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SERVICE MECHANISM CONTROL AND ARRIVAL CONTROL OF A TWO-STATION TANDEM QUEUE

Shōichi Nishimura
University of Tsukuba

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Abstract We analyze a controlled queueing system with two independent exponential service stations arranged in tandem. Our model is constructed by two control problems. The first one is arrival control in which the system is controlled by accepting or rejecting arriving customers. And second one is service mechanism control in which decision maker selects a station to be served. We formulate our model as a semi-Markov decision process and transition of state is represented by shift operator. Using the iteration method the monotomicity properties of the optimal policy are established.

1. Introduction

In this paper we consider a controlled queueing system with two independent exponential service stations arranged in tandem. There is one server and only one of both stations is served at the same time. Customers arrive according to Poisson process with arrival rate λ. Our model is constructed by two control problems. The first one is arrival control in which the system is controlled by accepting or rejecting arriving customers. And the second one is service mechanism control in which the decision maker selects a station to be served. We formulate our model as a semi-Markov decision process and transition of state is represented by shift operator. Using the iteration method the monotonicity of an optimal policy is established.

The arrival controlled queue has been studied in many papers (eg. Lippman and Stidham Jr. (1977), Stidham Jr. (1978), Nishimura (1982)). The characteristics of optimal policies, especially the monotonicity of critical number at an optimal policy are established. The controlled queue of service mechanism has many different problems, for example, removable server model in Heyman (1968) and Bell (1971), priority queue model in Harrison (1975), service rate control model in Doshi (1978), Gallish (1979), Sobel (1982), and Jo (1983). Our second control is one of optimal service rate problems, in which we restrict that service rate cost is not incurred and we select the station to be served.

Network queueing systems are applied to a production process and an information processing. In a production process a cost is an important factor and in an information processing it is natural to assume a capacity of processing customers. Therefore it is important to control network queueing systems. Optimal control with two service stations are studied by Rosberg, Varaiya and Walrand (1982) and Hajek (1984). Moreover if in a tandem queueing system the departure from intermediate stations is allowed, the service time distribution is a mixture of Erlang distributions, which is frequently called a hyper-Erlang distribution. Such distributions can be used to approximate general distributions and optimal control of a general distribution using phase method is studied by Langen (1982) and Jo and Stidham (1983).

Let i and j be the numbers of waiting customers to be served at the first and the second stations respectively. Partial order is defined on the set of current state $x = (i,j)$ being a nonnegative integer valued vector. We introduce shift operators on the state space and define a monotone function with respect to partial order. In section 3 we prove an arrival control monotonicity and a service mechanism monotonicity of optimal policies for the finite horizon problems. The customer who is accepted in x is also accepted in y if $x \geqq y$ where the inequality $\geqq$ is a partial order on the state space. If at state $x = (i,j)$ the first station is served then at current state $y = (i+1, j-1)$ the first station is also served, and if at $x = (i,j)$ the second station is served then at $y = (i,j+1)$ the second station is also served. In section 4 we consider infinite horizon problems both with and without discounting. The existence of optimal equation are proved and optimal policy for infinite horizon problems inherits monotomicities.

There does not seem to be any study of optimal policies when arrival control and service mechanism are concerned. In general arrival mechanism and service mechanism influence each other and such a control problem is complex. In our model, however, the monotonicity of optimal policy is obtained. In a single service station model the monotonicity of optimal policy is obtained by the convexity of value function. In our case the state space is two dimensional and the proof is closely related to general framework of submodular function on lattice for monotone optimal policies (see Topkis (1978) and Heyman and Sobel (1984)).

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2. Shift Operator

We consider controlled queueing systems denoted by $M/M/1 \to /M/1$. Customers arrive at the system from Poisson process with parameter λ. The rewards of successive customers are assumed to be independent random variables with finite mean and common distribution function $F(r)$. First the system is controlled by accepting or rejecting arriving customers. Each accepted customer goes to the first station and queues up for service at the first station. Having completed service there, he proceeds to a queue in front of the second station, and after completing service at the second station he leaves the system. It is assumed that service times at each station are mutually independent exponential random variables with service rate μ. There is one server who serves one of both stations at the same time. Second the system is controlled by selecting a station to be served depending upon the state of the system. We formulate our model as semi-Markov decision process. Decision points are those epoch at which either a service is completed or a customer arrives.

Let x be a two-dimensional integer-valued vector and $X = \{x = (i,j);\ i,j=\ldots, -1,0,1,\ldots\}$. Let $R \subset X$ be the set of customer numbers in the system, where i and j are numbers of customers at first and at second station, respectively. If $x = (i,j) \in R$, i and j are nonnegative integers. Moreover we may put three boundary constraints I, J and K where I and J are upper bounds of customer numbers at the first and second station respectively and K is an upper bounds of total number of customers in the system. It is, however, possible that these upper bounds are infinite. Then we have

(1) $$R = \{x = (i,\ j);\ i = 0,\ \ldots,\ I,\ j = 0,\ \ldots,\ J \text{ and } i + j = 0,\ \ldots,\ K\}.$$

For any $x = (i,\ j) \in X$, we define shift operators S, T_1, T_2, 1 such that

$$Sx = (i+1,\ j),\quad T_1x = (i-1,\ j+1),\ T_2x = (i,\ j-1),\quad 1x = (i,\ j)$$

$$S^{-1}x = (i-1,\ j),\quad T_1^{-1}x = (i+1,\ j-1),\quad T_2^{-1}x = (i,\ j+1).$$

The shift from x to Sx represents that an arriving customer is accepted and he enters the first queue, T_b ($b = 1,\ 2$) represents the completion of the service at bth station and 1 is the identity operator. In natural way we define the composition of shift operators, ST_1, T_2^2 and etc. where $ST_1x = (i,j+1)$ and $T_2^2x = (i,j-2)$. Now a binary relation $y \le x$ is defined on X if there exists T_1 or T_2 such that $T_1x = y$ or $T_2x = y$, or if there exists z such that $y \le z$ and $z \le x$. If $y = (k,\ \ell) \le x = (i,\ j)$, then we have $k+\ell \le i+j$ and $k \le i$. Therefore this binary relation $\le$ is reflexive, antisymmetric and transitive

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and X is a partially ordered set (see, TOPKIS (1978)).

Let $f(x)$ be any function on R and T be any shift operator on R. For $Tx \in R$ we define as

$$Tf(x) = f(Tx).$$

We assume the holding cost rate function $h(x)$ on R satisfying the following conditions.

a) If $x \in R$ and $T_b x \in R$, then
$$(1 - T_b)h(x) \geq 0 \qquad (b = 1, 2).$$

b) If $Sx \in R$ and $T_b x \in R$, then
$$(1 - S)(1 - T_b)h(x) \leq 0 \qquad (b = 1, 2).$$

c) If $T_{b^*}x \in R$ and $T_b^2 x \in R$, then
$$(T_b - T_{b^*})(1 - T_b)h(x) \leq 0 \qquad (b = 1,\ b^* = 2 \text{ or } b = 2,\ b^* = 1).$$

In this assumption, for example, it means that

$$(1 - S)(1 - T_b)h(x) = h(x) - h(Sx) - h(T_b x) + h(ST_b x).$$

Condition a) is monotonicity assumption and condition b) and c) correspond to convexity one in controlled queueing systems.

3. Optimal Policy for Finite Horizon

Let $V_{n,\alpha}(i, j, r)$ be the maximal expected α-discounted net benefit when a customer with reward r has just arrived to a system when there are i and j customers at first and second stations respectively and the horizon length is n. If the decision maker accepts the arriving customer, he gains reward r and the next state is $(i+1, j)$. If he rejects the customer the next state is (i, j) with no penalty cost. And let $V_{n,\alpha}(i, j)$ be the maximal expected α-discounted net benefit when the current state of a system is (i, j). The decision maker selects a station $b = 1$ or $b = 2$ to be served. Let Y_1 and Y_2 be a residual interarrival time of customers and a residual service time from a decision epoch, respectively. Then Y_1 and Y_2 are independent and exponential random variables with parameter λ and μ, erspectively. The random variable $Z = \min(Y_1, Y_2)$ is a time interval until the next decision and has exponential distribution with parameter $\Lambda \equiv \lambda + \mu$. The expected holding cost in this interval is $E\int_0^Z h(i,j)\, e^{-\alpha t}dt = h(i, j)/(\alpha+\Lambda)$. Moreover $P\{Y_1 = t|Z = t\} = \lambda/\Lambda$ and $P\{Y_2 = t|Z = t\} = \mu/\Lambda$ and $E[e^{-\alpha Z}] = \Lambda/(\alpha+\Lambda)$. Then we have the

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following recurseive equations for $n \geq 1 (V_{0,\alpha} \equiv 0)$.

(2) $$V_{n,\alpha}(i, j, r) = \max\{r + V_{n,\alpha}(i+1, j),\ V_{n,\alpha}(i,j)\}$$

(3) $$V_{n,\alpha}(i, j) = [-h(i, j) + \lambda\int V_{n-1,\alpha}(i, j, r)F(dr) + \mu \max\{V_{n-1,\alpha}(i-1, j+1),\ V_{n-1,\alpha}(i, j-1)\}]/(\alpha+\Lambda)$$

where $\Lambda = \lambda+\mu$. Using shift operators we rewrite Equation (2) and (3) as

(4) $$V_{n,\alpha}(x, r) = \max\{r+SV_{n,\alpha}(x),\ V_{n,\alpha}(x)\}$$

(5) $$V_{n,\alpha}(x) = [-h(x)+\lambda\int V_{n-1,\alpha}(x,r)F(dr) + \mu U_{n-1,\alpha}(x)]/(\alpha+\Lambda),$$

where $U_{n-1,\alpha}(x) = \max_{b=1,2} T_b V_{n-1,\alpha}(x)$. Equation (4) is arrival control and Equation (5) is service mechanism control.

Remark 1. In Equations (4) and (5) the current state $x = (i, j)$ should be contained in R. If $i = I$ or $i + j = K$ then an arriving customer can not be accepted and $V_{n,\alpha}(x, r) = V_{n,\alpha}(x)$ in Equation (4). If $i = 0$ and $j \neq 0$ then T_b in Equation (5) is equal to T_2 and if $i \neq 0$ and $j = 0$, T_b in Equation (5) is equal to T_1. Finally if $i = 0$ and $j = 0$ ($x = (0,0) = 0$), Equation (5) is equal to

(6) $$V_{n,\alpha}(0) = [-h(0) + \lambda\int V_{n-1,\alpha}(0,r)F(dr) + \mu V_{n-1,\alpha}(0)]/(\alpha+\Lambda).$$

We will discuss in this paper the monotonicity of an optimal policy given by Equations (4) - (5). First given n,α and r $V_{n,\alpha}(x)$ and $V_{n,\alpha}(x, r)$ are monotone nonincreasing functions of x. For $y \leq x (x, y \in R)$, $V_{n,\alpha}(y) \geq V_{n,\alpha}(x)$ and $V_{n,\alpha}(y, r) \geq V_{n,\alpha}(x, r)$, which is equivalent to that for $x \in R$ and $T_b x \in R$,

(7) $$(T_b-1)V_{n,\alpha}(x) \geq 0$$

and

(8) $$(T_b-1)V_{n,\alpha}(x, r) \geq 0.$$

Secondly if at the current state (x, r) an arriving customer is accepted under an optimal policy then for $y \leq x$ he is also accepted at the state (y, r). To prove this monotonicity of arrival control we will show that $r + SV_{n,\alpha}(x) \geq V_{n,\alpha}(x)$ implies $r + SV_{n,\alpha}(y) \geq V_{n,\alpha}(y)$. For this it is sufficient to prove

(9) $$(1-S)(1-T_b)V_{n,\alpha}(x) \geq 0.$$

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The third one is the service mechanism monotonicity: if at the current state x the station 1 (the station 2) is served under an optimal policy then at the current state $T_2x(T_1x)$ the station 1 (the station 2) is also served. In other words, we will prove that for $b = 1$, $b^* = 2$ or $b = 2$, $b^* = 1$

(10) $$(T_b - T_{b^*})(1-T_b)V_{n,\alpha}(x) \geq 0$$

Remark 2. We have to prove Equation (9) when all x, Sx, T_bx and ST_bx belong to R. From Equation (1) it is, however, easy to show that if negative terms Sx and T_bx in $(1-S)(1-T_b)x$ belong to R then positive terms x and ST_bx also belong to R. Then it is sufficient to prove Equation (9) for any x such that $Sx \in R$ and $T_bx \in R$. Similarly we will prove Equation (10) for any x such that negative terms in $(T_b-T_{b^*})(1-T_b)x$ belong to $R(T_b{}^2x \in R$ and $T_{b^*}x \in R)$.

Let $A_{n,\alpha}(x, r)$ represent an action in optimal policy when a customer with reward r arrives to the system at current state x, n periods remain and the discount rate is $\alpha \geq 0$:

(11) $$A_{n,}(x, r) = \begin{cases} 0 \text{ (reject)} & \text{if} \quad r \leq (1-S)V_{n,\alpha}(x) \\ 1 \text{ (accept)} & \text{if} \quad r > (1-S)V_{n,\alpha}(x). \end{cases}$$

Let $B_{n,\alpha}(x)$ be the optimal served station when the system is in state x, n periods remain and the discount rate is $\alpha \geq 0$:

(12) $$B_{n,\alpha}(x) = \begin{cases} 1 & \text{if} \quad (T_1-T_2)V_{n,\alpha}(x) \geq 0 \\ 2 & \text{if} \quad (T_1-T_2)V_{n,\alpha}(x) < 0. \end{cases}$$

Define as

(13) $$S_a = \begin{cases} 1 & \text{if} \quad a = 0 \\ S & \text{if} \quad a = 1. \end{cases}$$

The first theorem establishes the monotonicity of $V_{n,\alpha}(x)$ in x.

Theorem 1. Given n and α, $V_{n,\alpha}(x)$ and $V_{n,\alpha}(x, r)$ are monotone nonincreasing in x. That is, if for each $b = 1$ or 2, $x \in R$ and $T_bx \in R$, then

(14) $$(T_b-1)V_{n,\alpha}(x) \geq 0.$$

Proof: The proof is by induction on n. For $n = 0$ $V_{0,\alpha}(x) = 0$ and $V_{0,\alpha}(x, r) = \max\{r, 0\}$. The result follows immediately. To prove Theorem 1, we establish the following Lemma 1 and Lemma 2.

Lemma 1. Suppose that (14) is satisfied for n. If $x \in R$ and $T_bx \in R$, then

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(15) $\qquad (T_b-1)V_{n,\alpha}(x,\ r) \geq 0 \qquad\qquad (b = 1,2)$.

Proof: We put $a = A_{n,\alpha}(x,\ r)$ and S_a is defined in (13). From the definition of R in (1) it follows that if $x \in R$, $T_b x \in R$ and $S_a x \in R$, then $T_b S_a x \in R$. We have

(16) $\qquad (T_b-1)V_{n,\alpha}(x,\ r) \geq (T_b-1)S_a V_{n,\alpha}(x) \geq 0,$

where the first inequality comes from the fact that in negative term $V_{n,\alpha}(x,\ r)$ an action $a = A_{n,\alpha}(x,\ r)$ is chosen and the second inequality comes from the induction assumption (14).

Lemma 2. Suppose that (14) is satisfied for n. If $x \in R$ and $T_b x \in R$ then

$$(T_b-1)U_{n,\alpha}(x) \geq 0 \ .$$

Proof: Define $b' = T_b B_{n,\alpha}(x)$ and $b'' = B_{n,\alpha}(x)$, in which b' and b'' are numbers of stations to be served under optimal policy in current states $T_b x$ and x, respectively. If $b' = b''$, we easily have

(17) $\qquad (T_b-1)U_{n,\alpha}(x) = T_{b'}(T_b-1)V_{n,\alpha}(x) \geq 0$.

If $b' \neq b''$, then from $T_b T_{b'} x \in R$ and $T_{b''} x \in R$ we have that $T_b T_{b''} x \in R$ holds and

(18) $\qquad (T_b-1)U_{n,\alpha}(x) \geq T_{b''}(T_b-1)V_{n,\alpha}(x) \geq 0$.

Using Lemma 1, Lemma 2 and holding cost assumption a),

$$(T_b-1)V_{n+1,\alpha}(x) = [-(T_b-1)h(x) + \lambda\int(T_b-1)V_{n,\alpha}(x,\ r)F(dr)$$
$$+ \mu(T_b-1)U_{n,\alpha}(x)]/(\alpha+\Lambda) \geq 0 \ .$$

This completes the proof of Theorem 1.

To prove the monotonicity of optimal policy we need to establish the following Lemma 3 to Lemma 7.

Lemma 3. Suppose that $V_{n,\alpha}(x)$ satisfies (9), (10) and (14) for n. If $x \in R$ and $T_b^3 x \in R$, then

(19) $\qquad -(1-T_b)^2(1+T_b)V_{n,\alpha}(x) \geq 0 \qquad\qquad (b = 1,2)$.

Proof: In general we have

(20) $\qquad -(1-T_b)^2(1+T_b) = S^{-1}(1-S)(1-T_b) + T_b(T_b-T_{b^*})(1-T_b)$

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(21) $$= T_b^{\,2}(1-S)(1-T_b) + T_{b^*}^{\,-1}(T_b-T_{b^*})(1-T_b) ,$$

where $b = 1$, $b^* = 2$ or $b = 2$, $b^* = 1$. Four negative terms in right side of (20) are 1, $T_b^{\,3}$, $S^{-1}T_b$ and $T_{b^*}T_b$. From the assumption of this Lemma ($x \in R$, $T_b^{\,3}x \in R$) and Remark 2 if $S^{-1}T_bx = T_{b^*}T_b^{\,2}x \in R$ and $T_bT_{b^*}x \in R$, then all terms in (20) are contained in R. Using the induction assumptions (9) and (10) it follows from (20) that (19) holds. Using the same discussion if $ST_b^{\,2}x = T_{b^*}^{\,-1}T_bx \in R$ and $T_{b^*}^{\,-1}T_b^{\,2}x \in R$ then from (21) we also get (19). Since $x \in R$ and $T_b^{\,3}x \in R$ it can be shown that either ($T_{b^*}T_bx \in R$ and $T_{b^*}T_b^{\,2}x \in R$) or ($T_{b^*}^{\,-1}T_bx \in R$ and $T_{b^*}^{\,-1}T_b^{\,2}x \in R$) is satisfied. This completes the proof.

Lemma 3 $(-(1-T_b-T_b^{\,2}+T_b^{\,3})V_{n,\alpha}(x) \geq 0)$ is a weaker result than the concavity $(-1(1-T_b)^2V_{n,\alpha}(x) \geq 0)$.

Lemma 4. Suppose that $V_{n,\alpha}(x)$ satisfies (9), (10) and (14) for n. If $T_{b^*}x \in R$ and $T_b^{\,2}x \in R$ then

(22) $$(T_b-T_{b^*})(1-T_b)U_{n,\alpha}(x) \geq 0 ,$$

where $b = 1$, $b^* = 2$ or $b = 2$, $b^* = 1$.

Proof: In the case of $b = 1$ and $b^* = 2$ let $b' = T_1^{\,2}B_{n,\alpha}(x)$ and $b'' = T_2B_{n,\alpha}(x)$ then we have

$$\begin{aligned}
&(T_1-T_2)(1-T_1)U_{n,\alpha}(x)\\
&\quad = \{(1-T_1)T_1 - (1-T_1)T_2\}U_{n,\alpha}(x)\\
&\quad \geq \{(1-T_1)T_1T_{b'} - (1-T_1)T_2T_{b''}\}V_{n,\alpha}(x)
\end{aligned}$$

(23) $$= \{(T_{b'}-T_2)(1-T_1)T_1 + (T_1-T_{b''})(1-T_1)T_2\}V_{n,\alpha}(x) \geq 0 .$$

From assumption of $T_1^{\,2}x \in R$ and $T_2x \in R$ we get $T_1^{\,2}T_2x \in R$ and $T_1T_2x \in R$. It follows that all terms in (23) are contained in R. Using induction assumptions (9) and (10) we conclude the last inequality in (23). In the case of $b = 2$ and $b^* = 1$ we can prove (22) in the same way.

Lemma 5. Suppose that $V_{n,\alpha}(x)$ satisfies (9), (10) and (14) for n. If $Sx \in R$ and $T_bx \in R$ then

(24) $$(1-S)(1-T_b)U_{n,\alpha}(x) \geq 0 \qquad (b = 1,2) .$$

Proof: First in the case $b = 1$ we will prove (24). Negative terms in $(1-S)(1-T_1)$ are S and T_1. Let us put $b' = SB_{n,\alpha}(x)$ and $b'' = T_1B_{n,\alpha}(x)$. Since $b' = 1$ or 2, and $b'' = 1$ or 2, there are the following three cases:

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i) Case of $b' = b''$

$$(1-S)(1-T_1)U_{n,\alpha}(x) \geq (1-S)(1-T_1)T_{b'}V_{n,\alpha}(x) \geq 0.$$

ii) Case of $b' = 1$ and $b'' = 2$

$$(1-S)(1-T_1)U_{n,\alpha}(x) = \{(1-S) - (1-S)T_1\}U_{n,\alpha}(x)$$
$$\geq \{(1-S)T_1 - (1-S)T_1T_2\}V_{n,\alpha}(x) = T_1(1-S)(1-T_2)V_{n,\alpha}(x) \geq 0.$$

iii) Case of $b' = 2$ and $b'' = 1$

$$(1-S)(1-T_1)U_{n,\alpha}(x) = \{(1-T_1) - (1-T_1)S\}U_{n,\alpha}(x)$$
$$\geq \{(1-T_1)T_2 - (1-T_1)ST_1\}V_{n,\alpha}(x)$$
$$= -T_2^{-1}(1-T_1)^2(1+T_1)V_{n,\alpha}(x) \geq 0 ,$$

where the last inequality comes from Lemma 3. Using Remark 2, it follows from $T_{b'}Sx \in R$ and $T_{b''}T_1x \in R$ that all positive terms in i), ii) and iii) are contained in R.

In the case of $b = 2$ the proof is the same as this. However if $x = (0,1)$ then using (6) at $T_2x = (0,0)$ we have

$$(1-S)(1-T_2)U_{n,\alpha}(0,\ 1) = V_{n,\alpha}(0,\ 1) - \max\{V_{n,\alpha}(0,\ 2),\ V_{n,\alpha}(1,\ 0)\} \geq 0 \tag{25}$$

The last inequality in (25) comes from (14).

Lemma 6. Suppose that $V_{n,\alpha}(x)$ satisfies (9), (10) and (14) for n. If $Sx \in R$ and $T_bx \in R$ then

$$(1-S)(1-T_b)V_{n,\alpha}(x,\ r) \geq 0 \qquad (b = 1,\ 2) . \tag{26}$$

Proof: For $b = 1$ we will prove (26). In the case $b = 2$ the proof is the same as this. Define $a' = T_1A_{n,\alpha}(x,\ r)$ and $a'' = SA_{n,\alpha}(x,\ r)$.
Then

$$(1-S)(1-T_1)V_{n,\alpha}(x,\ r) \geq \{(1-T_1)S_{a'} - (1-T_1)SS_{a''}\}V_{n,\alpha}(x) . \tag{27}$$

We consider the following three cases:

i) Case of $a' = a''$

$$(1-S)(1-T_1)V_{n,\alpha}(x,\ r) = (1-S)(1-T_1)S_{a'}V_{n,\alpha}(x) \geq 0 .$$

ii) Case of $a' = 1$ and $a'' = 0$

$$(1-S)(1-T_1)V_{n,\alpha}(x,\ r) \geq \{S(1-T_1) - S(1-T_1)\}V_{n,\alpha}(x) = 0 .$$

iii) Case of $a' = 0$ and $a'' = 1$

$$(1-S)(1-T_1)V_{n,\alpha}(x,\ r) \geq \{(1-T_1) - S^2(1-T_1)\}V_{n,\alpha}(x)$$
$$= \{(1-S)(1-T_1) + S(1-S)(1-T_1)\}V_{n,\alpha}(x) \geq 0 .$$

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This completes the proof.

Lemma 7. Suppose that $V_{n,\alpha}(x)$ satisfies (9), (10) and (14) for n. If $T_b{}^2x = R$ and $T_{b^*}x \in R$, then

$$(28) \qquad (T_b - T_{b^*})(1-T_b)V_{n,\alpha}(x,\ r) \geq 0 \ .$$

Proof: In the case of $b = 1$ and $b^* = 2$ we will prove (28). Define $a' = T_1{}^2A_{n,\alpha}(x,\ r)$ and $a'' = T_2A_{n,\alpha}(x,\ r)$.

i) If $a' = a''$, we have

$$(T_1-T_2)(1-T_1)V_{n,\alpha}(x,\ r) \geq S_{a'}(T_1-T_2)(1-T_1)V_{n,\alpha}(x) \geq 0.$$

ii) If $a' = 0$ and $a'' = 1$, we have

$$\begin{aligned}(T_1-T_2)(1-T_1)V_{n,\alpha}(x,\ r) &= \{(1-T_1)T_1 - (1-T_1)T_2\}V_{n,\alpha}(x,\ r)\\ &\geq \{(1-T_1)T_1 - (1-T_1)T_2S\}V_{n,\alpha}(x)\\ &= (1-T_1)(T_1-T_1{}^{-1})V_{n,\alpha}(x)\\ &= -T_1{}^{-1}(1-T_1)^2(1+T_1)V_{n,\alpha}(x) \geq 0 \ .\end{aligned}$$

iii) If $a' = 1$ and $a'' = 0$ we have

$$\begin{aligned}(T_1-T_2)(1-T_1)V_{n,\alpha}(x,\ r) &= \{(T_1-T_2)-(T_1-T_2)T_1\}V_{n,\alpha}(x,\ r)\\ &\geq \{(T_1-T_2) - (T_1-T_2)T_1S\}V_{n,\alpha}(x)\\ &= T_2{}^{-1}(T_2-T_1)(1-T_2)V_{n,\alpha}(x) \geq 0 \ .\end{aligned}$$

The next theorem establishes arrival control monotonicity and service mechanism monotonicity in optimal policies.

Theorem 2. Let $b = 1$, $b^* = 2$ or $b = 2$, $b^* = 1$. If $Sx \in R$ and $T_bx \in R$ then

$$(29) \qquad (1-S)(1-T_b)V_{n,\alpha}(x) \geq 0 \ .$$

If $T_{b^*}x \in R$ and $T_b{}^2x \in R$ then

$$(30) \qquad (T_b-T_{b^*})(1-T_b)V_{n,\alpha}(x) \geq 0 \ .$$

Proof: The proof is by induction on n. For $n = 0$ it is easily shown to prove (29) and (30). Now suppose that for n (29) and (30) hold. Using Lemma 4 to Lemma 7 we have

$$\begin{aligned}(1-S)(1-T_b)V_{n+1,\alpha}(x) = [&-(1-S)(1-T_b)h(x)\\ &+ \lambda\int(1-S)(1-T_b)V_{n,\alpha}(x,\ r)F(dr)\\ &+ (1-S)(1-T_b)U_{n,\alpha}(x)]/(\alpha+\Lambda) \geq 0\end{aligned}$$

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and

$$(T_b-T_{b^*})(1-T_b)V_{n+1,\alpha}(x) = [-(T_b-T_{b^*})(1-T_b)h(x)$$
$$+ \lambda\int(T_b-T_{b^*})(1-T_b)V_{n,\alpha}(x, r)F(dr)$$
$$+ (T_b-T_{b^*})(1-T_b)U_{n,\alpha}(x)]/(\alpha+\Lambda) \geq 0 .$$

This completes the proof.

4. Infinite Horizon

In this section we consider infinite horizon problems, both with and without discounting. The existence of optimal equations is proved and the monotone optimal policies for infinite horizon problems are obtained as a limit of finite horizon problems.

Let $V_\alpha(x)$ be the maximal expected α - discounted net benefit when the current state of a system is x and horizon length is infinite. We can apply the same argument of Theorem 8 in Stidham (1978). Then we have

$$V_\alpha(x) = \lim_{n\to\infty} V_{n,\alpha}(x) \tag{31}$$

and the optimal equation is satisfied

$$V_\alpha(x) = [-h(x) + \lambda\int V_\alpha(x, r)F(dr) + \mu \max_{b=1,2} T_b V_\alpha(x)]/(\alpha+\Lambda), \tag{32}$$

where $\quad V_\alpha(x, r) = \max\{r+S\ V_\alpha(x),\ V_\alpha(x)\}$.

Theorem 3. Let $b = 1$, $b^* = 2$ or $b = 2$, $b^* = 1$. For any $\alpha > 0$, $V_\alpha(x)$ is monotone nonincreasing in $x((T_b-1)V_\alpha(x) \geq 0)$. Furthermore if $Sx \in R$ and $T_b x \in R$ then $(1-S)(1-T_b)V_\alpha(x) \geq 0$, and if $T_{b^*}x \in R$ and $T_b^2 x \in R$ then $(T_b-T_{b^*})(1-T_b)V_\alpha(x) \geq 0$.

Next we consider the infinite horizon undiscounted problem ($\alpha = 0$), in which the objective is to maximize long-run average return.

Theorem 4. Suppose $h(0,1) > h(0,0)$. Then there exist the optimal long-run average g and a function $v(x)$ such that for some sequence $\{\alpha_\nu\}(\alpha_\nu \to 0^+$ as $\nu \to \infty)$

$$v(x) = \lim_{\nu\to\infty}[V_{\alpha_\nu}(x) - V_{\alpha_\nu}(0,0)] \tag{33}$$

and

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(34) $$v(x) = [-h(x) + \lambda\int v(x, r)F(dr) + \mu \max_{b=1,2} T_b v(x) - g]/\Lambda,$$

where $v(x, r) = \max\{r+S\, v(x), v(x)\}$.
Furthermore, $v(x)$ is monotone nonincreasing in $x((T_b-1)v(x) \geq 0)$. If $Sx \in R$ and $T_b x \in R$ then $(1-S)(1-T_b)v(x) \geq 0$ and if $T_{b^*}x \in R$ and $T_b^2 x \in R$ then $(T_b-T_{b^*})(1-T_b)v(x) \geq 0$.

Proof: We prove the existence of a stationary optimal policy, which is obtained by the limit of a sequence of discounted optimal policies. When the system is in state x, and the discount rate is α, the acception rate is given by $\lambda\{1 - F((1-S)V_\alpha(x))\}$. If for all sufficiently large $|x|$ ($|x| \equiv 2i+j$) and all sufficiently small $\alpha > 0$ $2\lambda\{1 - F((1-S)V_\alpha(x))\} < \mu$ then the assumptions of Theorem 4 in Lippman (1973) are satisfied and then (33) and (34) hold. To prove this it suffices to show that

(35) $$\lim_{\alpha\to 0^+} \lim_{|x|\to\infty} (1-S)V_\alpha(x) = \infty .$$

Using the same discussion of (16) in Lemma 1, let us put $a = SA_{n,\alpha}(x)$, then we have

$$\begin{aligned}(1-S)V_{n,\alpha}(x, r) &\geq (1-S)S_a V_{n,\alpha}(x)\\ &\geq (1-S)V_{n,\alpha}(x) \geq \min_{b=1,2} (1-S)T_b V_{n,\alpha}(x),\end{aligned}$$

where the second and the third inequalities come from (29) in Theorem 2. Also from (17) and (18)

$$(1-S)U_{n,\alpha}(x) \geq \min_{b=1,2} T_b(1-S)V_{n,\alpha}(x) .$$

Hence

$$\begin{aligned}(\alpha+\Lambda)(1-S)V_{n,\alpha}(x) &= -(1-S)h(x) + \lambda\int(1-S)V_{n-1,\alpha}(x, r)F(dr)\\ &\quad + \mu(1-S)U_{n-1,\alpha}(x)\\ &\geq \delta + \Lambda \min_{b=1,2} T_b(1-S)V_{n-1,\alpha}(x),\end{aligned}$$

where $\delta = h(1, 0) - h(0, 0) > 0$. From iteration we have

$$(1-S)V_{n,\alpha}(x) \geq \delta \sum_{k=0}^{\min[n,|x|]} [\Lambda/(\alpha+\Lambda)]^k/(\alpha+\Lambda).$$

Therefore

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$$\lim_{|x|\to\infty} (1-S)V_\alpha(x) = \lim_{|x|\to\infty} \lim_{n\to\infty} (1-S)V_{n,\alpha}(x)$$

$$\geq \delta \sum_{k=0}^{\infty} [\Lambda/(\alpha+\Lambda)]^k/(\alpha+\Lambda) = \delta/\alpha,$$

and (35) holds. The remainder of this theorem is immediately obtained from Theorem 3.

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References

[1] Bell, C.: Characterization and Computation of Optimal Policies for Operating an M/G/1 Queue with Removable Server. *Opns. Res.*, Vol.19, (1971), 208-218.

[2] Doshi, B.: Optimal Control of the Service Rate in an M/G/1 Queueing System. *Adv. Appl. Prob.*, Vol.10, (1978), 682-701.

[3] Gallisch, E.: On Monotone Optimal Policies in a Queueing Model of M/G/1 TYPE with Controllable Service Time Distribution. *Adv. Appl. Prob.*, Vol.11, (1979), 870-887.

[4] Hajek, B.: Optimal Control of Two Interacting Service Stations. *IEEE Trans. Automat. Contr.*, Vol. AC-29, (1984), 491-499.

[5] Harrison, J. M.: Dynamic Scheduling of a Multi-class Queue, Discount Optimality. *Opns. Res.*, Vol.23, (1975), 270-282.

[6] Heyman, D.: Optimal Operating Policies for M/G/1 Queueing Systems. *Opns. Res.*, Vol.16, (1968), 362-382.

[7] Heyman, D. P. and Sobel, M. J.: *Stochastic Models in Operations Research*, Volume II. McGraw-Hill 1984.

[8] Jo, K.: Optimal Service-Rate Control of Exponential Queueing Systems. *J.O.R.S. of Japan*, Vol.26, (1983), 147-165.

[9] Jo, K. and Stidham, S.: Optimal Service-Rate Control of M/G/1 Queueing Systems Using Phase Method. *Adv. Appl. Prob.*, Vol.15, (1983), 616-637.

[10] Langen, H. J.: Applying the Method of Phases in the Optimization of Queueing Systems. *Adv. Appl. Prob.*, Vol.14, (1982), 122-142.

[11] Lippman, S. A.: Semi-Markov Decision Processes with Unbounded Rewards. *Management Sci.*, Vol.19, (1973), 717-731.

[12] Lippman, S. A. and Stidham, S.: Individual versus Social Optimization in

Copyright © by ORSJ. Unauthorized reproduction of this article is prohibited.

Exponential Congestion Systems. *Opns. Res.*, Vol.25, (1977), 233-247.

[13] Nishimura, S.: Monotone Optimal Control of Arrivals Distinguished by Reward and Sercice Time. *J.O.R.S. of Japan*. Vol.25, (1982), 205-218.

[14] Rosberg, z., Varaiya, P. P. and Walrand, J. C.: Optimal Control of Service in Tandem Queues. *IEEE Trans. Automat. Contr.*, Vol. AC-27, (1982), 600-609.

[15] Sobel, M.: The Optimality of Full Service Policies. *Opns. Res.*, Vol.30, (1982), 636-649.

[16] Stidham, S.: Social and Individually Optimal Control of Arrivals to a GI/M/1 Queue. *Management Sci.*, Vol.24, (1978), 1598-1610.

[17] Topkis, D.: Minimizing a Submodular Function on a Lattice. *Opns. Res.*, Vol.26, (1978), 305-321.

Shōichi NISHIMURA: Institute of
Socio-Economic Planning
University of Tsukuba, Sakura,
Ibaraki, 305, Japan.

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