Journal of the Operations Research
Society of Japan
Vol. 29, No. 4, December 1986

REPLACEMENT POLICY FOR COMPONENTS IN A MARKOVIAN DETERIORATING SYSTEM WITH MINIMAL REPAIR

Mamoru Ohashi
Ehime University

(Received May 27, 1983: Final July 7, 1986)

Abstract This paper deals with a discrete time replacement model for a system with minimal repair. The system consists of n components under Markovian deterioration. The transition probability of its component is not independent each other and the cost of replacing several components jointly is less than the sum of the costs of separate replacements. Then we investigate the structural properties of the optimal replacement policy which minimizes the expected total discounted cost, and propose the simple replacement policy which lets easily implementable policy. Also a numerical example is presented.

1. Introduction

In this paper, we consider a discrete time replacement model for a system with minimal repair. The system consists of n components under Markovian deterioration. The transition probability of each component is not independent each other, and the cost of replacing several components is less than the sum of the costs of the separate replacement. Furthermore, when the system failure is observed, we allow a controller to carry out minimal repair for such a system. The system is observed at the beginning of discrete time periods and is classified into one of the possible number of states. If a system failure is observed then it is classified into one of the possible number of states showing the degree of the system failure. Then the possible actions are "no action", "replacement of each component" and "minimal repair of the system". A replacement of each component means changing the component for a new one, and minimal repair means mending the system failure, namely, this repair brings the failure state of the system back to the operating state.

The replacement policies for stochasically independent two-component system were studied by Sethi[8], Berg[2], and Radner and Jorgenson[6]. Ohashi and Nishida[4] have considered the replacement policy for components in two-component system possessing stochastic and economic dependence, and shown that the optimal replacement policy has the form of a control limit policy. Furthermore, Ohashi[5] have discussed the replacement policy for the coherent system consisting of n repairable components with two states. The objective of this paper is to clarify the structure of the optimal replacement policy which minimizes the expected total discounted cost for the n-component system with minimal repair. We show that the optimal replacement policy has the control limit type, and discuss the properties of the optimal region of the action. A numerical example is also presented. Furthermore, from these results we propose the simple replacement policy, called (ABC)-policy, which lets to easily implementable policy and which is a generalization of (n,N) policy and (t,T) policy.

2. Model Formulation

We consider a system consisting of n components under Markovian deterioration. Let $N=\{1,2,\dots,n\}$ be the set of components and E_i be the set of deterioration levels of component i, $i\varepsilon N$. We assume that E_i is partially ordered set with relation $\geq$, and lattice with minimal and maximal elements. The minimal element 0 of E_i represents the best state of component i, and the maximal element e_i represents the worst state. Let $(\Omega,\mathfrak{J},P)$ be the probability space and (E_i,β_i) be the measurable space of component i where β_i contains all singleton events $\{x\}$, $x\varepsilon E_i$. For each $t\varepsilon T=\{0,1,\dots\}$, let $X_i(t)$ be a measurable function which maps from $(\Omega,\mathfrak{J})$ to (E_i,β_i). Then the stochastic process $\{X_i(t);\ t\varepsilon T\}$ represents the behavior of deterioration levels of component i, and $X_c(t)=(X_1(t),\dots,X_n(t))$ denotes the deterioration levels of components. Similarly, let E_0 be the set of damage levels of the system failure. It is assumed that E_0 is partially ordered set with relation $\geq$ and lattice with minimal and maximal elements. Specially the minimal element 0 of E_0 represents the operating state of the system and other states denote the system failure. Let $X_0(t)$ be a measurable function which maps from $(\Omega,\mathfrak{J})$ to (E_0,β_0), where (E_0,β_0) is a measurable space and β_0 contains all singleton events $\{x\}$, $x\varepsilon E_0$. Then the stochastic process $\{X_0(t);\ t\varepsilon T\}$ represents the behavior of the state showing the degree of the system failure, and $X(t)=(X_0(t),X_c(t))$ denotes the deterioration levels of the whole system. Therefore the evolution of the state of the system is discribed by the stochastic process $Z=\{X(t);\ t\varepsilon T\}$ with state space $(E,\beta)=(\prod_{i=0}^{n} E_i, \prod_{i=0}^{n} \beta_i)$. For simplisity we assume that E_i is subset of nonnegative real number R_+.

Next we consider a discrete time replacement model for the system with minimal repair. The minimal repair brings the failure state, $x_0\neq 0$, back to the operating state, $x_0=0$. However, the deterioration level x_c of its components is not disturbed after performing minimal repair. Therefore when this system fails at state $x=(x_0,x_c)$, $x_0\neq 0$, the minimal repair brings the system to state $x=(0,x_c)$. For example, we consider an automobile whose components are a tire, a battery, a body, an engine and so on. When the tire punctures we consider that the system failure occurs, and the minimal repair means mending the puncture. Then we consider that $X_0(t)$ represents the minimal repair cost and $X_c(t)$ represents the deterioration levels of its components.

The system is observed at the beginning of discrete time periods $t\varepsilon T$, and classified into one of the possible number of E. Then the possible actions are "no action", "replacement of each component" and "minimal repair of the system". Let $a=(a_1,\ldots,a_n)$ represent the action taken for components, where $a_i\varepsilon D_i=\{0,1\}$ is an action taken for component i and $D_c=\prod_{i=1}^{n} D_i$. Here $a_i=1$ means replacing component i and $a_i=0$ means keeping it. Let $D=D_c\cup\{m\}$ be an action space of this replacement model, where $a=m$ means carrying out minimal repair of the system. When an action $a\varepsilon D^0=\{a\varepsilon D|\ a\neq 0\}$ is taken on the system with state $x=(x_0,x_1,\ldots,x_n)$, the time consumption required for replacement or minimal repair, $T(x,a)$, has a probability distribution $G(t;x,a)$ with a finite mean.

Let $Z=\{X(t);\ t\varepsilon T\}$ represent the behavior of the deterioration levels of the system under no action. Let $x_c=(x_1,\ldots,x_n)$ and $x_c^a=(x_1^{a_1},\ldots,x_n^{a_n})$, where $x_i^{a_i}=x_i$ if $a_i=0$, and $x_i^{a_i}=0$ if $a_i=1$. If an action $a\varepsilon D^1=\{a\varepsilon D_c|\ a\neq 0\}$ is taken on the system with state $X(t)=x$, then we have $X(t+T(x,a))=(0,x_c^a)$, and if $a=m$ then $X(t+T(x,a))=(0,x_c)$. We are interested in the state of the system. Thus we introduce the following stochastic process $Z^\pi=\{Z^\pi(t);\ t\varepsilon T\}$ under a stationary replacement policy $\pi(.)$:

$$Z^\pi(t)=X(T_t) \tag{2.1}$$

where $0=T_0<T_1<\ldots$, $T_t=T_{t-1}+S(Z^\pi(t-1),\pi)$ and

$$S(Z^\pi(t-1),\pi)=\begin{cases} 1 & \textit{if } \pi(Z^\pi(t-1))=0, \\ T(Z^\pi(t-1),\pi) & \textit{otherwise}. \end{cases}$$

Then the transition probability $P^{\pi}(x,U)$ of the stochastic process Z^{π} is given by for each $U\varepsilon\beta$

$$P^{\pi}(x,U)=P[Z^{\pi}(t+1)\varepsilon U|\ Z^{\pi}(t)=x,\ \pi(x)=a]$$

$$(2.2)\qquad =\begin{cases} Q(x,U) & \textit{if } a=0, \\ 1 & \textit{if } a=m \textit{ and } (0,x_c)\varepsilon U, \\ 1 & \textit{if } a\varepsilon D^1 \textit{ and } (0,x_c^a)\varepsilon U, \\ 0 & \textit{otherwise}, \end{cases}$$

where $Q(x,U)$ is a probability measure on (E,β) for each $x\varepsilon E$.

For the costs associated with the discrete time replacement model of the system with minimal repair, we consider a replacement cost $C_i(x_i)$ of component i per period, a set up cost $K(x)$ of replacement per period, an operating cost $B(x)$ per period and minimal repair cost $M(x)$ per period when the system is in state x at the beginning of the period. We assume that all costs and transition probabilities are known, and all costs are bounded and nonnegative. Now let $V_\alpha(x)$ be the minimum expected total discounted cost with discounted factor $\alpha\varepsilon[0,1)$ when the state of the system is x at the beginning. Then $V_\alpha(x)$ obeys the following functional equation:

$$(2.3)\qquad \begin{aligned} V_\alpha(x)=\min[&B(x)+\alpha\int V_\alpha(u)Q(x,du), \\ &\int\{M(x)\frac{1-\alpha^t}{1-\alpha}+\alpha^t V_\alpha(0,x_c)\}dG(t;x,m), \\ &\min_{a\varepsilon D^1}\int\{(K(x)+\sum_{i\varepsilon A(a)}C_i(x_i))\frac{1-\alpha^t}{1-\alpha} \\ &\qquad +\alpha^t V_\alpha(0,x_c^a)\}dG(t;x,a)] \end{aligned}$$

where $A(a)=\{i\varepsilon N|\ a_i=1,\ a\varepsilon D^1\}$.

3. Structure of Optimal Replacement Policy

Our aim is to examine the structural properties of the optimal replacement policy for components in the system with minimal repair, under the criterion of the expected total discounted cost. First we seek the structural property of the optimal expected total discounted cost function. Let B(E) be the set of all bounded real valued β-measurable function on E, and F(E) be the subset of B(E) such that for $f\varepsilon B(E)$, $x'\geq x$ in E implies $f(x')\geq f(x)$. Furthermore, let S(E) be the family of all increasing set U

on E. The following theorem shows the structure of $V_\alpha(x)$ under the following condition.

Condition 3.1. (1) $Q(x,U)\varepsilon F(E)$ for all $U\varepsilon S(E)$,
(2) $B(x)\varepsilon F(E)$, $K(x)\varepsilon F(E)$, $M(x)\varepsilon F(E)$ and $\sum_{i\varepsilon N} C_i(x_i)\varepsilon F(E)$,
(3) $1-G(t;x,a)\varepsilon F(E)$ for each $a\varepsilon D^0$ and $t\varepsilon T$.

This condition (1) means that the system under no action has a tendency for monotonically deterioration. Condition (2) means that the operating cost, the set up cost of replacement, the minimal repair cost and the replacement cost of components increase as a function of deterioration levels of the system. Condition (3) means that the replacement time and minimal repair time have a tendency for monotonically increase as a function of deterioration levels of the system.

Theorem 3.1. If Condition 3.1 holds, then the optimal expected total discounted cost function $V_\alpha(x)$ is a member of F(E).

Proof: The proof is carried out by the method of successive approximations. Let $V_0(x)=0$ and define recurrsively:

$$
\begin{aligned}
V_k(x)=\min[&B(x)+\alpha\int V_{k-1}(u)Q(x,du),\\
&\int\{M(x)\frac{1-\alpha^t}{1-\alpha}+\alpha^t V_{k-1}(0,x_c)\}dG(t;x,m),\\
&\min_{a\varepsilon D^1}\int\{(K(x)+\sum_{i\varepsilon A(a)} C_i(x_i))\frac{1-\alpha^t}{1-\alpha}\\
&\qquad+\alpha^t V_{k-1}(0,x_c^a)\}dG(t;x,a)].
\end{aligned}
\tag{3.1}
$$

We first show $V_k(x)\varepsilon F(E)$ for each k. We have $\frac{1-\alpha^t}{1-\alpha}dG(t;x,a)\varepsilon F(E)$ from Condition 3.1 (3). Therefore for $k=1$ it follows trivially from Condition 3.1 (2). Suppose that for some k $V_k(x)\varepsilon F(E)$, $V_k(x)\leqq(K(x)+C_i(x_i))/(1-\alpha)$ for each $i\varepsilon N$ and $V_k(0,x_c)\leqq M(x)/(1-\alpha)$. Then under Condition 3.1, we obtain that

$$B(x)+\alpha\int V_k(u)Q(x,du)$$

is a member of F(E) by the induction hypothesis. On the other hand, we obtain for each $a\varepsilon D^1$

$$(K(x)+\sum_{i\varepsilon A(a)} C_i(x_i))\frac{1-\alpha^t}{1-\alpha}+\alpha^t V_k(0,x_c^a)$$

is increasing in t from the induction hypothesis. Thus

$$\int\{(K(x)+\sum_{i\varepsilon A(a)} C_i(x_i))\frac{1-\alpha^t}{1-\alpha}+\alpha^t V_k(0,x_c^a)\}dG(t;x,a)$$

is a member of F(E). Similarly, for $a=m$ we obtain that

$$\int\{M(x)\frac{1-\alpha^t}{1-\alpha}+\alpha^t V_k(0,x_c)\}dG(t;x,m)$$

is a member of F(E). Therefore we have $V_{k+1}(x)\varepsilon F(E)$ from equation (3.1). Next we show that $V_{k+1}(x)\leqq(K(x)+C_i(x_i))/(1-\alpha)$ and $V_{k+1}(0,x_c)\leqq M(x)/(1-\alpha)$. From the equation (3.1) we have for each $i\varepsilon N$

$$V_{k+1}(x)\leqq\int\{(K(x)+C_i(x_i))\frac{1-\alpha^t}{1-\alpha}+\alpha^t V_k(0,(0_i,x_c))\}dG(t;x,(1_i,0))$$
$$\leqq(K(x)+C_i(x_i))/(1-\alpha).$$

The last inequality is true since $V_k(0,(0_i,x_c))\leqq V_k(x)\leqq(K(x)+C_i(x_i))/(1-\alpha)$. Similarly, we have

$$V_{k+1}(x)\leqq\int\{M(x)\frac{1-\alpha^t}{1-\alpha}+\alpha^t V_k(0,x_c)\}dG(t;x,m)$$
$$\leqq M(x)/(1-\alpha).$$

Then we have $V_{k+1}(0,x_c)\leqq M(x)/(1-\alpha)$ from $V_{k+1}(x)\varepsilon F(E)$. Thus we obtain that $V_{k+1}(x)$ is a member of F(E). Then we obtain $V_k(x)\varepsilon F(E)$ for each k. Since $\alpha<1$ and all costs are bounded, it is easy to see that $V_k(x)\to V_\alpha(x)$ as $k\to\infty$ for each $x\varepsilon E$. Therefore we have $V_\alpha(x)\varepsilon F(E)$. ||

Next, the structural properties of the optimal replacement policy are studied. Let $F(D^1)$ be a set of all bounded real valued increasing function on D^1.

Condition 3.2. (1) $M(x)\leqq K(x)+C_i(x_i)$ for each $i\varepsilon N$,
(2) $G(t;x,a)\leqq G(t;x,m)$ for each $a\varepsilon D^1$,
(3) $1-G(t;x,a)\varepsilon F(D^1)$ for each t and x.

This condition (1) states that the minimal repair cost is not larger than the replacement cost. Similarly, condition (2) states that the minimal repair time is not stochastically larger than the replacement time. Condition (3) means that the replacement time has a trend for monotonously increase as a function of the number of the replacement components.

The following theorem shows a simple property of the optimal replacement policy.

Theorem 3.2. Assume that Conditions 3.1 and 3.2 hold. If the deterioration level of component i is in the best state 0, then the action to keep component i is optimal.

Proof: Concerning with the action to keep component i we define

$$[V_\alpha(x)]_k^i=\min[B(x)+\alpha\int V_\alpha(u)Q(x,du),$$
$$\int\{M(x)\frac{1-\alpha^t}{1-\alpha}+\alpha^t V_\alpha(0,x_c)\}dG(t;x,m),$$
$$\min_{a\in D_0^i}\int\{(K(x)+\sum_{j\in A(a)}C_j(x_j))\frac{1-\alpha^t}{1-\alpha}$$
$$+\alpha^t V_\alpha(0,x_c^a)\}dG(t;x,a)],$$

and concerning with the action to replace component i we define

$$[V_\alpha(x)]_r^i=\min_{a\in D_1^i}\int\{(K(x)+\sum_{j\in A(a)}C_j(x_j))\frac{1-\alpha^t}{1-\alpha}+\alpha^t V_\alpha(0,x_c^a)\}dG(t;x,a),$$

where $D_j^i=\{a\in D^1 \mid a_i=j\}$ for each $j\in D_i$. Then we have for $D_0^i\neq\phi$

$$[V_\alpha(x)]_k^i-[V_\alpha(x)]_r^i$$
$$\leqq\min_{a\in D_0^i}\int\{(K(x)+\sum_{j\in A(a)}C_j(x_j))\frac{1-\alpha^t}{1-\alpha}+\alpha^t V_\alpha(0,x_c^a)\}dG(t;x,a)$$
$$-\min_{a\in D_1^i}\int\{(K(x)+\sum_{j\in A(a)}C_j(x_j))\frac{1-\alpha^t}{1-\alpha}+\alpha^t V_\alpha(0,x_c^a)\}dG(t;x,a)$$

and for each $a\in D_0^i$ and $x_i=0$

$$\int\{(K(x)+\sum_{j\in A(a)}C_j(x_j))\frac{1-\alpha^t}{1-\alpha}+\alpha^t V_\alpha(0,x_c^a)\}dG(t;x,a)$$
$$-\int\{(K(x)+\sum_{j\in A(1_i,a)}C_j(x_j))\frac{1-\alpha^t}{1-\alpha}+\alpha^t V_\alpha(0,x_c^{(1_i,a)})\}dG(t;x,(1_i,a))$$
$$\leqq\int\{(K(x)+\sum_{j\in A(a)}C_j(x_j))\frac{1-\alpha^t}{1-\alpha}+\alpha^t V_\alpha(0,x_c^a)\}dG(t;x,a)$$
$$-\int\{(K(x)+\sum_{j\in A(a)}C_j(x_j)+C_i(x_i))\frac{1-\alpha^t}{1-\alpha}$$
$$+\alpha^t V_\alpha(0,x_c^{(1_i,a)})\}dG(t;x,a)$$
$$=\int\{-C_i(x_i)\frac{1-\alpha^t}{1-\alpha}+\alpha^t(V_\alpha(0,x_c^a)-V_\alpha(0,x_c^{(1_i,a)}))\}dG(t;x,a)$$
$$\leqq 0.$$

The first inequality is true from Condition 3.2 (3) and the proof of Theorem 3.1, and the second inequality follows from $V_\alpha(0,(0_i,x_c)^a)=V_\alpha(0,(0_i,x_c)^{(1_i,a)})$. Furthermore from Condition 3.2 (1) (2), we have for $D_0^i=\phi$ and $x_i=0$

$$\min[B(x)+\alpha\int V_\alpha(u)Q(x,du),$$
$$\int\{M(x)\frac{1-\alpha^t}{1-\alpha}+\alpha^t V_\alpha(0,x_c)\}dG(t;x,m)]$$

$$-\int\{(K(x)+C_i(x_i))\frac{1-\alpha^t}{1-\alpha}+\alpha^t V_\alpha(0,(0_i,x_c))\}dG(t;x,(1_i,0))$$
$$\leqq 0.$$

Thus we can easily obtain that the difference $[V_\alpha(x_0,(0_i,x_c))]^i_k-[V_\alpha(x_0,(0_i,x_c))]^i_r\leqq 0$. Therefore the result directly follows. ||

Let $E^i=E_1\times\ldots\times E_{i-1}\times E_{i+1}\times\ldots\times E_n$ and $J_i(x_0,x_c^i)$ be a subset of E_i for each $(x_0,x_c^i)\varepsilon E_0\times E^i$. Let $S(E_i)$ be the family of all increasing set in E_i.

Definition 3.1. Let π_i be a stationary replacement policy for component i in the system. Then π_i is said to be a *control limit policy* with respect to component i if and only if there exists a *replacement set* $J_i(x_0,x_c^i)\varepsilon S(E_i)$ for each $(x_0,x_c^i)\varepsilon E_0\times E^i$ such that if state x_i is in the set $J_i(x_0,x_c^i)$, replace component i, otherwise, do not replace.

The structure of the optimal replacement policy will be clarified under the following additional conditions.

Condition A. (1) $C_i(x_i)=C_i$ and $G(t;x,a)=G(t)$,
(2) $M(x)-K(x)\varepsilon F(E)$ and $M(x)-K(x)\leqq 0$,
(3) $B(x)-M(x)\int\frac{1-\alpha^t}{1-\alpha}dG(t)\varepsilon F(E)$.

Theorem 3.3. Assume that Conditions 3.1 and 3.2 hold. If Condition A is satisfied, then there exists a control limit policy π_i with respect to component i.

Proof: Under Condition A (1), we have

$$\begin{aligned}[V_\alpha(x)]^i_k-[V_\alpha(x)]^i_r=\min[&B(x)-K(x)\int\frac{1-\alpha^t}{1-\alpha}dG(t)+\alpha\int V_\alpha(u)Q(x,du),\\ &\int\{(M(x)-K(x))\frac{1-\alpha^t}{1-\alpha}+\alpha^t V_\alpha(0,x_c)\}dG(t),\\ &\min_{a\varepsilon D_0^i}\int\{\sum_{j\varepsilon A(a)}C_j\frac{1-\alpha^t}{1-\alpha}+\alpha^t V_\alpha(0,x_c^a)\}dG(t)]\\ &-\min_{a\varepsilon D_1^i}\int\{\sum_{j\varepsilon A(a)}C_j\frac{1-\alpha^t}{1-\alpha}+\alpha^t V_\alpha(0,x_c^a)\}dG(t).\end{aligned}$$

Then from Condition A (2) and (3) we can easily obtain that the difference $[V_\alpha(x)]^i_k-[V_\alpha(x)]^i_r$ is a member of $F(E_i)$. Because

$$B(x)-K(x)\int\frac{1-\alpha^t}{1-\alpha}dG(t)\varepsilon F(E)$$

is obtained from Condition A (2) and (3). Thus the result follows from the definition of the control limit policy. ||

Remark 3.1. Let $J_i^*(x_0,x_c^i)$ be the optimal replacement set minimizing the expected total discounted cost. Then since the state space of component i is a subset of nonnegative real number R_+, there exists a control limit $x_i^*(x_0,x_c^i)\varepsilon E_i \cup\{\infty\}$ such that $J_i^*(x_0,x_c^i)=[x_i^*(x_0,x_c^i),\infty)\cap E_i$, where $x_i^*(x_0,x_c^i)=\infty$ for $J_i^*(x_0,x_c^i)=\phi$.

Remark 3.2. This theorem remains true even if we suppose that $M(x)-K(x)\varepsilon F(E_c)$ and $B(x)-M(x)\int\frac{1-\alpha^t}{1-\alpha}dG(t)\varepsilon F(E_c)$ in place of Condition A (2) and (3).

Corollary 3.1. Under conditions of Theorem 3.3 if the action to replace component i with the worst state e_i is optimal, then we have $x_i^*(x_0,e^i)=x_i^*(0,e^i)$ for each $x_0\varepsilon E_0$, where $e^i=(e_1,\ldots,e_{i-1},e_{i+1},\ldots,e_n)$.

Proof: Since the action to replace component i with the worst state e_i is optimal, we have for each $x_0\varepsilon E_0$

$$\begin{aligned}
&[V_\alpha(x_0,(x_i,e))]_k^i-[V_\alpha(x_0,(x_i,e))]_r^i \\
&=\int\{(K(x_0,(x_i,e))+\sum_{j\varepsilon N}C_j-C_i)\frac{1-\alpha^t}{1-\alpha}+\alpha^t V_\alpha(0,(x_i,0))\}dG(t) \\
&\quad-\int\{(K(x_0,(x_i,e))+\sum_{j\varepsilon N}C_j)\frac{1-\alpha^t}{1-\alpha}+\alpha^t V_\alpha(0)\}dG(t) \\
&=\int\{\alpha^t(V_\alpha(0,(x_i,0))-V_\alpha(0))-C_i\frac{1-\alpha^t}{1-\alpha}\}dG(t).
\end{aligned}$$

Thus the result follows from the definition of $x_i^*(x_0,x_c^i)$ and Theorem 3.3. ||

Corollary 3.2. Under conditions of Theorem 3.3 if the action to replace component i with the worst state e_i is optimal, then we have $x_i^*(x_0,e^i)\leqq x_i^*(x_0,0^i)$ for each $x_0\varepsilon E_0$.

Proof: From Theorem 3.2 we have

$$\begin{aligned}
&[V_\alpha(x_0,(x_i,0))]_k^i-[V_\alpha(x_0,(x_i,0))]_r^i \\
&=\min[B(x_0,(x_i,0))+\alpha\int V_\alpha(u)Q((x_0,(x_i,0)),du), \\
&\qquad\int\{M(x_0,(x_i,0))\frac{1-\alpha^t}{1-\alpha}+\alpha^t V_\alpha(0,(x_i,0))\}dG(t)] \\
&\quad-\int\{(K(x_0,(x_i,0))+C_i)\frac{1-\alpha^t}{1-\alpha}+\alpha^t V_\alpha(0)\}dG(t) \\
&\leqq\int\{M(x_0,(x_i,0))-K(x_0,(x_i,0))\}\frac{1-\alpha^t}{1-\alpha}dG(t) \\
&\qquad+\int\{\alpha^t(V_\alpha(0,(x_i,0))-V_\alpha(0))-C_i\frac{1-\alpha^t}{1-\alpha}\}dG(t).
\end{aligned}$$

And from the proof of Corollary 3.1 we have

$$[V_\alpha(x_0,(x_i,e))]_k^i-[V_\alpha(x_0,(x_i,e))]_r^i$$
$$=\int\{-C_i\frac{1-\alpha^t}{1-\alpha}+\alpha^t(V_\alpha(0,(x_i,0))-V_\alpha(0))\}dG(t).$$

Then from $K(x)-M(x)\geqq 0$, we have

$$[V_\alpha(x_0,(x_i,0))]_k^i-[V_\alpha(x_0,(x_i,0))]_r^i$$
$$\leqq[V_\alpha(x_0,(x_i,e))]_k^i-[V_\alpha(x_0,(x_i,e))]_r^i$$

Thus we can easily obtain that $x_i^*(x_0,e^i)\leqq x_i^*(x_0,0^i)$ by the definition of $x_i^*(x_0,x_c^i)$. ||

Remark 3.3. This corollary is concerned with (n,N) policy introduced by Radner and Jorgenson[6]. If the system consists of two components and the state space of component 2 is $E_2=\{0,1\}$, then this corollary asserts that the optimal replacement policy for the system with minimal repair is an (n,N) policy with $n=x_1^*(x_0,1)$ and $N=x_1^*(x_0,0)$ for each $x_0\varepsilon E_0$.

Let $F(E_0)$ be a set of all bounded real increasing function on E_0, and $\overline{F}(E_0)$ be a set of all bounded real decreasing function on E_0.

Theorem 3.4. Assume that Conditions 3.1 and 3.2 hold. If Condition A is satisfied, then a control limit $x_i^*(x_0,x_c^i)$ is a member of $\overline{F}(E_0)$.

Proof: Under Condition A (1), we have

$$\begin{aligned}[V_\alpha(x)]_k^i-[V_\alpha(x)]_r^i=\min[&B(x)-K(x)\int\frac{1-\alpha^t}{1-\alpha}dG(t)+\alpha\int V_\alpha(u)Q(x,du),\\ &\int\{(M(x)-K(x))\frac{1-\alpha^t}{1-\alpha}+\alpha^tV_\alpha(0,x_c)\}dG(t),\\ &\min_{a\varepsilon D_0^i}\int\{\sum_{j\varepsilon A(a)}C_j\frac{1-\alpha^t}{1-\alpha}+\alpha^tV_\alpha(0,x_c^a)\}dG(t)]\\ &-\min_{a\varepsilon D_1^i}\int\{\sum_{j\varepsilon A(a)}C_j\frac{1-\alpha^t}{1-\alpha}+\alpha^tV_\alpha(0,x_c^a)\}dG(t).\end{aligned}$$

Then from Condition A (2) and (3) we can easily obtain that the difference $[V_\alpha(x)]_k^i-[V_\alpha(x)]_r^i$ is a member of $F(E_0)$. Thus the result follows from the definition of $x_i^*(x_0,x_c^i)$. ||

Remark 3.4. This theorem remains true even if we suppose that $M(x)-K(x)\varepsilon F(E_0)$ and $B(x)-M(x)\int\frac{1-\alpha^t}{1-\alpha}dG(t)\varepsilon F(E_0)$ in place of Condition A (2) and (3). Also, this theorem is concerned with (t,T) policy introduced by Tahara and Nishida[9] in the case of single-component system.

Theorem 3.5. Assume that Conditions 3.1 and 3.2 hold. If Condition A is satisfied, then there exists a control limit $x_0^*(x_c)$ for each $x_c \varepsilon E_c$ such that the action to carry out minimal repair for the system is optimal if and only if the failure damage x_0 exceeds $x_0^*(x_c)$.

Proof: From the functional equation (2.3), we have for each $x_c \varepsilon E_c$

$$[V_\alpha(x)]_{a=0}-[V_\alpha(x)]_{a=m}$$
$$=B(x)+\alpha\int V_\alpha(u)Q(x,du)-\int\{M(x)\frac{1-\alpha^t}{1-\alpha}+\alpha^t V_\alpha(0,x_c)\}dG(t)$$
$$=\alpha\int V_\alpha(u)Q(x,du)+\int\{B(x)-M(x)\frac{1-\alpha^t}{1-\alpha}-\alpha^t V_\alpha(0,x_c)\}dG(t).$$

Then we have $[V_\alpha(x)]_{a=0}-[V_\alpha(x)]_{a=m} \varepsilon F(E_0)$ by Condition A (3). Also we have

$$[V_\alpha(x)]_{a=0}-[V_\alpha(x)]_{a\varepsilon D^1}$$
$$=B(x)+\alpha\int V_\alpha(u)Q(x,du)$$
$$-\min_{a\varepsilon D^1}\int\{(K(x)+\sum_{j\varepsilon A(a)} C_j)\frac{1-\alpha^t}{1-\alpha}+\alpha^t V_\alpha(0,x_c^a)\}dG(t)$$
$$=\int\{B(x)-K(x)\frac{1-\alpha^t}{1-\alpha}\}dG(t)+\alpha\int V_\alpha(u)Q(x,du)$$
$$-\min_{a\varepsilon D^1}\int\{\sum_{j\varepsilon A(a)} C_j\frac{1-\alpha^t}{1-\alpha}+\alpha^t V_\alpha(0,x_c^a)\}dG(t).$$

Then we have $[V_\alpha(x)]_{a=0}-[V_\alpha(x)]_{a\neq 0} \varepsilon F(E_0)$. Thus we can easily find the result. ||

Remark 3.5. In this replacement model, it is assumed that the action to replace several components in the state $x=(x_0,x_c)$ $(x_0\neq 0)$ contains the action to carry out minimal repair.

The following properties clarify the structure of the *optimal region* $G(a)=\{x\varepsilon E \mid \pi(x)=a\}$.

Property 3.1. Assume that Conditions 3.1 and 3.2 hold. If Condition A is satisfied, then the optimal region $G(0)$ is closed in the sence that $x^1 \wedge x^2 \varepsilon G(0)$ for all x^1 and x^2 in $G(0)$.

Proof: For x^1 and x^2 in $G(0)$, we have $x_i^1 \wedge x_i^2 \leq x_i^1$ or $x_i^1 \wedge x_i^2 \leq x_i^2$ for each $i\varepsilon N$. Thus the result follows from Theorem 3.3 and 3.5. ||

Property 3.2. Assume that Conditions 3.1 and 3.2 hold. If Condition A is satisfied, then the optimal region $G(\mathbb{1})$ is closed in the sence that $x^1 \vee x^2 \varepsilon G(\mathbb{1})$ for all x^1 and x^2 in $G(\mathbb{1})$.

Proof: The proof is similar to that of property 3.1. ||

Property 3.3. Assume that Conditions 3.1 and 3.2 hold. If Condition A is satisfied, then the optimal region $G(0_i,\mathbb{1})$ is closed in the sence that

$x^1 \vee x^2 \in G(0_i, \mathbb{1})$ for all x^1 and x^2 in $G(0_i, \mathbb{1})$.

Proof; First if $x^1 \vee x^2 = x^1$ or $x^1 \vee x^2 = x^2$, then the result is obvious. Let $x^1 \vee x^2 = (x_0^1 \vee x_0^2, \ldots, x_n^1 \vee x_n^2)$, then we have $x^1 \vee x^2 \in G(0_i, \mathbb{1}) \cup G(\mathbb{1})$ from $x_i^1 \vee x_i^2 \leqq x_i^1$ $x_i^1 \vee x_i^2 \leqq x_i^2$ for each $i \in N$ and Theorem 3.3. Then we have

$$\begin{aligned}
&[V_\alpha(x^1 \vee x^2)]_{a=\mathbb{1}} - [V_\alpha(x^1 \vee x^2)]_{a=(0_i, \mathbb{1})} \\
&= \int \{ (K(x^1 \vee x^2) + \sum_{j \in N} C_j) \frac{1-\alpha^t}{1-\alpha} + \alpha^t V_\alpha(0) \} dG(t) \\
&\quad - \int \{ (K(x^1 \vee x^2) + \sum_{j \neq i} C_j) \frac{1-\alpha^t}{1-\alpha} + \alpha^t V_\alpha(0, (x_i^1 \vee x_i^2, 0)) \} dG(t) \\
&= \int \{ C_i \frac{1-\alpha^t}{1-\alpha} + \alpha^t (V_\alpha(0) - V_\alpha(0, (x_i^1 \vee x_i^2, 0))) \} dG(t) \\
&\geqq 0.
\end{aligned}$$

The last inequality is true since $x_i^1 \vee x_i^2 = x_i^1$ or $x_i^1 \vee x_i^2 = x_i^2$, and x^1 and x^2 in $G(0_i, \mathbb{1})$. Then the result is obvious. ||

Property 3.4. Assume that Conditions 3.1 and 3.2 hold. If Condition A is satisfied, then the optimal region $G(\mathbb{1})$ is a member of $S(E)$.

Proof: The result is easily obtained by Theorems 3.3 and 3.5. ||

4. Example

In this section we consider a two-component system with minimal repair. Let $E_0 = \{0, 1\}$ and $E_1 = E_2 = \{0, 1, \ldots, 7\}$ be the state space. The transition probability is given by

$$Q(x, y) = P^0_{xy_0} \cdot P^1_{x_1 y_1} \cdot P^2_{x_2 y_2}$$

where $P^0_{xy_0}$ is the transition probability of the system failure damage, and $P^i_{x_i y_i}$ $(i=1,2)$ is the transition probability of component i. To illustrate the optimal replacement policy, we consider a numerical example. The transition probability matrix P^0 of the system failure damage is given in Table 4.1, and the transition probability matrix P^i of component i is given in Table 4.2. The operating cost $B(x) = B_0(x_0) + B_1(x_1) + B_2(x_2)$, the replacement cost $C_i(x_i)$, the set up cost $K(x) = K(x_0)$, and the minimal repair cost $M(x) = M(x_0)$ are given in Table 4.3. Furthermore the replacement time and minimal repair time are one period. Then Conditions 3.1, 3.2 and A are satisfied.

Table 4.1. Transition probability matrix $P^0=\{P_{(x_0,x_1,x_2)y_0}\}$

(a) $x_0=0$ and $y_0=0$

$x_1 \setminus x_2$	0	1	2	3	4	5	6	7
0	1.0	0.9	0.8	0.7	0.6	0.5	0.3	0.1
1	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.1
2	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
3	0.7	0.6	0.5	0.4	0.3	0.2	0.2	0.1
4	0.6	0.5	0.4	0.3	0.3	0.2	0.1	0.1
5	0.5	0.4	0.3	0.2	0.2	0.2	0.1	0.0
6	0.3	0.3	0.2	0.2	0.1	0.1	0.1	0.0
7	0.1	0.1	0.1	0.1	0.1	0.0	0.0	0.0

(b) $x_0=0$ and $y_0=1$

$x_1 \setminus x_2$	0	1	2	3	4	5	6	7
0	0.0	0.1	0.2	0.3	0.4	0.5	0.7	0.9
1	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.9
2	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
3	0.3	0.4	0.5	0.6	0.7	0.8	0.8	0.9
4	0.4	0.5	0.6	0.7	0.7	0.8	0.9	0.9
5	0.5	0.6	0.7	0.8	0.8	0.8	0.9	1.0
6	0.7	0.7	0.8	0.8	0.9	0.9	0.9	1.0
7	0.9	0.9	0.9	0.9	0.9	1.0	1.0	1.0

(c) $x_0=1$ and $y_0=0$

$P_{(x_0,x_1,x_2)y_0}=0$ for all x_1 and x_2,

(d) $x_0=1$ and $y_0=1$

$P_{(x_0,x_1,x_2)y_0}=1$ for all x_1 and x_2.

Table 4.2. Transition probability matrix $P^i=\{P_{x_iy_i}\}$

$x_i \setminus y_i$	0	1	2	3	4	5	6	7
0	0.00	0.20	0.20	0.15	0.15	0.10	0.05	0.05
1	0.00	0.25	0.20	0.15	0.15	0.10	0.10	0.05
2	0.00	0.10	0.20	0.20	0.15	0.15	0.10	0.10
3	0.00	0.05	0.10	0.15	0.25	0.20	0.15	0.10
4	0.00	0.05	0.05	0.10	0.25	0.20	0.20	0.15
5	0.00	0.00	0.05	0.10	0.15	0.25	0.25	0.20
6	0.00	0.00	0.00	0.05	0.05	0.10	0.40	0.40
7	0.00	0.00	0.00	0.00	0.00	0.00	0.00	1.00

Table 4.3. Costs $B_i(x_i)$, $C_i(x_i)$, $B_0(x_0)$, $K(x_0)$ and $M(x_0)$

x_i	0	1	2	3	4	5	6	7
$B_i(x_i)$	0	5	10	15	20	25	30	35
$C_i(x_i)$	70	70	70	70	70	70	70	70

x_0	0	1
$B_0(x_0)$	0	300
$K(x_0)$	100	160
$M(x_0)$	100	160

The optimal replacement policy for components in a two-component system with minimal repair is shows in Figure 4.1 in the case of α=0.95. This example shows that the optimal replacement policy is similar to the (n,N) policy with n=5 and N=6 in the case of $x_0=0$, and is fairly close to the (n,N) policy with n=5 and N=5 in the case of $x_0=1$.

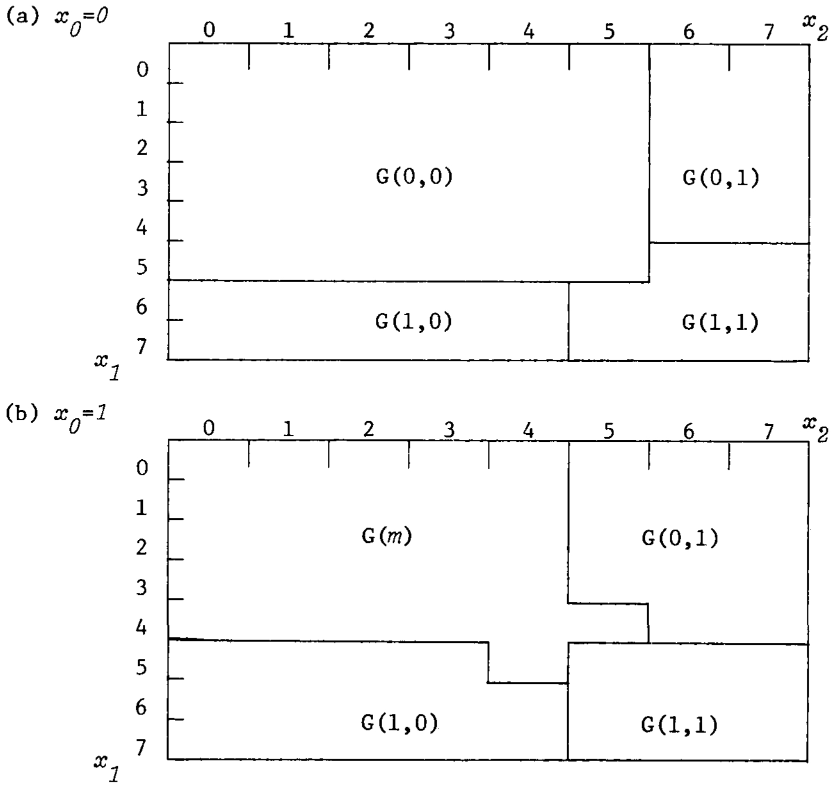

Figure 4.1. Optimal replacement policy

5. Concluding Remarks

we are interesting in an elegant and simple replacement policy which lets to easily implementable policy. The optimal replacement policy, however, is not so simple, and requires large scale computations for implementation. Thus from previous theorems and corollaries we consider a simple replacement policy for component i, called (ABC)-policy, such that ;

(1) if $0 \leqq x_i < A$, keep component i,

(2) if $A \leqq x_i < C$, replace component i concurrently if other components are replaced,

(3) if $B \leqq x_i < C$, replace component i if the system fails,

(4) if $C \leqq x_i$, replace component i at once,

where x_i is the state of component i, and $A=x_i^*(x_0, e^i)$ for all x_0, $C=x_i^*(0, 0^i)$ and $B=x_i^*(x_0, 0^i)$ for some x_0 $(x_0 \neq 0)$. $A \leqq C$ and $A \leqq B$ are followed from Corollaries 3.1 and 3.2, and $B \leqq C$ is followed from Theorem 3.4. This (ABC)-policy is simple structure and easily implementable policy. However, to determinie the values of A, B and C for each component, in general, is not easy. These values can be determined by using the method of successible approximations or policy improvement. It is a furture problem to find effective algorithm to determine the values of A, B and C by using the simple structure of (ABC)-policy.

If the failure of the system is not considered, then this (ABC)-policy is similar to (n,N) policy, introduced by Radner and Jorgenson[6], with A=n and C=N. Furthermore if the opportunistic replacement is not considered, then this (ABC)-policy is similar to (t,T) policy, introduced by Tahara and Nishida[9], with B=t and C=T. Thus (ABC)-policy is a generalization of (n,N) policy and (t,T) policy. In the previous example, it can be seen that the optimal replacement policy is fairly close to the (ABC)-policy with A=5, B=5 and C=6. Thus in some cases it might be better to use a simple (ABC)-policy than a more complex one.

Acknowledgement

The author wishes to express his thanks to referees for their very helpful comments on the paper.

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Mamoru OHASHI: Department of Information Sciences, Faculty of General Education, Ehime University, Matsuyama, 790, Japan.