

TWO MACHINE OPEN SHOP SCHEDULING PROBLEM WITH CONTROLLABLE MACHINE SPEEDS

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Abstract This paper considers a scheduling problem in which the objective is to determine an optimal machine speed pair and an optimal schedule. There are two machines A, B and n jobs each of which consists of two operations. One operation is to be processed on machine A and the other on machine B. All jobs are open shop type, i.e., processing order of two operations is not specified and so processing of each job can be started on either machine. Of course each machine processes at most one job and each job is processed on at most one machine, simultaneously. Further it is assumed that speeds of machines A, B are controllable. In the situation, the total sum of costs associated with the maximum completion time and machine speeds is to be minimized.

The problem is a generalization of two machine open shop problem in a sense that machine speeds are not fixed but variables. This paper proposes an $O(n \log n)$ algorithm which finds an optimal speed of each machine and optimal schedule.

1. Introduction

This paper considers a generalized two machine open shop problem specified as follows.

- (1) There are two machines A, B and n jobs J_1, J_2, \dots, J_n , each of which consists of two operations. All jobs are open shop type, i.e., the processing order of two operations of each job is not specified.
- (2) Each machine processes at most one job and each job is processed on at most one machine, simultaneously. While, preemptions are not allowed.
- (3) Speed of each machine is controllable.
- (4) The objective is to determine an optimal speed of each machine and optimal schedule minimizing the total sum of costs associated with the maximum completion time and speed of each machine.

Section 2 formulates the problem P and introduces subproblem P_h of P. Section 3 proposes an algorithm and clarifies its time complexity. Further Section 3 gives an illustrative example. Finally, Section 4 discusses further research problems.

2. Problem Formulation

First some notations are defined which are used throughout this paper.

s' : Speed of machine A, $s \triangleq 1/s'$. t' : Speed of machine B, $t \triangleq 1/t'$.

a_i : Job processing amount (standard processing time at unit speed) to be processed on machine A.

b_i : Job processing amount (standard processing time at unit speed) to be processed on machine B.

$$T_1 \triangleq \sum_{i=1}^n a_i, \quad T_2 \triangleq \sum_{i=1}^n b_i.$$

$t_{max}(s', t')$: The maximal completion time of optimal schedule subject to machine speeds s' , t' (if any confusion does not occur, simplified notation t_{max} is used).

Note that actual processing times on machines A, B are sa_i and tb_i respectively.

This paper considers the following problem P.

$$\begin{aligned} P: \quad & \text{Minimize } c_0 t_{max}^{q_1} + c_1 (s')^{q_2} + c_2 (t')^{q_2} \\ & \text{subject to } s' > 0, \quad t' > 0. \end{aligned}$$

where c_0, c_1, c_2 are positive constants, and q_1, q_2 are positive integers.

$$\begin{aligned} (2.1) \quad t_{max} &= \max \{ \max_{1 \leq i \leq n} (sa_i + tb_i), sT_1, tT_2 \} \\ &= t \times \max \{ \max_{1 \leq i \leq n} (\gamma a_i + b_i), \gamma T_1, T_2 \} \end{aligned}$$

where $\gamma = s/t = t'/s'$. An optimal schedule giving t_{max} under fixed s', t' can be found by the algorithm due to [1].

Now let $a_{n+1} \triangleq T_1$, $a_{n+2} \triangleq 0$, $b_{n+1} \triangleq 0$, $b_{n+2} \triangleq T_2$ and define $(n+2)$ linear functions of γ ,

$$(2.2) \quad y_j \triangleq \gamma a_j + b_j, \quad j=1, 2, \dots, n, n+1, n+2 \text{ and } y_{\max} \triangleq \max(y_1, \dots, y_n, y_{n+1}, y_{n+2}).$$

Then y is a piecewise linear function and further it is increasing and convex one. Using y , $t_{max} = t \times y$. Megiddo's algorithm in [6] can be utilized to determine y , i.e., t_{max} in at most $O(n \log n)$ computational time. Arranging breaking points of y in an increasing order, let

$$\gamma_0 = 0 < \gamma_1 < \dots < \gamma_p < \gamma_{p+1} = \infty$$

where p is the number of breaking points. Note that on an interval $[\gamma_h, \gamma_{h+1}]$, $y = y_\ell$ for a certain ℓ , $1 \leq \ell \leq n+2$. Thus the following subproblem P_h is introduced.

$$P_h: \text{Minimize } C^h = c_0(s a_\ell + t b_\ell)^{q_1} + c_1(1/s)^{q_2} + c_2(1/t)^{q_2}$$

$$\text{subject to } \gamma (=s/t) \in [\gamma_h, \gamma_{h+1}], s, t > 0$$

where ℓ is the index of y_ℓ that gives y on the interval $[\gamma_h, \gamma_{h+1}]$.

By solving all P_h explicitly or implicitly (refer to Remark in the next page) and taking the best solution among optimal solutions of them, P can be solved, i.e., each optimal speed of A , B and an optimal schedule can be found.

3. An Algorithm

First solution procedure for solving P_h is proposed. By the theorem of the arithmetic and geometric means, it holds that

$$C^h = c_0(s a_\ell + t b_\ell)^{q_1} + c_1(1/s)^{q_2} + c_2(1/t)^{q_2}$$

$$= c_0 t^{q_1} (\gamma a_\ell + b_\ell)^{q_1} + (1/t)^{q_2} \{c_1(1/\gamma)^{q_2} + c_2\}$$

(dividing the first term into q_2 equal components and second q_1 equal components, and applying the theorem of the arithmetic and geometric means ([2]) to these $q_1 + q_2$ components)

$$\geq (q_1 + q_2)^{q_1 + q_2} \sqrt[q_1 + q_2]{(c_0/q_2)^{q_2} (a_\ell \gamma + b_\ell)^{q_1 q_2} (1/q_1)^{q_1} \{c_1(1/\gamma)^{q_2} + c_2\}^{q_1}}$$

where equality holds if and only if

$$t = \frac{q_1 + q_2}{\sqrt[q_1 + q_2]{\{q_2/(c_0 q_1)\} (\gamma a_\ell + b_\ell)^{-q_1} (c_1 \gamma^{-q_2} + c_2)}}$$

Thus in order to solve P_h , it suffices to find a minimizer γ_h^* of

$$f_h(\gamma) \triangleq (\gamma a_\ell + b_\ell)^{q_2} (c_1 \gamma^{-q_2 + c_2}) (1/q_2)$$

on the interval $[\gamma_h, \gamma_{h+1}]$. Once γ_h^* is found, an optimal solution (s_h^*, t_h^*) of P_h is constructed as follows.

$$t_h^* = \frac{q_1 + q_2}{\sqrt{\{q_2 / (c_0 q_1)\} (\gamma_h^* a_\ell + b_\ell)^{-q_1} (c_1 \gamma_h^*)^{-q_2 + c_2}}}$$

$$s_h^* = t_h^* \gamma_h^*.$$

Differentiating $f_h(\gamma)$ with respect to γ ,

$$f'_h(\gamma) = (\gamma a_\ell + b_\ell)^{q_2 - 1} \gamma^{-q_2 - 1} \{a_\ell c_2 \gamma^{q_2 + 1} - b_\ell c_1\}.$$

Note that the sign of $f'_h(\gamma)$ is determined by that of $\gamma^{q_2 + 1} - (b_\ell c_1) / (a_\ell c_2)$.

Thus $f'_h(\gamma)$ changes its sign at most once and so γ_h^* is determined as follows.

- (i) If $(a_\ell c_2)(\gamma_h)^{q_2 + 1} \geq (b_\ell c_1)$, then $\gamma_h^* = \gamma_h$.
- (ii) If $(a_\ell c_2)(\gamma_{h+1})^{q_2 + 1} \leq (b_\ell c_1)$, then $\gamma_h^* = \gamma_{h+1}$.
- (iii) If $(a_\ell c_2)(\gamma_{h+1})^{q_2 + 1} > (b_\ell c_1) > (a_\ell c_2)(\gamma_h)^{q_2 + 1}$, then $\gamma_h^* = \sqrt[q_2 + 1]{(b_\ell c_1) / (a_\ell c_2)}$.

Remark: Further let y_ℓ , gives y on the right interval $[\gamma_{h+1}, \gamma_{h+2}]$ to the interval $[\gamma_h, \gamma_{h+1}]$. Then it holds that $a_\ell \geq a_\ell$ and $b_\ell \leq b_\ell$, and so $(b_\ell c_1) / (a_\ell c_2) \geq (b_\ell c_1) / (a_\ell c_2)$ since y is piecewise linear and increasing.

By Remark, we have the following theorem.

Theorem 1. If case (i) occurs in a certain interval, then only case (i) occurs in its all righter intervals. Similarly, if case (ii) occurs in a certain interval, then only case (ii) occurs in its all lefter intervals.

Further case (iii) occurs at most once and if case (iii) occurs in a certain interval, its optimal speed pair is also an optimal speed pair of P .

Proof: Theorem 1 is easily deduced from Remark and so its proof is omitted.

Q. E. D.

Now we are ready to describe our algorithm for solving P .

Algorithm

Step 1: Calculate breaking points γ_h , $h=1, \dots, p$, and set $\gamma_0=0$ and $\gamma_{p+1}=M$.

Step 2: Find a minimizer of

$$c_h^*(s_{h*}^*, t_{h*}^*) \triangleq \min_h \{c_h^*(s_h^*, t_h^*) \mid h=0,1,\dots,p\}$$

by any binary search technique and set optimal machine speeds s_x', t_x' of A, B as follows:

$$s_x' = 1/s_{h*}^*, \quad t_x' = 1/t_{h*}^*.$$

For this speed pair, construct an optimal schedule by the algorithm in [1]. Terminate.

Theorem 2. Above algorithm finds an optimal speed pair s_x', t_x' and optimal schedule in $O(n \log n)$ computational time for fixed q_1 and q_2 if any power and root can be calculated in a constant time.

Proof: Validity of the algorithm is clear from preceding discussions and so it is omitted.

Calculation of γ_h , that is, construction of y takes $O(n \log n)$ computational time by using Megiddo's algorithm in [6]. Thus Step 1 takes $O(n \log n)$ computational time. For Step 2, s_x', t_x' are determined by solving $O(\log n)$ P_h 's using a binary search technique, and solving each P_h takes $O(n)$ computational time if any power and root can be calculated in a constant time. Thus determination of s_x', t_x' takes $O(n \log n)$ computational time in total.

Once optimal speed of each machine A, B is determined, an optimal schedule can be found in $O(n \log n)$ computational time by the algorithm in [1]. Thus Step 2 takes $O(n \log n)$ computational time. In total, our algorithm finds optimal speed of each machine A, B and optimal schedule in $O(n \log n)$ computational time.

Q. E. D.

Example. consider an example given by the following data:

$$n=3; a_1=4, a_2=5, a_3=6; b_1=24, b_2=4, b_3=2; q_1=q_2=1; c_0=4, c_1=54, c_2=100.$$

Then $T_1=15$, $T_2=30$ and $t_{\max} = t \times \max(4\gamma+24, 5\gamma+4, 6\gamma+2, 15\gamma, 30)$, $y_1=4\gamma+24$, $y_2=5\gamma+4$, $y_3=6\gamma+2$, $y_4=15\gamma$, $y_5=30$. Applying Megiddo's algorithm in [6] to these y_1, y_2, y_3, y_4, y_5 , $y = \max(y_1, y_2, y_3, y_4, y_5)$ is determined by the following process:

Renumbering indices of y_1, y_2, y_3, y_4, y_5 according to lexicographic ordering of (a_i, b_i) , $i=1,2,3,4,5$, results $y_1=30$, $y_2=4\gamma+24$, $y_3=5\gamma+4$, $y_4=6\gamma+2$, $y_5=15\gamma$. Let $g^i(\gamma) \triangleq \max(y_1, \dots, y_i)$, $i=1, 2, \dots, 5$. $g^1(\gamma)=30$.

$$g^2(\gamma) = \begin{cases} 30 & (0 < \gamma \leq 1.5) \\ 4\gamma + 24 & (1.5 < \gamma) \end{cases} \quad \text{and breaking point } \gamma = 1.5.$$

Since $5\gamma + 4 < 4\gamma + 24$ at breaking point $\gamma = 1.5$, $y_3 = 5\gamma + 4$ intersects $g^2(\gamma)$ at the part $4\gamma + 24$ and new breaking point is $\gamma = 20$, that is

$$g^3(\gamma) = \begin{cases} 30 & (0 < \gamma \leq 1.5) \\ 4\gamma + 24 & (1.5 < \gamma \leq 20) \\ 5\gamma + 4 & (20 < \gamma) \end{cases}.$$

Since $6\gamma + 2 > 5\gamma + 4$ at the rightmost breaking point $\gamma = 20$ and $6\gamma + 2 < 4\gamma + 24$ at the leftmost breaking point $\gamma = 1.5$, y_4 intersect $g^3(\gamma)$ at the part $4\gamma + 24$ and new breaking point is $\gamma = 11$. Thus

$$g^4(\gamma) = \begin{cases} 30 & (0 < \gamma \leq 1.5) \\ 4\gamma + 24 & (1.5 < \gamma \leq 11) \\ 6\gamma + 2 & (11 < \gamma) \end{cases}.$$

Similarly,

$$y = g^5(\gamma) = \begin{cases} 30 & (0 < \gamma \leq 1.5) \\ 4\gamma + 24 & (1.5 < \gamma \leq 24/11) \\ 15\gamma & (24/11 < \gamma) \end{cases}.$$

Based on y and breaking points $\gamma_0 = 0$, $\gamma_1 = 1.5$, $\gamma_2 = 24/11$, P is decomposed into the corresponding subproblems P_0 , P_1 and P_2 . By Theorem 1, we first check P_1 .

$$P_1: \text{Minimize } C^1 = 4 \times (4s + 24t) + 54/s + 100/t \\ \text{subject to } \gamma = s/t \in [1.5, 24/11], s, t > 0.$$

Note that $a_\ell = 4$, $b_\ell = 24$, $f_1(\gamma) = (4\gamma + 24)(54/\gamma + 100)$ and $f'_1(\gamma) = -1296/\gamma^2 + 400$.

Since

$a_\ell c_2 \gamma_2^2 = 4 \times 100 \times (24/11)^2 = 1904.13 > b_\ell c_1 = 1296 > a_\ell c_2 \gamma_1^2 = 900$, that is, case (iii) occurs,

$\gamma_1^* = (b_\ell c_1 / a_\ell c_2)^{\frac{1}{2}} = 1.80$, $t_1^* = 1.02$ and $s_1^* = 1.84$. By Theorem 1, an optimal solution (s_1^*, t_1^*) of P is the reciprocal of (s_1^*, t_1^*) , that is, $(0.54, 0.98)$.

Next we must determine corresponding optimal schedule. Actual processing times are:

$$J_1 \text{---} s_1^* a_1 = 7.36, t_1^* b_1 = 24.48; J_2 \text{---} s_1^* a_2 = 9.20, t_1^* b_2 = 4.08; J_3 \text{---} s_1^* a_3 = 11.04, \\ t_1^* b_3 = 2.04. \text{ We divide job set } \{J_1, J_2, J_3\} \text{ into two subsets}$$

$J^{(1)}_{\Delta} \{J_i | s^*_i a_i \geq t^*_i b_i\} = \{J_2, J_3\}$ and $J^{(2)}_{\Delta} \{J_i | s^*_i a_i < t^*_i b_i\} = \{J_1\}$, and choose two distinct jobs J_r, J_ℓ satisfying

$$J_r; s^*_r a_r \geq \max\{t^*_i b_i | J_i \in J^{(1)}\} \text{ and } J_\ell; t^*_\ell b_\ell \geq \max\{s^*_i a_i | J_i \in J^{(2)}\}.$$

In our example, we can choose $J_r = J_2$ and $J = J_1$. Further let

$$\bar{J}^{(1)}_{\Delta} \{J_r, J_\ell\} = \{J_3\} \text{ and } \bar{J}^{(2)}_{\Delta} \{J_r, J_\ell\} = \emptyset.$$

First $\bar{J}^{(2)}_{\cup\{J_\ell\}} = \{J_1\}$ and $\bar{J}^{(1)}_{\cup\{J_r\}} = \{J_2, J_3\}$ are scheduled as Figure 1 and 2 respectively. An optimal schedule is constructed by concatenating $\bar{J}^{(1)}_{\cup\{J_r\}}$ after $\bar{J}^{(2)}_{\cup\{J_\ell\}}$ and moving J_ℓ to the last in A since

$$s^*_1 a_2 + s^*_1 a_3 = 20.24 < t^*_1 b_1 + t^*_1 b_2 = 28.56. \text{ Figure 3 shows this schedule.}$$

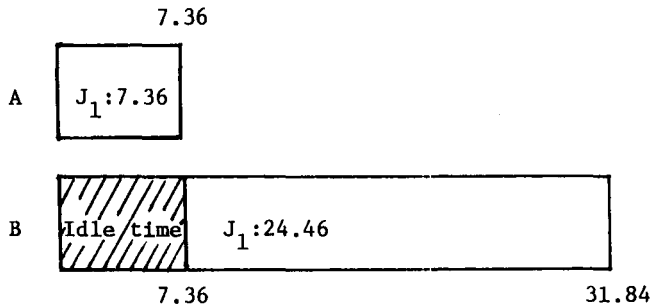


Figure 1. Schedule of $\bar{J}^{(2)}_{\cup\{J_\ell\}} = \{J_1\}$.

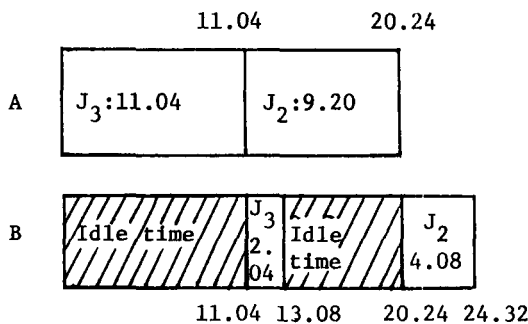


Figure 2. Schedule of $\bar{J}^{(1)}_{\cup\{J_r\}} = \{J_2, J_3\}$.

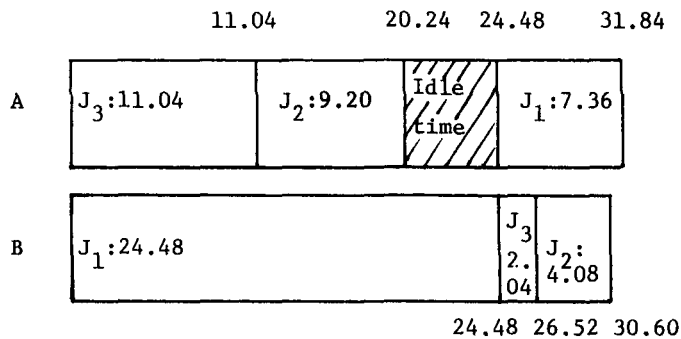


Figure 3. An optimal schedule.

4. Discussion

Up to now, there are very few papers dealing with machine constraints or machine costs explicitly. Only exception is Nakajima et al [7], which considers a machine cost. Models with variable machine speeds, however, are none.

We have already investigated the generalized uniform processor system in [3] and flow shop case in [4], where machine speeds are variables. Moreover, we are preparing the paper treating the mixed shop case. For the ordinary mixed shop scheduling problem, see [5].

Generally speaking, for the success of generalized cases with controllable machine speeds, tractability of the ordinary one is necessary. In this sense, generalized m machine open shop scheduling problem with preemptions may be a promising one and its research is left as one of further research problems. Another is investigation of discrete machine speed cases, though it may be difficult as is seen in [3].

Finally, investigation of more general cost cases is important but may be also difficult.

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