3 - \mathrm{rank}(M(G, p)) = 0$$

$$\Longleftrightarrow 2n - 3 - |E| = 0 \Longleftrightarrow 2r(E) - 1 - |E| = 0.$$

((2)$\Longleftrightarrow$(3)) It is easy to see the following:

$$2r(E) = |E| + 1$$

$$\Longleftrightarrow 2r(E \cup \{e'\}) = 2r(E) = |E| + 1 \Longleftrightarrow L \ni E \cup \{e'\} \Longleftrightarrow E^- = \emptyset$$

where $L = \arg\min\{2r(A) + |E \cup \{e'\} - A| : A \subseteq E \cup \{e'\}\}$.

Since (G, p) is independent, we have

$$2r(A) \geq 2r(A') \geq |A'| + 1 \geq |A|$$

for any $A \subseteq E \cup \{e'\}$ with $A' = A - \{e'\}$. This implies $L \ni \emptyset$, so that $E^+ = \emptyset$.

((3)$\Longleftrightarrow$(4)) The proof is not difficult, but rather lenghy. So we omit it here and refer to [13][15].

4. Inner Structures of Link Systems and Examples

Suppose (G, p) to be a generic link system and let r denote the rank function of the cycle matroid $M(G)$. A set-function h defined by

(4.1) $\quad h(X) = 2r(X) - 1 - |X| \quad$ for $X \subseteq E$

is obviously a submodular function. We define

(4.2) $\quad H = \{A \subseteq E : h(A) = a,\ |A| \geq 2\}.$

(where $a = \min\{h(X) : X \subseteq E,\ |X| \geq 2\}$)

As in (2.15), H is a 2-intersecting family. In case that (G, p) is independent and rigid, $A \in H$ implies that the sub-system spanned by A is rigid. That is, the family H coincides with the collection of all the rigid sub-systems of (G, p) except singleton sets.

Fix any $e \in E$ and let $E' = E - \{e\}$. r' denotes the rank function of $M(G)/e$ defined by (2.17). And let

$$H(e) = \{A \in H : e \in A\}, \tag{4.3}$$

$$C(e) = \begin{cases} \cap A : A \in H(e) & \text{if } H(e) \neq \emptyset, \\ E & \text{if } H(e) = \emptyset. \end{cases} \tag{4.4}$$

It is easy to see that

$$a \underset{H}{<} b \Longleftrightarrow C(a) \subseteq C(b) \Longleftrightarrow a \in C(b). \tag{4.5}$$

Hence a quasi-order $\underset{H}{<}$ is determined if all the $C(a)$ for $a \in E$ are established. Furthermore,

$$C(a) = \{b \in E : b \underset{H}{<} a\}. \tag{4.6}$$

Especially $C(a)$ is an ideal of the quasi-order $\underset{H}{<}$.

Hereafter we suppose (G, p) to be independent. Then

$$\begin{aligned} &H(e) \neq \emptyset \\ &\Longleftrightarrow \min\{2r(X) + |E - X| : X \subseteq E,\ e \in X,\ |X| \geq 2\} = |E| + 1 \\ &\Longleftrightarrow \min\{2r'(X') + |E' - X'| : X' \subseteq E',\ |X'| \geq 1\} = |E'|. \end{aligned} \tag{4.7}$$

We define

$$L(e) = \arg\min\{2r'(X') + |E' - X'| : X' \subseteq E'\}. \tag{4.8}$$

By definition and assumption, we have

$$\emptyset \in L(e), \qquad L(e) - \{\emptyset\} = H(e). \tag{4.9}$$

We write the partition (2.21) of E' derived from $L(e)$ as follows:

$$(E^+(e),\ \{A : A \in F(e)\},\ E^-(e)). \tag{4.10}$$

Note here that $E^+(e)$ is necessarily an empty set since $\emptyset \in L(e)$ and that (4.10) is a partition associated with the union matroid $(M(G)/e) \vee (M(G)/e)$. Once (4.10) is established, $C(e)$ is constructed by the following way.

(i) In case where $F(e) = \emptyset$ and $E^-(e) = E'$. Then let $C(e) = E$.

(ii) In case where $F(e) \neq \emptyset$. Take a minimal element A in $F(e)$ (minimal with respect to the partial order on $F(e)$). If there exists no minimal element other than A then let $C(e) = A \cup \{e\}$. Otherwise,

i.e., if there exist a multiple number of minimal elements in F(e) then let C(e) = e .

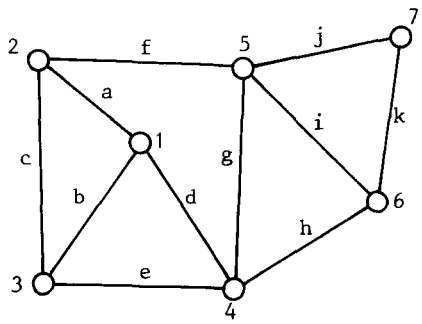
Following the above, each C(e) is found and from this the quasi-order is determined as is stated in (4.5).

Let us consider the meaning of the obtained quasi-order $\underset{H}{<}$. H is the collection of all the nontrivial rigid sub-systems of (G, p) and each subset in H is realized as an ideal of the quasi-order $\underset{H}{<}$. So it can be said that the quasi-order $\underset{H}{<}$ gives a representation of the inner structures of a link system, more precisely a representation of the collection of all the rigid sub-systems of a link system. In general if the given link system becomes larger, the cardinality of H will tremendously increase and it will be almost impossible to count out all the elements of H. In contrast with this, the quasi-order $\underset{H}{<}$ gives a simple way of representing the structure of a link system and evenmore it can be constructed by an efficient algorithm as is stated before.

Sugihara[18][19] described a decomposition of a plane link system, which is equivalent to that in this paper, in a rather intuitive way. He did not either referred to the relation between his decomposition and the principal partition of graphs nor established an effective algorithm which provides the decomposition. Fujishige[6] proposed a similar decomposition method for sub-modular systems.

Example 2. Let (G_1, p_1) be the link system shown in Fig. 2. This link system is both independent and rigid. The family H of (4.2) associated with this link system is the following:

$$H = \Big\{\{a,b,c\},\ \{b,d,e\},\ \{g,h,i\},\ \{i,j,k\},\ \{a,b,c,d,e\},\ \{g,h,i,j,k\},\ \{a,b,c,d,e,f,g\},\ \{a,b,c,d,e,f,g,h,i\},\ \{a,b,c,d,e,f,g,h,i,j,k\}\Big\}.$$



$G_1 = (E_1, V_1)$

$E_1 = \{a, b, \ldots, k\}$

$V_1 = \{1, 2, 3, 4, 5\}$

Fig. 2

Let us try to construct C(x) for some $x \in E$.

Case of x = b. Fig. 3 is the graph G_1/b. The partition (4.10) and the partial order corresponding to G_1/b is shown in Fig. 4. In Fig. 4, there are two minimal subsets {a,c} and {d,e}, so that C(b) = {b}.

Case of x = f. Fig. 5 is G_1/f. The partition and the partial order derived from G_1/f is Fig. 6. There exists a unique minimal subset {a,b,c,d,e,g} in Fig. 6. Hence C(f) = {a,b,c,d,e,f,g}.

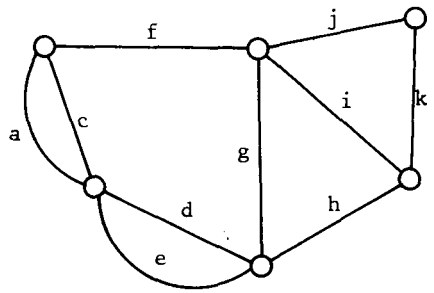


Fig. 3

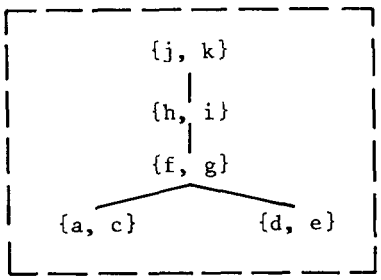


Fig. 4

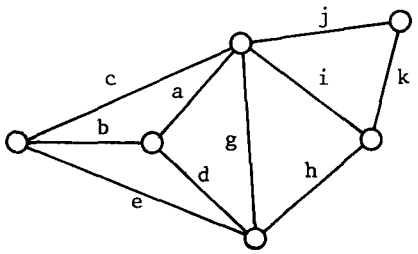


Fig. 5

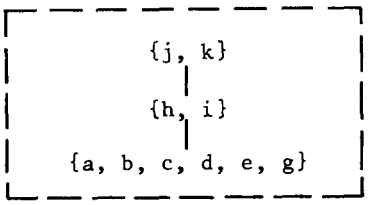
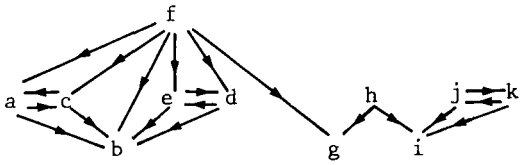


Fig. 6

Likewise all the other C(x) are determined. The resultant C(x) for each $x \in E$ is shown in Table 1. From this table, the quasi-order $\underset{H}{<}$ will be fixed as in Fig. 7. Fig. 8 shows the partition with the partial order which is equivalent to the quasi-order $\underset{H}{<}$ by means of (2.1), (2.2) and (2.3).

x	C(x)	x	C(x)
a	{a b c}	g	{g}
b	{b}	h	{g h i}
c	{a b c}	i	{i}
d	{b d e}	j	{i j k}
e	{b d e}	k	{i j k}
f	{a b c d e f g}		

Table 1



(where $p \leftarrow q$ implies $p \underset{H}{<} q$)

Fig. 7

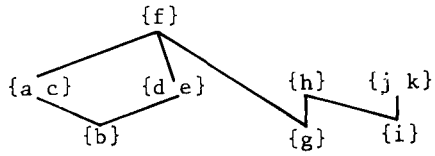


Fig. 8

Example 3. Next we shall consider the link system (G, p) in Fig. 9. This link system is rigid, but not independent, i.e.

$$a = \min \{h(X) : X \subseteq E, |X| \geq 2\} = -1 < 0.$$

Even though this link system is not independent, an analogous argument is possible using

$$H' = \{A \subseteq E : h(A) = -1, |A| \geq 2\}$$

in place of H. Fig. 10 shows the partition with the partial order which is equivalent to the quasi-order $\underset{H'}{<}$. The resultant partition identifies that (G_2, p_2) is not independent since it contains a non-independent sub-system spanned by {a,b,c,d,e,f}. The partial order in Fig. 10 indicates the process to construct the whole system from the kernel of {a,b,c,d,e,f} under keeping the rigidity.

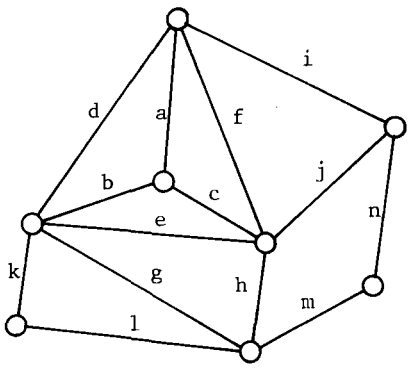


Fig. 9

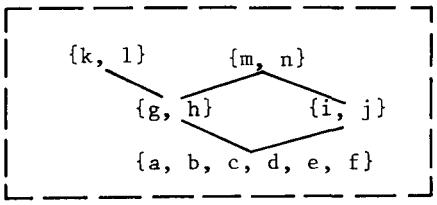


Fig. 10

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