

A CAPACITY-CONSTRAINED SINGLE-FACILITY MULTI-PRODUCT PRODUCTION PLANNING MODEL

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Abstract A single-facility multi-product production planning model with time-variant capacity constraints is analyzed, in which known demands must be satisfied. In the model, in every production period the single facility produces a fixed number of distinct products each taking a prespecified part of the involved production activity (or input resource quantity). Concave production and inventory costs are assumed. Both the cases of nonbacklogging and backlogging are considered, and piecewise concave inventory costs are assumed for the case of backlogging. The structure of an optimal solution is characterized and this is used in developing a tree-search algorithm.

1. Introduction

Wagner and Whitin [6] have analyzed a single-product single-facility production and inventory problem under the assumptions that demands for the products were known, and that the production and inventory cost functions were concave over a finite planning horizon of N periods. Zangwill [7] has considered a similar problem where backlogging is allowed. Florian and Klein [2] have also studied the problem under the assumption that there are period-dependent capacity-limits. They devised a dynamic programming, shortest-path algorithm only for the case of constant capacity. Love [4] has considered a somewhat more general model than that of Florian and Klein [2]. Recently, Sung [5] studied a single-facility multi-product problem, for which no capacity constraint was involved.

In this paper, we consider a production planning model for a single-facility multi-product problem with dynamic capacity constraints, where a single input resource is employed. In every production period, the facility manufactures a number of products each taking a fixed part of the involved

production activity to satisfy its own known demands. The finite planning horizon is N periods.

Such a problem frequently occurs in two functionally-distinctive classes of manufacturing industries; one in the chemical industry and the other one in the machinery industry. Each of these cases is described in more detail. In the first case, the involved input resource contains various components in a fixed quantity proportion which are distinctively converted into their corresponding commercial products in the fixed proportion. Therefore, all the products are distinctively variant in both quality and quantity. As an example, an oil refinery problem can be considered, where each unit of crude oil is refined to produce two distinctive products, say gasoline and a fine chemical resource, in the fixed quantity ratios, of $\alpha_1 > 0$ and $\alpha_2 > 0$, respectively. As another example, consider a production system, for which each operation generates by-product in a fixed ratio, say α . In the other case, the employed facility (plant) with a certain number of distinctive sub-operation lines attached operates to supply a resource (fixed) simultaneously to all the attached lines, each of which generates its own commercial products. Therefore, all the products are in the same quality but their quantities are distinctively dependent on the capacities (fixed) of the involved sub-operation lines. As an example, a steel processing system can be considered, where each unit of steel is processed to produce nail and wire simultaneously in a prespecified weight proportion.

We will analyze two cases, nonbacklogging and backlogging cases. In nonbacklogging case, both the production and inventory costs are assumed concave, while in the backlogging case piecewise concave inventory costs are considered.

According to Florian, Lenstra and Rinnooy Kan [3] the problem that we consider here is NP-complete, making it doubtful that any good algorithm (in the worst case sense) exists. Baker, Dixon, Magazine and Silver [1] have exploited the special structure of the capacitated single-product single-facility problem without backlogging to develop an optimal tree-search procedure more efficient than the usual dynamic programming approach.

The objective of this paper is to find a useful characterization of optimal plans. In both the nonbacklogging and backlogging cases, solutions consist of independent subplans, called "at-most-one partial sequence", in which each positive production level is at capacity, except for at most one period. Also, the inventory level is nonzero in every period except the last. This characterization is to be further specified in terms of the computational load reduction, so that the usual combinatorial approaches

can be more efficiently applied for a solution search. Then, it is shown how the tree search procedure of Baker et al. [1] can be directly applied to our nonbacklogging case.

2. Model Formulation without Backlogging

Consider a M -product problem with each product i , ($i=1,2,\dots,M$), taking $\alpha_i > 0$ of the production amount in every period. That is, the amount produced in every period $(x_1, x_2, \dots, x_N) = X$ satisfies the relations

$$(1) \quad x_t = \sum_{i=1}^M x_{ti} \quad \text{and}$$

$$x_{ti} = x_t \cdot (\alpha_i / M_\alpha)$$

where $M_\alpha = \sum_{i=1}^M \alpha_i$, and x_{ti} is the production amount of product i in period t ($t=1,2,\dots,N$). Let r_1, r_2, \dots, r_N represent the known demands over the planning horizon N , where r_t , ($t=1,2,\dots,N$), is the demand vector in period t , $r_t = (r_{t1}, \dots, r_{tM})$. Each component $r_{ti} \geq 0$ represents the demand for product i in period t . Note that each demand r_{ti} is not necessarily required to take α_i part of the total demand $\sum_{j=1}^M r_{tj}$ in each period t .

In this section, we assume that demands must always be satisfied from either production or inventory storage, hence

$$(2) \quad I_{ti} \geq 0 \quad \text{for all } t \text{ and } i,$$

where I_{ti} represents the inventory quantity of product i at the end of period t and is the i^{th} component of the inventory vector I_t . There is no loss in generality in assuming that both the initial and final inventories are zero.

The reason is that if the equation of " $R_{1N}(i) / \sum_{j=1}^M R_{1N}(j) = \alpha_i / M_\alpha$ (where $R_{hm}(i) = \sum_{t=h}^M r_{ti}$)" does not hold for any i , some additional (artificial) demands can be added to each of the last demands to obtain equality. Furthermore, as discussed earlier, we assume capacity restrictions on each period's production, i.e.,

$$(3) \quad 0 \leq x_t \leq c_t, \quad t=1,2,\dots,N.$$

Evidently, a feasible solution and consequently an optimal solution will exist iff

$$(4) \quad \sum_{h=1}^t c_{hi} \geq \sum_{h=1}^t r_{hi}, \text{ for all } t \text{ and } i,$$

where $c_{hi} = c_h \cdot (\alpha_i / M_\alpha)$. Henceforth, we shall assume that eq. (4) is satisfied.

The problem is then to minimize the total costs of production and inventory, say

$$Z(X) = \sum_{t=1}^N P_t(x_t) + \sum_{t=1}^N \sum_{i=1}^M H_{ti}(I_{ti})$$

subject to,

$$\begin{aligned} I_{ti} &= X_{1t}(i) - R_{1t}(i), \\ 0 &\leq x_t \leq c_t, \\ (A) \quad x_{ti} &= (\alpha_i / \alpha_1) x_{t1}, \\ I_{ti} &\geq 0, \\ I_{0i} &= 0 = I_{Ni}, \text{ for all } i \text{ and } t, \end{aligned}$$

where $\sum_{h=1}^t c_{hi} \geq \sum_{h=1}^t r_{hi}$, $X_{hm}(i) = \sum_{t=h}^m x_{ti}$, ($h=1, \dots, N-1; m=h+1, \dots, N$), and P_t is the concave production cost function for period t , and H_{ti} gives the concave cost of storing the inventory quantity I_{ti} from period t to period $t+1$.

To constraints of problem (A) define a closed bounded convex set. Since Z is concave, it attains its minimum at an extreme point of this set. Let D be the set of all extreme points of the solution set. A characterization of D will be made in the next section, which will facilitate finding an optimal production plan.

3. Characterization of Extreme Points

Let us introduce the concept of an "at-least-one exact requirement" sequence that will form the basis for our characterization of D .

Definition. Period n is called an "at-least-one exact requirement point" if $I_{ni} = 0$ for some $i \in \{1, 2, \dots, M\}$, and a production sequence (x_1, x_2, \dots, x_N) is called an "at-least-one exact requirement" sequence if for every $n (1 \leq n \leq N)$ such that $I_{ni} = 0$ for some $i \in \{1, 2, \dots, M\}$, $\sum_{t=1}^n x_t = L(n)$, where $L(n) = \max_i \{ (M_\alpha / \alpha_i) R_{1n}(i) \}$. Further, let L_{mn} be called a "production sequence", which

represents a subset of a feasible production plan X that includes the components of X for all periods between the two successive at-least-one exact requirement points m and n , i.e.,

$$L_{mn} = \left\{ \begin{array}{l} x_t, \\ t=m+1, \dots, n \end{array} \middle| \begin{array}{l} I_{mi} = 0 = I_{nj}, \text{ for some } i, j \in \{1, 2, \dots, M\}; \\ I_{tk} > 0 \text{ for all product } k \text{ and } m < t < n \end{array} \right\}$$

where $0 \leq m < n \leq N$. Each L_{mn} then produces the total quantity of $L(n) - L(m)$.

Clearly, any feasible production plan can be decomposed into one or more production sequences, and since $I_0 = 0 = I_N$, at least one production sequence L_{0N} exists. Furthermore, all at-least-one exact requirement points of a feasible production plan X are shared by two other distinctive feasible plans which are specified in Lemma 1.

Lemma 1. If X' and X'' are distinct feasible plans and $X = \frac{1}{2}(X' + X'')$, then X' and X'' share all the at-least-one exact requirement points of X .

Proof: Suppose that period s is any at-least-one exact requirement point of X and that the demand for product i is exactly satisfied in period s . Then, by hypothesis

$$\sum_{t=1}^s x_t = \frac{1}{2} \left[\sum_{t=1}^s x'_t + \sum_{t=1}^s x''_t \right], \text{ so that}$$

$$\sum_{t=1}^s x_{ti} = \frac{1}{2} \left[\sum_{t=1}^s x'_{ti} + \sum_{t=1}^s x''_{ti} \right]. \text{ If } \sum_{t=1}^s r_{ti} \text{ is subtracted from each side,}$$

then

$$\sum_{t=1}^s (x_{ti} - r_{ti}) = \frac{1}{2} \left[\sum_{t=1}^s (x'_{ti} - r_{ti}) + \sum_{t=1}^s (x''_{ti} - r_{ti}) \right]$$

$$\text{and } I_{si} = \frac{1}{2}(I'_{si} + I''_{si}).$$

Since $I_{si} = 0$, I'_{si} and I''_{si} must both be zero and so the three plans share the at-least-one exact requirement point s . Otherwise, one of them is negative, and the associated plan is not feasible. This completes the proof.

Now let us introduce a production sequence that will form the basis for our characterization of D . Denote by $\pi(m, n)$ the number of partial (positive but less than capacity) production periods between the two successive at-least-one exact requirement points m and n .

Definition. A production sequence L_{mn}^* is "at-most-one partial sequence" if the production level in at most one period d , $(m+1 \leq d \leq n)$, is partial and all other production levels are either zero or at their capacities, i.e.,

$$L_{mn}^* = \left\{ \begin{array}{l} x_t, \\ t=m+1, \dots, n \end{array} \left| \begin{array}{l} I_{mi}=0=I_{nj}, \text{ for some } i, j \in \{1, 2, \dots, M\}; \\ I_{tk} > 0 \text{ for all } k \text{ and } m < t < n; \\ \pi(m, n) \leq 1 \end{array} \right. \right\}$$

Then each feasible production plan consisting only of at-most-one partial sequences is characterized as the correspondence to an element of D .

Theorem 1. A feasible production plan X consists only of at-most-one partial sequences iff it is in D .

Proof: Suppose $X \in D$, L_{mn} is part of X and L_{mn} is not at-most-one partial sequence. Then, there are at least two periods, say b and d , $m+1 \leq b < d \leq n$, in which $0 < x_b < c_b$ and $0 < x_d < c_d$. Without loss of generality, we suppose that there are just two such periods.

Let $\delta = \frac{1}{2} \min[x_b, c_b - x_b, x_d, c_d - x_d, \min_{b \leq t \leq d} \{(M/\alpha_i)I_{ti}, \text{ for all } i\}]$

and let U_t be a $(n-m)$ component vector with a unity element in the t^{th} position, and zeroes elsewhere.

Now define the distinct production sequences

$$L'_{mn} = L_{mn} - \delta U_b + \delta U_d \text{ and } L''_{mn} = L_{mn} + \delta U_b - \delta U_d.$$

Since $\delta > 0$, these sequences are easily seen to be feasible. However, $L_{mn} = \frac{1}{2}[L'_{mn} + L''_{mn}]$, contradicting our assumption that X is an extreme point. This proves "if part".

Suppose on the contrary that $X \notin D$. Then, there are feasible distinct plans X' and X'' such that $X = \frac{1}{2}(X' + X'')$.

From Lemma 1, X' and X'' share all of the at-least-one exact requirement points of X . Let m and n be two such successive points, and let L_{mn}^* , X'_{mn} , and X''_{mn} be the associated distinct subplans in X , X' and X'' . Evidently, $L_{mn}^* = \frac{1}{2}(X'_{mn} + X''_{mn})$, and neither X'_{mn} nor X''_{mn} is an at-least-one exact requirement sequence. This implies that there are at least two periods b and d , $m+1 \leq b < d \leq n$, such that for $\delta > 0$ and sufficiently small, we can write

$$X'_{mn} = L_{mn}^* - \delta U_b + \delta U_d \text{ and}$$

$$X''_{mn} = L_{mn}^* + \delta U_b - \delta U_d.$$

We now show that X'_{mn} and X''_{mn} cannot simultaneously be feasible, hence neither can X' and X'' , thus contradicting our initial assumption that $X \notin D$.

(a) If $x_b = 0$ and $x_d = 0$, or if $x_b = c_b$ and $x_d = c_d$, the result is immediate.

- (b) If $x_b = 0$ and x''_{mn} is feasible, then $x'_b = x_b - \delta < 0$ and so x'_{mn} is not feasible.
- (c) If $0 < x_b < c_b$ and $x_d = c_d$ and if x''_{mn} is not feasible, then $x'_d = x_d + \delta > c_d$ and so x'_{mn} is not feasible.

This completes the proof.

Theorem 1 implies that the set, D , of the extreme points is identified with all the feasible solutions each consisting only of at-most-one partial sequences. Therefore, one of these feasible solutions has the minimum cost. This is an optimal solution.

4. An Algorithm

Based on the results of Theorem 1, we devise a dynamic programming, shortest path algorithm similar to that of Florian and Klein [2]. It will be used in section 5 for the backlogging case. However, for large problems (i.e., N or M is large) the algorithm may be slow. In fact, according to Florian et al. [3], the problem considered here is in the class of NP -complete problems. Therefore, we will adapt the empirical, tree-search algorithm studied by Baker et al. [1], which is for problems with constant production and inventory costs.

Baker et al. [1] have shown that in seeking an optimal solution to the single-product single-facility problem with capacity constraints it suffices to consider only plans in which the last production quantity is equal to capacity or to demand for the periods remaining in the problem. Similarly, we have the next Theorem (which can easily be proved by contradiction).

Theorem 2(a). If (x_1, \dots, x_N) represents an optimal plan, then for every t

$$\sum_{i=1}^M I_{t-1,i} \cdot (c_t - x_t) x_t = 0.$$

(b). Let $t = \max \{ h \mid x_h > 0 \}$. Then if (x_1, \dots, x_N) is an optimal plan, $x_t = \min [c_t, \min_{1 \leq i \leq M} \{(M/\alpha_i) R_{t,n}(i)\}]$.

Theorem 2(a) indicates that if there is positive inventory ($I_{t-1,i} > 0$, \forall_i) carried over from a previous period, then production is either at capacity or zero. On the other hand, if there is positive production at a level less than capacity, then incoming inventory for at least one product is zero. Theorem 2(b) states that if $x_t = \min_{1 \leq i \leq M} \{(M/\alpha_i) R_{t,n}(i)\}$, then for at least one product k " $I_{t-1}(k) = 0$ " is required, so that it remains to determine the optimal plan for reaching period $(t-1)$ with the final inventory level of " $r(j) =$

$R_{t,N}(j) - x_{t,N}(j)$, $\forall j$ " which is the so-called backshifted demand. On the other hand, if $x_t = c_t$, it remains to determine the optimal plan for reaching period $(t-1)$ with a final inventory equal to $r(j) = R_{t,N}(j) - x_{t,N}(j)$, $\forall j$. Thus, either of the subproblems is a smaller version of the original with the backshifted demand $r(j) \geq 0$, $(j=1,2,\dots,M)$, in period $(t-1)$. This implies that the problem decomposes at the end of the period immediately preceding the last production period.

The above description of the solution decomposition implies that the tree-search procedure suggested by Baker et al. [1] can be directly applied to our problem, with additional work required to test feasibility and to compute inventory costs for M different products. The three elimination conditions (infeasibility, partial cost, dominance) proposed by them also allow our solution procedure to examine considerably fewer than 2^N nodes (the total number of possible partial and complete production plans) in determining an optimum. The effectiveness of such conditions depends on the data in a given problem.

The solution procedure is illustrated with a 5-period two-product single-facility problem with the production ratios of " $\alpha_1:\alpha_2 = 3:5$ ". The production and inventory cost functions are given as follows: for all t ($t=1,2,\dots,N$),

$$P_t(x_t) = 150\delta(x_t) + 7x_t,$$

$$H_{t1}(I_{t1}) = 2I_{t1}, \text{ and}$$

$$H_{t2}(I_{t2}) = 3I_{t2},$$

$$\text{where } \delta(x_t) = \begin{cases} 1, & \text{if } x_t > 0 \\ 0, & \text{otherwise.} \end{cases}$$

The demands for $\{r_{t1}\}$ are $(6,4,8,5,7)$, and for $\{r_{t2}\}$ are $(8,10,11,11,10)$. The capacity bound c_t is 20 for all periods.

Following Baker et al. [1], we consider a set of indices, $\sigma = \{\sigma_1, \sigma_2, \dots, \sigma_s\}$, that specify periods when production runs occur from period σ_1 to the end of the problem, so that in subproblems A_σ we have $x_t > 0$ for $t = \sigma_1, \sigma_2, \dots, \sigma_s$. From this specification we can determine the costs incurred over the interval from σ_1 to N . Moreover, we can determine $r(i)$, $(i=1,2,\dots,M)$, the amount of the backshifted demand (if any) occurring in period σ_1 to N which must be met by production prior to σ_1 . Therefore, subproblems A_σ can be decomposed into smaller problems $A_{t\sigma}$, $(t=1,2,\dots,\sigma_1-1)$, where t denotes the last period prior to σ_1 where production occurs, and $r(i)$'s are added to

each associated demand in period $\sigma_1 - 1$.

Thus, a complete solution to the original problem is represented by any feasible subproblem A_σ for which $\sigma_1 = 1$. Hence, the optimal cost will be given by

$$Z^* = \min_{\sigma} \{ Z(A_\sigma) \mid \sigma_1 = 1 \text{ and } A_\sigma \text{ is feasible} \},$$

where $Z(A_\sigma)$ represents the total production (setup) and holding cost associated with each A_σ .

Based upon the above decomposition procedure and the results of Theorem 2, it is easily found that the example problem has only two feasible plans A_{1234} and A_{12345} . A_{1234} is the best with the total cost of 1286 and the optimal plan is (20, 20, 20, 20, 0).

5. Production Planning with Backlogging

The model described in section 2 may be extended so that backlogging is permitted. We shall assume as in Zangwill [7] that all demands must be satisfied within β_j , ($j=1,2,\dots,M$), periods after the specified delivery date for each product j . Thus, the constraints $I_{tj} \geq 0$ in the problem (A) is replaced by

$$I_{tj} \geq - \sum_{h=t-\beta_j+1}^t r_{hj}, \text{ for all } j \text{ and } t, (t=\beta_j, \beta_{j+1}, \dots, N).$$

With backlogging, we assume that for each product j a penalty cost is charged on the amount backlogged in any period and the cost is concave, so that the total inventory cost is piecewise concave.

Although the objective function of (A) is piecewise concave rather than concave, all of the results obtained for the nonbacklogging case hold. Since the arguments are, with slight modifications, the same as for the nonbacklogging problems, we shall not repeat them in detail.

Following Zangwill [7], the set of all feasible solutions can be partitioned into disjoint subsets, called "basic sets", which are characterized by whether the quantity I_{tj} is nonpositive or positive for each product j in each period t , $t=1,2,\dots,N-1$. Letting Ω be the set of all feasible solutions X that satisfy the constraints of the extended problem, then Ω is the union of all $2^{M(N-1)}$ basic sets. Each such subset is closed, bounded and convex with a finite number of extreme points and further the objective function, $Z(X)$, is concave on each given basic set. Thus for a fixed basic set, $Z(X)$

attains its minimum at one of its extreme points. Thereupon, letting D be the union of all extreme points of all basic sets, then the minimum of $Z(X)$ occurs at a point in D .

Consider two feasible solution vectors X' and X'' , and the respective inventories I' and I'' . Then it is easy to see that Lemma 1 holds, because otherwise X' and X'' belong to different basic sets. Furthermore, each minimum-allowed backlog point (i.e., in period t and product i , $I_{ti} =$

$-\sum_{h=t-\beta_i+1}^t r_{hi}$) is also shared by the two vectors X' and X'' .

The above properties lead to our conclusion that the characterization of D for the extended problem is still given by Theorem 1 with the modified production sequences, Q_{mn}^* , defined as follows;

$$Q_{mn}^* = \left\{ \begin{array}{l} x_t, \\ t=m+1, \dots, n \end{array} \left| \begin{array}{l} I_{mi}=0 \text{ or } -\sum_{h=m-\beta_i+1}^m r_{hi}, \text{ and } I_{nj}=0 \text{ or } -\sum_{h=n-\beta_j+1}^m r_{hj} \\ \text{for some } i, j \in \{1, 2, \dots, M\}; \text{ in each period } t, \\ m < t < n, \text{ and for all } i, I_{ti} \neq 0 \text{ but} \\ I_{ti} > -\sum_{s=t-\beta_i+1}^t r_{si}; \pi(m, n) \leq 1. \end{array} \right. \right\}$$

Each sequence Q_{mn}^* is a subset of a feasible production vector X which satisfies the constraints of the extended problem, and it has at most one positive production at a level less than capacity and all other productions either at capacity or zero. This implies that the production sequence Q_{mn}^* results in the total quantity $Q(n)-Q(m)$, where $Q(t) \in \{L(t), L_\beta(t)\}$ for each $t (t=m, n)$, and $L_\beta(t) = \max_i \{(M/\alpha_i)R_{1, t-\beta_i}(i)\}$. Furthermore, it corresponds to a subset of a feasible vector X contained in D .

Theorem 3. A feasible production plan X consists only of production sequences Q_{mn}^* , ($0 \leq m < n \leq N$), iff it is in D .

Proof: The proof can be completed easily by following the proof steps of Theorem 1 with the newly defined δ' ;

$$\delta' = \frac{1}{2} \min [x_b, c_b - x_b, x_d, c_d - x_d, \min_{b \leq g \leq d} \{(M/\alpha_i)(I_{gi} + \sum_{t=g-\beta_i+1}^g r_{ti}), \forall i\}].$$

The solution to the original problem by use of these optimal subplan Q_{mn}^* 's will now be discussed.

Suppose m and n are selected as two successive "at-least-one exact requirement" points. The results of Theorem 3 can then be used to facilitate

finding each optimal subplan Q_{mn}^* , ($m=0,1,\dots,n-1$; $n=1,2,\dots,N$). Let d_{mn} denote the cost associated with following an optimal plan over periods $m+1,\dots,n$. Then it follows that

$$d_{mn} = \sum_{t=m+1}^n P_t(x_t) + \sum_{t=m+1}^n \sum_{j=1}^M H_{tj}(I_{tj}),$$

where I_{tj} 's are associated with the subplan Q_{mn}^* , given an optimal subplan over periods 0, 1, ..., m .

In fact, each d_{mn} value can be computed by a tree-search procedure. The result of Theorem 3 imply that given an optimal production plan over periods 0, 1, ..., m , each d_{mn} value is obtained from an optimal combination (feasible and minimum cost combination) of some capacities $c_{t_s} \in \{c_{m+1}, \dots, c_n\}$, ($m+1=t_1 < t_2 < \dots < t_W \leq n$; $1 \leq W \leq n-m$), and at most one partial production ϵ_t , $0 < \epsilon_t < c_t$, at period $t \in \{m+1, \dots, n\}$ (but $t \notin \{t_1, \dots, t_W\}$) such that $\sum_{s=1}^W c_{t_s} + \epsilon_t = Q(n) - Q(m)$. Each such d_{mn} value may then be evaluated recursively in determining a minimum-cost route (shortest-route) corresponding to an optimal plan X^* . In other words, such a shortest route may be found by a dynamic programming recursion using the periods 0, 1, ..., N as states, among which each pair of two periods corresponds to a d_{mn} value.

Let F_t be the cost associated with an optimal production plan over periods 0, 1, ..., t , ($t=0, 1, \dots, N$), given that for at least one product i , $I_{ti}=0$. Then, we have

$$(5) \quad F_n = \min_{0 \leq m \leq n-1} \{F_m + d_{mn}\}, \quad (n=1, 2, \dots, N),$$

and $F_0 = 0$.

The recursion (5) indicates that the minimum cost for the first n periods comprises all the production and inventory costs over periods $m+1, \dots, n$ and the cost of adapting an optimal policy over periods 0 through m taken by themselves. Theorem 3 guarantees that at period n we shall find an optimal plan of this type.

As discussed above, d_{mn} values needed for the dynamic programming recursion (5) are associated with optimal at-most-one partial sequences Q_{mn}^* . However, finding optimal Q_{mn}^* 's is, in general, a tedious combinational problem.

We now illustrate the algorithm with a 2-product 4-period problem having the production ratios of " $\alpha_2:\alpha_1 = 2:3$ ", and the associated backlog periods of " $\beta_1=1$ and $\beta_2=1$ ". The production and inventory cost functions are given

as follows:

$$c_t(x_t) = 25(6-t)\delta(x_t) + (6-0.5t)x_t,$$

$$H_{t1}(I_{t1}) = \begin{cases} 10 I_{t1}, & \text{if } I_{t1} \geq 0 \\ -20 I_{t1}, & \text{if } I_{t1} < 0, \end{cases}$$

$$H_{t2}(I_{t2}) = \begin{cases} 5 I_{t2}, & \text{if } I_{t2} \geq 0 \\ -10 I_{t2}, & \text{if } I_{t2} < 0, \text{ for all } t, t=1,2,3,4, \end{cases}$$

$$\text{where for each } t, \delta(x_t) = \begin{cases} 1, & \text{if } x_t > 0 \\ 0, & \text{if } x_t = 0. \end{cases}$$

The demands for $\{x_{t1}\}$ are (5,4,3,8) and for $\{x_{t2}\}$ are (5,8,7,10).

The capacity bound c_t is 20 for all periods.

For the example problem, notations x_t , d_{mn} and F_n will be expressed with superscripts (i,j) such as x_t^{ij} , d_{mn}^{ij} and F_n^{ij} , respectively, where both i and j have 0 or 1 to indicate nonbacklogging or backlogging, respectively. Then, for all the possible production sequences $\{Q_{mn}^*\}$, d_{mn}^{ij} and F_n^{ij} are computed as follows:

- 1) For Q_{01}^* ; ($x_1^{00}=12.5$; $d_{01}^{00}=206.25=F_1^{00}$), ($x_1^{11}=0$; $d_{01}^{11}=150=F_1^{11}$) and ($x_1^{10}=25/3$; $d_{01}^{10}=204.17=F_1^{10}$),
- 2) For Q_{02}^* ; ($x_1^{00}=20$, $x_2^{00}=2.5$; $d_{02}^{00}=415=F_2^{00}$), ($x_1^{11}=12.5$, $x_2^{11}=0$; $d_{02}^{11}=341.25=F_2^{11}$) and ($x_1^{10}=20$, $x_2^{10}=5/3$; $d_{02}^{10}=415=F_2^{10}$),
- 3) For Q_{12}^* (given Q_{01}^*); ($x_2^{00}=10$; $d_{12}^{00}=152.5$, and $F_2^{00}=d_{01}^{00}+d_{12}^{00}=358.75$), ($x_2^{11}=0$; $d_{12}^{11}=135$, and $F_2^{11}=d_{01}^{11}+d_{12}^{11}=341.25$) and ($x_2^{10}=55/6$; $d_{12}^{10}=152.5$, and $F_2^{10}=d_{01}^{10}+d_{12}^{10}=358.75$),
- 4) For Q_{03}^* ; ($x_1^{00}=20$, $x_2^{00}=0$, $x_3^{00}=40/3$; $d_{03}^{00}=478.33=F_3^{00}$), ($x_1^{11}=20$, $x_2^{11}=2.5$, $x_3^{11}=0$; $d_{03}^{11}=540=F_3^{11}$) and ($x_1^{01}=20$, $x_2^{01}=0$, $x_3^{01}=10$; $d_{03}^{01}=470=F_3^{01}$),
- 5) For Q_{13}^* (given Q_{01}^*); ($x_2^{00}=20$, $x_3^{00}=40/3$; $d_{13}^{00}=378.33$, and $F_3^{00}=d_{01}^{00}+d_{13}^{00}=528.33$), ($x_2^{11}=10$, $x_3^{11}=0$; $d_{13}^{11}=277.5$, and $F_3^{11}=d_{01}^{11}+d_{13}^{11}=483.75$) and ($x_2^{01}=35/2$, $x_3^{01}=0$; $d_{13}^{01}=262.5$, and $F_3^{01}=d_{01}^{01}+d_{13}^{01}=468.75$),

- 6) For Q^*_{23} (given Q^*_{01} and Q^*_{12}) ; $(x_3^{00}=65/6; d_{23}^{00}=137.08, \text{ and } F_3^{00}=d_{01}^{00}+d_{12}^{00}+d_{23}^{00}=495.83), (x_3^{11}=0; d_{23}^{11}=125, \text{ and } F_3^{11}=d_{01}^{00}+d_{12}^{00}+d_{23}^{11}=483.75) \text{ and } (x_3^{01}=15/2; d_{23}^{01}=128.75, \text{ and } F_3^{01}=d_{01}^{00}+d_{12}^{00}+d_{23}^{01}=487.5),$
- 7) For Q^*_{04} ; $(x_1^{00}=20, x_2^{00}=0, x_3^{00}=10, x_4^{00}=20; d_{04}^{00}=600=F_4^{00}),$
- 8) For Q^*_{14} (given Q^*_{01}) ; $(x_2^{00}=17.5, x_3^{00}=0, x_4^{00}=20; d_{14}^{00}=392.5, \text{ and } F_4^{00}=d_{01}^{00}+d_{14}^{00}=598.75),$
- 9) For Q^*_{24} (given Q^*_{01} and Q^*_{12}) ; $(x_3^{00}=7.5, x_4^{00}=20; d_{24}^{00}=258.75, \text{ and } F_4^{00}=d_{01}^{00}+d_{12}^{00}+d_{24}^{00}=617.5), \text{ and}$
- 10) For Q^*_{34} (given Q^*_{03}) ; $(x_4^{00}=50/3; d_{34}^{00}=116.67, \text{ and } F_4^{00}=d_{03}^{00}+d_{34}^{00}=595).$

Thus, the sub-plan sequence (Q^*_{03}, Q^*_{34}) is identified as the optimal plan, for which $X=(20, 0, 40/3, 50/3)$ and the associated total cost is 595.

6. Concluding Remarks

In this paper, we have found a useful description of the structure of optimal plans which consist of independent subplans. In each subplan, the positive production level is at capacity, except for at most one period in which it is less than capacity. This characterization has been shown to hold both for the model with nonbacklogging and for the case with backlogging. For problems where backlogging is allowed, the characterization may be used to find an optimal solution efficiently for small problems.

We conclude that the results of this paper can be useful for a variety of managerial problems (e.g., system design, production planning, cash-flow control, etc.) with capacity constraints in manufacturing (or service) industries, for which whenever a managerial decision on a single resource is implemented, a number of outcomes (products) are generated.

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