Journal of the Operations Research Society of Japan
Vol. 29, No. 2, June 1986

A SEQUENTIAL EVASION-SEARCH GAME WITH A GOAL

Teruhisa Nakai
Osaka University

(Received January 30, 1985: Final January 27, 1986)

Abstract Infinite boxes 0, 1, 2, ... are arranged in a row in this order. An evader starts from a certain box and chooses at each period one of three decisions: to stay in the current box and to move to the nearest box either to the right or to the left. A searcher looks in any one box except box 0 which is a safe region (goal) for the evader because it is unsearchable. Two types of conditional detection probability are given, that is, one is used in the case that the evader stays and another is used in the case that he moves. It is assumed that the searcher is informed of the evader's position at the end of each period. The evader maximizes the probability that he is not detected for given periods and the searcher minimizes it. This two-person zero-sum sequential game is solved recursively. The evader's optimal strategy indicates to run into the goal as soon as possible if his position is near the goal, to go ahead or stay if he is somewhat away from the goal, and to go back with a positive probability if he is far away from the goal.

1. Introduction

In this paper we consider the following evasion-search problems with a goal, formulate them as a discrete-time two-person zero-sum sequential game and solve it recursively.

(i) Displaced persons, who have been burned out in a war, wish to move toward their safe region with avoiding their enemy. Which of the following three actions should they choose?

  (a) They should run into the safe region as soon as possible.

  (b) They should stay in the current place because the movement quickens detection.

  (c) They should go back for putting the enemy off the scent and wait the end of the war.

(ii) A transport ship, which was detected by the enemy in a war, intends to run away to the safe port in his own country. How should he run away?

(iii) How should the pursuit for the criminal intending to run away into his safe region be carried out?

The study in the area of the discrete-time sequential evasion-search game has begun recently and contains only a few papers. In the Washburn's model [5], both players can move with their own free wills, but the searcher suffers a travelling cost as well as a searching cost. The payoff is the expected total cost to detect the evader. Under the assumption of a noisy searcher and perfect detection, he obtains the optimal strategy for the searcher. In the Nakai's model [2], the evader can go ahead as he likes, but cannot go back. The payoff is the expected total reward (reward minus search cost) for the searcher in a given length of periods. Under the assumption of a noisy evader, he obtains the optimal strategies. In the Lee's model [1], the evader is noisy and can move to only one of the neighboring boxes of the current box at each period. There is a safe region (goal) for the evader. The payoff is the number of detections before the evader runs into the goal. Under the assumption of perfect detection, he obtains the optimal strategies for both players. The models of Ruckle [3] and Stewart [4] are sequential in the real development of games, but are one-stage games in the mathematical treatment.

2. Model And Formulation

Infinite boxes 0,1,2, are arranged in a row in this order. The evader (player I) states from some box and when he is in box n ($\neq 0$), he must choose one of the three decisions : (i) to stay in box n, (ii) to go to ahead to box (n-1) and (iii) to go back to box (n+1). The searcher (player II) can at each period look in any one box except box 0. Since box 0 is unsearchable, it is a safe region for the evader and is called the "goal". When the evader has come from the neighboring box at the preceding period, let $\alpha (0<\alpha<1)$ be the conditional detection probability given that the current position of the evader is looked. When the evader stays in the same box from the preovious period, let $\beta (0<\beta<1)$ be the conditional detection probability. Note that α and β are independent of the number of the box. We assume that the evader is noisy, that is, the searcher is informed of the evarder's position at the end of each period. The payoff is given by the probability that the evader is not detected in the given $K(\geq 1)$ periods. Player I (II) intends to maximize (minimize) it.

The state of the process can be denoted by a pair (k,n) where k is the number of the remaining periods and n is the current position of the evader. Let $v(k,n)$ be the game value of the sequential game which starts from a state(k,n). Since $\alpha,\beta < 1$, it is evident that $0<v(k,n)\leq 1$ for any k,n. Moreover by the definition of $v(k,n)$, $v(0,n)=v(k,0)=1$ for any k,n. If the evader is in box 1, he can escape from the searcher by running away into box 0 and therefore $v(k,1)=1$ for any k. When the state is (k,n) ($k\geq 1$, $n\geq 2$), player I has three pure strategies: H_i (to hide in box i) : $i=n-1$, n, $n+1$, and player II has infinite numbers of pure strategies : L_j (to look in box j) : $j=1,2,\cdots$. But it is useless to look in a box which does not contain the evader, that is, pure stragegies L_j ($j\neq n-1,n$, $n+1$) are dominated. If the evader hides in box $n-1$ and the searcher looks in the same box, with probability α the detection occurs and the process stops with the payoff 0, but with probability $1-\alpha$ the evader is not detected and the process transfers to the next period at which the state is $(k-1, n-1)$ and therefor in this case the payoff for the evader is given by $(1-\alpha)v(k-1,n-1)$. By the similar considerations, we can obtain the following recurrent relation.

$$
(2.1) \qquad v(k,n)=\mathrm{Val}\ \begin{array}{c} \\ H_{n-1} \\ H_n \\ H_{n+1} \end{array}\!\!\begin{array}{c} \begin{array}{ccc} L_{n-1} & L_n & L_{n+1} \end{array} \\ \begin{bmatrix} (1-\alpha)v(k-1,n-1) & v(k-1,n-1) & v(k-1,n-1) \\ v(k-1,n) & (1-\beta)v(k-1,n) & v(k-1,n) \\ v(k-1,n+1) & v(k-1,n+1) & (1-\alpha)v(k-1,n+1) \end{bmatrix} \end{array}
$$
$$(k=1,2,3,\cdots;\ n=2,3,4,\cdots)$$

where Val denotes the game value. Let $\Gamma(k,n)$ be the matrix game in the right-hand side of the equation (2.1).

3. Optimal solution

The following theorem states that the increase of the number of the remaining periods k (or the evader's position n or conditional detection probabilities α,β) is of no benefit to the evader.

Theorem 1. The game value $v(k,n)$ is nonincreasing in k, n, α and β.

Proof: The proof on k. By induction on k, we prove that $v(k,n)\geq v(k+1,n)$ for any n. The case of $k=0$ is clear since $v(0,n)=1$ and $v(1,n)\leq 1$. Assume that the assertion is satisfied for $k=0,1,2,\cdots,\ell-1$. Comparing each component of $\Gamma(\ell,n)$ with that of $\Gamma(\ell+1,n)$, the former is larger by the assumption of induction and therefore $v(\ell,n)\geq v(\ell+1,n)$ for any n which states that the assertion is satisfied for $k=\ell$. Therefore the proof on k is completed. Similarly by induction on k we can prove that $v(k,n)\geq v(k,n+1)$ for any n. Finally we prove the nonincreasing property in α and β by induction on n. The case of $k=0$ is clear since $v(0,n)=1$ for any α and β. Assume that $v(k,n)$ is nonincreasing in α and β for $k=0,1,2,\cdots,\ell-1$. Each component of $\Gamma(\ell,n)$ is nonincreasing in α and β and therefore the assertion is satisfied for $k=\ell$. (q.e.d.)

As a matter of convenience, in the below discussion we often use a notation $w(n)=v^{-1}(k-1,n)$ for fixed k.

The following theorem expresses the value and the optimal strategies at the first period of the sequential game starting from the state (k,n) by values of $(k-1)$-period sequential games. Hence the solution of the sequential game with any number of periods can be calculated recurrently.

Theorem 2. The optimal strategy $x^*(y^*)$ for player I (II) at the first period and the value of the sequential game starting from the state (k,n) $(k\geq 1, n\geq 2)$ are given as follows:

(i) If $w(n-1)\leq(1-\alpha)w(n)$, then

$$x^*=y^*=\langle 1,0,0\rangle \text{ and } v(k,n)=(1-\alpha)v(k-1,n-1). \tag{3.1}$$

(ii) If $w(n-1)>(1-\alpha)w(n)$ and $\beta w(n-1)+\alpha w(n)\leq(\alpha+\beta-\alpha\beta)w(n+1)$, then

$$x^*=\left\langle \frac{\beta w(n-1)}{\beta w(n-1)+\alpha w(n)}, \frac{\alpha w(n)}{\beta w(n-1)+\alpha w(n)}, 0\right\rangle \tag{3.2}$$

$$y^*=\left\langle \frac{w(n)-(1-\beta)w(n-1)}{\beta w(n-1)+\alpha w(n)}, \frac{w(n-1)-(1-\alpha)w(n)}{\beta w(n-1)+\alpha w(n)}, 0\right\rangle \tag{3.3}$$

$$v(k,n)=\frac{\alpha+\beta-\alpha\beta}{\beta w(n-1)+\alpha w(n)}. \tag{3.4}$$

(iii) If $w(n-1)>(1-\alpha)w(n)$ and $\beta w(n-1)+\alpha w(n)>(\alpha+\beta-\alpha\beta)w(n+1)$, then

$$x^*=\left\langle \frac{\beta w(n-1)}{\beta w(n-1)+\alpha w(n)+\beta w(n+1)}, \frac{\alpha w(n)}{\beta w(n-1)+\alpha w(n)+\beta w(n+1)}, \frac{\beta w(n+1)}{\beta w(n-1)+\alpha w(n)+\beta w(n+1)}\right\rangle \tag{3.5}$$

$$(3.6)\quad y^*=\left\langle \frac{\alpha w(n)+\beta w(n+1)-(\alpha+\beta-\alpha\beta)w(n-1)}{\alpha\{\beta w(n-1)+\alpha n(n)+\beta w(n+1)\}},\ \frac{w(n-1)+w(n+1)-(2-\alpha)w(n)}{\beta w(n-1)+\alpha w(n)+\beta w(n+1)},\right.$$

$$\left.\frac{\beta w(n-1)+\alpha w(n)-(\alpha+\beta-\alpha\beta)w(n+1)}{\alpha\{\beta w(n-1)+\alpha w(n)+\beta w(n+1)\}}\right\rangle$$

$$(3.7)\quad v(k,n)=\frac{\alpha+2\beta-\alpha\beta}{\beta w(n-1)+\alpha w(n)+\beta w(n+1)}\ .$$

Proof: In case (i), the result is clear since the matrix game $\Gamma(k,n)$ has a saddle point at (1,1)-element. In case (ii), $x_i^*\geq 0$ ($=1,2,3$), $\sum_{i=1}^{3} x_i^*=1$ and $\sum_{j=1}^{3} y_j^*=1$ are clear. $w(n)-(1-\beta)w(n-1)=w(n-1)w(n)\times [v(n-1)-(1-\beta)v(n)]>0$ by Theorem 1 and $\beta>0$, and hence $y_1^*>0$. Also $w(n-1)-(1-\alpha)w(n)=w(n-1)w(n)[v(k-1,n)-(1-\alpha)v(k-1,n-1)]>0$ by the assumption, and hence $y_2^*>0$. Next by simple calculation, we can check $[x^*\Gamma(k,n)]_1=[x^*\Gamma(k,n)]_2=v(k,n)$ and $[x^*\Gamma(k,n)]_3=(\alpha+\beta)/[\beta w(n-1)+\alpha w(n)]>v(k,n)$. Similarly we can check $[\Gamma(k,n)y^*]_1=[\Gamma(k,n)y^*]_2=v(k,n)$ and $[\Gamma(k,n)y^*]_3=v(k-1,n+1)\leq v(k,n)$. Hence the solution is given by (3.2), (3.3) and (3.4). In case (iii), $x_i^*>0$ ($=1,2,3$), $\sum_{i=1}^{3} x_i^*=1$ and $\sum_{j=1}^{3} y_l^*=1$ are clear. By the assumption and Theorem 1, $\alpha w(n)+\beta w(n+1)-(\alpha+\beta-\alpha\beta)w(n-1)>\{\alpha+\beta(2-\alpha)\}\times\{w(n+1)-w(n-1)\}\geq 0$, and hence $y_1^*>0$. Also $w(n-1)+w(n+1)-(2-\alpha)w(n)>w(n+1)-w(n)\geq 0$ and hence $y_2^*>0$. $y_3^*>0$ is clear by the assumption. Next by simple calculation, we can check $[x^*\Gamma(k,n)]_j=[\Gamma(k,n)y^*]_i=v(k,n)$ for any i and j. Hence the solution is given by (3.5), (3.6) and (3.7).

(q.e.d.)

Note that by the property of the problem the solution in the case of $n=1$ is given by $x^*=\langle 1,0,0\rangle$, y^* is arbitrary, and $v(k,1)=1$.

The assumption of the case (i) in Theorem 2 states that even if the searcher looks in box $(n-1)$, moving to box $(n-1)$ is not worse than staying in box n (of course, than moving to box $(n+1)$). On the other hand, if the searcher looks in box n or $n+1$, it is clear that moving to box $(n-1)$ is better. Hence the result of the case (i) is reasonable.

In the following lemma, the game value $v(k,n)$ is shown to be convex in n.

Lemma 1. $v(k,n+1)/v(k,n)$ is nondecreasing in n for any k.

Proof: By induction on k. The case of $k=0$ is clear since $v(0,n)=1$ for any n. Assume that $v(k,n+1)/v(k,n)$ is nondecreasing in n for $k=0,1,2,\cdots,\ell$.

We show that the assertion is satisfied for $k=\ell+1$. There are nine cases according to types (in Theorem 2) of the game values $v(\ell+1,n)$ and $v(\ell+1,n+1)$. For example, if $v(\ell+1,n)$ and $v(\ell+1,n+1)$ are given by cases (ii) and (iii) in Theorem respectively,

$$\frac{v(\ell+1,n+1)}{v(\ell+1,n)}=\frac{\alpha+2\beta-\alpha\beta}{\alpha+\beta-\alpha\beta}\left[\frac{\alpha+\beta v(\ell,n)/v(\ell,n-1)}{\beta+\alpha v(\ell,n)/v(\ell,n+1)+\beta v(\ell,n)/v(\ell,n+2)}\right]$$

which is nondecreasing in n by the assumption of induction. Similarly in other eight cases, we can check that $v(\ell+1,n+1)/v(\ell+1,n)$ is nondecreasing in n. Hence the proof is completed. (q.e.d.)

In the following lemma, we clarify the meanings of conditions of three cases in Theorem 2.

Lemma 2. For a fixed k, we put

(3.8) $n_1{}^*=\max\{n\,|\,w(n-1)\leq(1-\alpha)w(n)\}$

(3.9) $n_2{}^*=\max\{n\,|\,\alpha w(n-1)+\alpha w(n)\leq(\alpha+\beta-\alpha\beta)w(n+1)\}$.

For any n (≥ 2),

(i) $n\left\{\begin{matrix}\leq\\ >\end{matrix}\right\}n_1{}^* \Longleftrightarrow w(n-1)\left\{\begin{matrix}\leq\\ >\end{matrix}\right\}(1-\alpha)w(n)$

(ii) $n\left\{\begin{matrix}\leq\\ >\end{matrix}\right\}n_2{}^* \Longleftrightarrow \beta w(n-1)+\alpha w(n)\left\{\begin{matrix}\leq\\ >\end{matrix}\right\}(\alpha+\beta-\alpha\beta)w(n+1)$

(iii) $0\leq n_1{}^*\leq n_2{}^*\leq\infty$.

Proof: By Lemma 1, $w(n-1)/w(n)=v(k-1,n)/v(k-1,n-1)$ is nondecreasing in n for any k, and hence the result of case (i) is clear. Similarly $[\beta w(n-1)+\alpha w(n)]/w(n+1)=\beta v(k-1,n+1)/v(k-1,n-1)+\alpha v(k-1,n+1)/v(k-1,n)$ is nondecreasing in n for any k, and hence the result of case (ii) is clear. To prove (iii), we show that if $w(n-1)\leq(1-\alpha)w(n)$, then $\beta w(n-1)+\alpha w(n)\leq(\alpha+\beta-\alpha\beta)w(n+1)$.

$$\frac{\beta w(n-1)+\alpha w(n)}{w(n+1)}=\frac{w(n)}{w(n+1)}\left[\beta\,\frac{w(n-1)}{w(n)}+\alpha\right]\leq\alpha+\beta-\alpha\beta$$

by the assumption and Theorem 1. (q.e.d.)

Taking Theorem 2 and Lemma 2 into consideration, we can obtain the following theorem without proof.

Theorem 2. Define $n_1{}^*$ and $n_2{}^*$ by (3.8) and (3.9) respectively. The solution of the sequential game starting from the state (k,n) is given as follows:

(i) If $2\leq n\leq n_1{}^*$, then the solution is given by (2).

(ii) If $n_1{}^*<n\leq n_2{}^*$, then the solution is given by (3.2), (3.3) and (3.4).
(iii) If $n_2{}^*<n$, then the solution is given by (3.5), (3.6) and (3.7).

When the evader is near the goal, he must go ahead with probability 1. When he is somewhat away from the goal, he must go ahead or stay, but never go back. When he is far away from the goal, he must go back with a positive probability, and therefore his optimal stragegy is completely mixed. Namely, the further the evader's position is away from the goal, the more complicated his actions become, and it is optimal to wait the end of the process with putting the searcher off the scent. On the other hand, the searcher must look in the box for which the evader's optimal strategy allocates a positive probability. The author cannot clarify the more concrete meaning of the probability in optimal strategies (3.2), (3.3), (3.5) and (3.6).

Corollary 1. If $n>k$ (=1,2,$\cdots$), then the solution is given as follows:

$$x^*=y^*=\left\langle \frac{\alpha^{-1}}{2\alpha^{-1}+\beta^{-1}},\ \frac{\beta^{-1}}{2\alpha^{-1}+\beta^{-1}},\ \frac{\alpha^{-1}}{2\alpha^{-1}+\mu^{-1}} \right\rangle \tag{3.10}$$

$$v(k,n)=a^k \tag{3.11}$$

where $a=(\alpha+2\beta-\alpha\beta)/(\alpha+2\beta)$.

Proof: By induction on k. In the case of $k=1$, we can check that the case (iii) in Theorem 2 occurs and therefore the results can be derived from (3.5), (3.6) and (3.7). Assume that the results (3.10) and (3.11) are satisfied for $k=1,2,\cdots,\ell$. For any $n(>\ell+1)$, $v(\ell,n)-(1-\alpha)v(\ell,n-1)=\alpha a^\ell>0$ and $\beta v^{-1}(\ell,n-1)+\alpha v^{-1}(\ell,n)-(\alpha+\beta-\alpha\beta)v^{-1}(\ell,n+1)=\alpha\beta a^\ell>0$, and hence the case (iii) in Theorem 2 occurs. Then the results can be derived from (3.5), (3.6) and (3.7).

(q.e.d.)

When the evader's position n is larger than the number k of the remaining periods, the evader cannot reach to the goal during k periods even if he runs into the goal as soon as possible. Therefore he gives up running into the goal and hides near the current position optimally as if the goal does not exist. Namely, he hides in his nearest boxes with the rates proportional to the inverses of the conditional detection probabilities. On the other hand, the searcher's optimal strategy is to look in those boxes with the same rates.

4. Discussion

We consider some modifications of our model.

(i) Assume that when the evader is in box n, he can move to one of the $(m_1 + m_2+1)$ boxes numbered $n-m_1, \cdots n-1$, n, $n+1 \cdots$, and $n+m_2$. Also in this case, it seems that the fundamental property of our model holds still. Namely, near the goal the evader must run into the goal at full speed and in the very far position from the goal he must behave as if there is no goal.

(ii) When the conditional detection probability depends on the number of the looked box, it seems that with only slight modification our result remains valid.

(iii) The following two modifications of our model are valuable open problems.

a. Introducing the detection reward and the search cost, we define the payoff as the expected total loss (cost minus reward) until the searcher stops the search.

b. Assume that the evader is silent, that is, the searcher is not informed of the evader's position at each period. The model is a sequential game with incomplete information.

Acknowledgement

The author would like to thank the referee for his valuable comments made on this paper.

References

[1] Lee, K. T. : A Firing Game with Time Lag. *Journal of Optimization Theory and Applications*, Vol.41, No.4 (1983), 547-558.

[2] Nakai, T. : A Sequential Evasion-Search Game. *Mathematica Japonica*, Vol.28, No.3 (1983), 315-326.

[3] Ruckle, W. H. : Pursuit on a Cyclic Graph-The Symmetric Stochastic Case. *International Journal of Game Theory*, Vol.10, No.2 (1981), 91-99.

[4] Stewart, T. J. : A Two-Cell Model of Search for an Evading Target. *European Journal of Operational Research*, Vol.8 (1981), 369-378.

[5] Washburn, A. R. : Search-Evasion Game in a Fixed Region. *Operations Research*, Vol.28, No.6 (1980), 1290-1298.

Teruhisa NAKAI : Department of Applied
Mathematics, Faculty of Engineering
Science, Osaka Univeristy,
Machikaneyama 1-1, Toyonaka
Osaka, 560, Japan