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PERIODIC PROPERTY OF STREETCAR CONGESTION AT THE FIRST STATION

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Abstract This analysis explores the periodic property of congestion in the streetcars arriving at a station with an equal time interval, assuming the Poisson arrivals of passengers. We obtain a stationary condition and some properties of the spectral density about the number of passengers on board and consider the periodicity of the congestion in terms of numerical examples which show the existences of three types of spectral density.

1. Introduction

In the essay in 1922, a Japanese physicist Torahiko Terada [11] described the congestion phenomenon of the streetcar, which is called *chin-chin densha* in Japan. His argument can be briefly summarized as follows. Streetcars arrive at the first station at the regular intervals and leave there for the next station as soon as the boarding of passengers is completed. Since the number of passengers varies from time to time, the staying time of each streetcar fluctuates. This fluctuation generates the periodicity of congestion. For example, when many passengers happen to be waiting at the first station, this streetcar must stay there for a long time and arrives at the next station with delay. During the prolonged time interval at the second station between the departure of the preceding one and the arrival of this streetcar, the number of passengers tends to increase and it needs somewhat long time again for boarding. In sequence, the similar phenomenon occures at the third and the fourth stations and consequently this streetcar becomes congested more and more. Conversely the time interval between the departure of this relatively congested streetcar and the arrival of the next one is shorter than the mean time interval and a few passengers board this succeeding streetcar. This makes the interval between two streetcars shorter and shorter.

Thus this succeeding streetcar becomes congested less and less.

Terada found based on his observation of actual schedule at a certain point that the congested one comes after the long interval and that the less congested one comes after the short interval. Since he was sick, he used the above observation for boarding the streetcar with vacant seats.

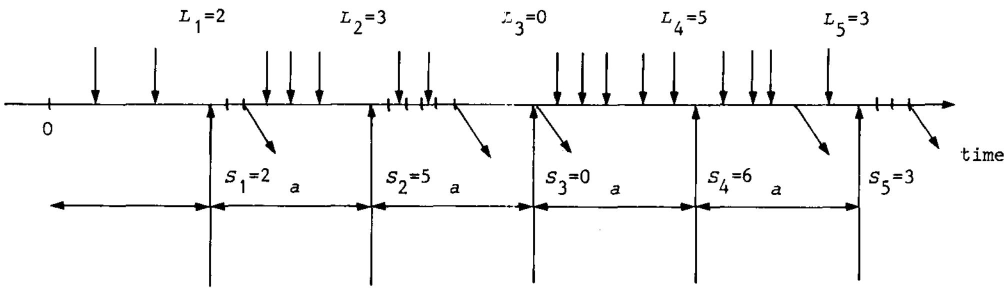
Although Terada's essay is famous in Japan and a member of Logergist [7] considered the congestion of cars on the highway by analogy with the congestion of streetcars, the present author has found no published literature which rigorously analyzed this phenomenon. The purpose of this paper is to represent the above phenomenon at the first station as the spectral density.

We build the model in the next section and provide a stationary condition of the model in section 3. For the study of periodicity in a stationary process, the spectral distribution function or the spectral density function is useful. For example, if the spectral distribution function of a stationary process has the jump on a certain point x in the interval $[0,\pi]$, it has periodic element with period $2\pi/x$. We notice the number of passengers on board and discuss the general properties of its spectral density function in sections 4 and 5. Section 6 gives numerical examples. These examples show the existence of three types of spectral density.

2. Model

We consider the following simple model. Passengers arrive at a station at the time instants τ_1^e, $\tau_1^e + \tau_2^e$, $\cdots$. We assume that this input sequence follows the Poisson process with intensity λ. The streetcars arrive at the station at the time instants a, $2a$, $3a$, $\cdots$ without passengers, where a is a constant. Let N be a boarding capacity of the streetcar. Although several authors (e.g.[2][10]) study about the variation of time when passengers need to board, we assume for brevity that each passenger's boarding time is constant and we represent it as b. Let L_n be the number of the waiting passengers at the arrival epoch of the nth streetcar, and let S_n be the number of passengers boarding the nth streetcar, which represents the congestion of the streetcar. If $L_n = 0$, the streetcar does not stop at this station and $S_n = 0$. If $L_n \geqq N$, the streetcar leaves at time $na + Nb$ and $S_n = N$. When $0 < L_n < N$, the streetcar stays there till no waiting passengers exist or till the time $na + Nb$. Let r_n be the number of passengers remaining at the station when the nth streetcar leaves. We assume that $Nb \leqq a$, which means that the streetcar starts before the arrival of the next streetcar. Figure 1 illustrates an example for $N = 6$.

Arrivals of passengers



Arrivals of streetcars

Figure 1.

3. Stationary Condition

In our model $\{L_n,\ n\geq 1\}$ and $\{(S_n,\ r_n),\ n\geq 1\}$ are Markov chains. Since we are interested in the S_n, we consider about the sequence $\{(S_n,\ r_n)\}$. We call the state $\{S_n=i,\ r_n=0\}$ as state i and the state $\{S_n=N,\ r_n=i\}$ as state $N+i$. Let P_{ij} be the transition probability from state i to state j and $P = (P_{ij})_{j=0,1,\cdots}^{i=0,1,\cdots}$ be the transition matrix. Since $\{(S_n,\ r_n),\ n\geq 1\}$ is an irreducible Markov chain, there is a limit u_j for each j such that $\lim_{n\to\infty} P_{ij}^{(n)} = u_j$.

Theorem 3.1. There is a stationary distribution of the Markov chain $\{(S_n,\ r_n)\}$ if and only if $\lambda a < N$. When $\lambda a < N$, the mean of S_n in the stationary Markov chain $\{(S_n,\ r_n)\}$ is λa.

Proof: Let f_n be the number of total passengers arriving at a station on the time interval $(0,\ na)$. The numbers of passengers arriving at the time interval $(ia,\ (i+1)a]$, $i=0,\cdots,n-1$ are mutually independent so that by the strong law of large numbers,

$$\lim_{n\to\infty} f_n/n = \lambda a \qquad w.p.1. \tag{3.1}$$

(Sufficiency) We assume that $\lambda a < N$. When each state is nonrecurrent, the Markov chain will be in each state only finitely often ([1] p.20, Corollary of Theorem 3), so that $\lim_{n\to\infty} r_n = \infty$ with probability one. Hence $\lim_{n\to\infty} f_n/n \geq \lim_{n\to\infty} \sum_{i=1}^{n-1} S_i/n = N$. This contradicts (3.1).

When each state is recurrent null, we let $(0=k_0^{(j)}<)\ k_1^{(j)}<k_2^{(j)}<\cdots$ be the streetcar numbers carrying j passengers. If $j<N$, then $\{X_i^{(j)}=k_i^{(j)}-k_{i-1}^{(j)},\ i\geqq 2\}$ is the mutually independent positive sequence. Therefore $\lim_{p\to\infty}\sum_{i=2}^{p} X_i^{(j)}/p = EX_i^{(j)} = \infty$ *w.p.1*. Let $i(j,n)$ be the integer such that $k_{i(j,n)}^{(j)} \leqq n < k_{i(j,n)+1}^{(j)}$. Since $\lim_{n\to\infty} i(j,n) = \infty$ *w.p.1*,

$$\frac{i(j,n)}{n} \leqq i(j,n)/k_{i(j,n)}^{(j)} = i(j,n)/\sum_{q=1}^{i(j,n)} X_q^{(j)} \xrightarrow{n\to\infty} 0.$$

Hence

$$\begin{aligned}\lim f_n/n &\geqq \liminf_{n\to\infty} \sum_{i=1}^{n-1} S_i/n \\ &= \liminf \{N(n-1-\sum_{j=0}^{N-1} i(j,n)) + \sum_{j=0}^{N-1} j i(j,n)\}/n \\ &= N.\end{aligned}$$

This contradicts (3.1).

When each state is nonnull recurrent, the stationary distribution exists by the well known theorem ([1] I §6 and §7).

(Necessity) We assume that $u_i > 0$ for all i. Let ξ_n be the random variable which means the number of elements of the set $\{i:\ i \leqq n,\ S_i < N\}$. Then

$$\lim \xi_n/n = \sum_{i=0}^{N-1} u_i > 0 \quad w.p.1,$$

(see [1] p.93, Corollary 1) and therefore $\limsup_{n\to\infty} \sum_{i=1}^{n-1} S_i/n < N$ with probability one. If we select the sequence n_i such that $S_{n_i} < N$,

$$\lambda a = \lim_{n\to\infty} f_n/n = \lim_{i\to\infty} f_{n_i}/n_i \leqq \liminf_{i\to\infty} \sum_{j=1}^{n_i} S_j/n_i < N \quad w.p.1.$$

Lastly we will consider the mean ES_n of the stationary distribution. If $\lambda a < N$, there is an infinite sequence n_j such that $S_{n_j} = 0$. Hence we obtain

$$\begin{aligned}ES_n &= \sum_{i=0}^{N-1} i u_i + N(1-\sum_{i=0}^{N-1} u_i) = \lim_{j\to\infty} \sum_{i=1}^{j-1} S_i/j, \quad w.p.1. \\ &= \lim_{j\to\infty} \sum_{i=0}^{n_j-1} S_i/n_j = \lim_{j\to\infty} f_{n_j}/n_j = \lambda a.\end{aligned}$$

Q.E.D.

4. Spectral Density of s_n

Strictly speaking, s_n in Figure 1 is not the stationary process, because the initial condition L_0 is assumed to be zero. In what follows, we assume that $\lambda a < N$. Since there is the stationary distribution under this assumption, we can make the stationary process s_n by giving the stationary distribution to the dummy streetcar (s_0, r_0) or L_0.

We will show the absolute continuity of the spectral distribution of the stationary s_n process. As the preparation, we will find the transition matrix P and the rate of convergence of each state. Let $A = (\alpha_{ij})_{j=0,1,\cdots}^{i=0,1,\cdots}$ and $B = (\beta_{ij})_{j=0,1,\cdots}^{i=0,1,\cdots}$ be the matrices whose components are respectively

$$\alpha_{ij} = Pr\{L_n = j \mid s_{n-1} = \min(i, N),\ r_{n-1} = \max(i-N, 0)\}$$

$$\beta_{ij} = Pr\{s_n = \min(j, N),\ r_n = \max(j-N, 0) \mid L_n = i\}.$$

Then P is given as $P = AB$. The α_{ij} is given as

$$\alpha_{ij} = \begin{cases} \dfrac{\{\lambda(a-ib)\}^j}{j!} e^{-\lambda(a-ib)} & : i < N \\[2ex] \dfrac{\{\lambda(a-Nb)\}^{j-i+N}}{(j-i+N)!} e^{-\lambda(a-Nb)} & : i \geqq N,\ j \geqq i-N \\[2ex] 0 & : i \geqq N,\ j < i-N. \end{cases}$$

To obtain β_{ij}, consider the case that $N = \infty$ and $a = \infty$. Under the condition that there are i waiting persons just before the arrival of a streetcar, we define $P(j,k|i)$ as the probability that not less than j persons board and k persons remain just after the j-th person's boarding. That is,

$$\text{(4.1)} \quad \begin{cases} P(j,k|0) = \begin{cases} 1 : j = k = 0 \\ 0 : \text{otherwise} \end{cases} \\[2ex] P(j,k|i) = \displaystyle\sum_{n=1}^{j+k-i} \frac{(i\lambda b)^n}{n!} e^{-i\lambda b} P(j-i,k|n) & : 1 \leqq i < j \\[2ex] P(j,k|i) = \dfrac{(j\lambda b)^{j+k-i}}{(j+k-i)!} e^{-j\lambda b} & : j \leqq i \leqq j+k \\[2ex] P(j,k|i) = 0 & : j+k < i. \end{cases}$$

Then, returning to the model with finite N and a, the β_{ij} is represented by

$$\beta_{ij} = \begin{cases} P(j,0|i) & : i \leqq j < N \\ P(N,j-N|i) & : j \geqq N \\ 0 & : \text{otherwise.} \end{cases}$$

Since $P(j,k|i)$ is obtained recursively by (4.1), we can obtain β_{ij} and the transition matrix.

Lemma 4.1. The radius of convergence of the power series $\sum_{j=0}^{\infty} P_{ij}x^j$ is infinite.

Proof: If $s \geqq 1$ and $t \geqq 0$, then from (4.1)

$$P(t+s,m|t) = \sum_{j=1}^{m+s} \frac{(t\lambda b)^j}{j!} \exp(-t\lambda b)P(s,m|j)$$

so that by the mathematical induction method with respect to s, we obtain

$$P(t+s,m|t) \leqq \frac{(t+s)^{m+s}}{(m+s)!} (\lambda b)^{m+s} \exp\{-(t+s)\lambda b\}.$$

Hence when $0 \leqq t \leqq N-1$ and x is positive,

$$\sum_{j=t}^{\infty} \beta_{tj}x^j = \sum_{j=t}^{N-1} P(j,0|t)x^j + \sum_{j=N}^{\infty} P(N,j-N|t)x^j$$

$$\leqq \sum_{j=t}^{N-1} P(j,0|t)x^j + x^t \exp(-N\lambda b+N\lambda bx) - \sum_{j=t}^{N-1} \frac{(N\lambda b)^{j-t}}{(j-t)!} \exp(-N\lambda b)x^j.$$

When $t \geqq N$,

$$\sum_{j=t}^{\infty} \beta_{tj}x^j = \sum_{j=t}^{\infty} P(N,j-N|t)x^j = x^t \exp(-N\lambda b+N\lambda bx).$$

Since, for a positive x,

$$\lim_{r\to\infty} \sum_{j=0}^{r} P_{ij}x^j = \lim_{r\to\infty} \sum_{t=0}^{r} \alpha_{it} \sum_{j=t}^{r} \beta_{tj}x^j \leqq \lim_{r\to\infty} \sum_{t=0}^{r} \alpha_{it} \sum_{j=t}^{\infty} \beta_{tj}x^j < \infty,$$

$\sum_{j=0}^{\infty} P_{ij}x^j$ converges absolutely for any complex number x.

Q.E.D.

Lemma 4.2. For each positive integer d there is a constant K_{ij} such that $|P_{ij}^{(n)}-u_j| < K_{ij}/n^d \quad (i,j=0,1,\cdots)$.

Proof: Popov ([9] Theorem 2 and 3) showed that this lemma holds if there are numbers $g_j^{(0)}=1$, $g_j^{(t)} \geqq 0$ $(t=1,2,\cdots,d : j=0,1,2,\cdots)$ such that for $t=1,\cdots,d$

$$(4.2) \qquad \sum_{j=0}^{\infty} p_{ij} \sum_{s=0}^{t} {}_tC_s g_j^{(s)} \qquad \begin{cases} \leqq g_i^{(t)} & : i > 0 \\ < \infty & : i = 0 \end{cases}$$

where ${}_tC_s = \frac{t!}{s!(t-s)!}$. For $t > 0$, we will find $g_j^{(t)}$ with the form

$$g_j^{(t)} = \begin{cases} 0 & : j = 0,\ t > 0 \\ M_t \eta_t^j & : j > 0,\ t > 0 \end{cases}$$

where M_t and η_t are positive numbers. Then (4.2) becomes

$$(4.3) \qquad p_{io} + \sum_{s=0}^{t} {}_tC_s M_s \sum_{j=1}^{\infty} p_{ij} \eta_s^j \qquad \begin{cases} \leqq M_t \eta_t^i & : i > 0 \\ < \infty & : i = 0 \end{cases}$$

where $M_o = \eta_o = 1$.

When $i = 0$, (4.3) holds for any M_s and η_s by Lemma 4.1. We consider the case $i \geqq 1$. Since $p_{ij} = \frac{(\lambda a)^{j-i+N}}{(j-i+N)!} e^{-\lambda a}$ $(i \geqq 2N,\ j \geqq i-N)$ and $p_{ij} = 0$ $(i \geqq 2N,\ j < i-N)$,

$$p_{io} + \sum_{s=0}^{t} {}_tC_s M_s \sum_{j=1}^{\infty} p_{ij} \eta_s^j - M_t \eta_t^i$$

$$= p_{io} + \sum_{s=0}^{t-1} {}_tC_s M_s \sum_{j=1}^{\infty} p_{ij} \eta_s^j + \begin{cases} M_t \left(\sum_{j=1}^{\infty} p_{ij} \eta_t^j - \eta_t^i \right) & : 1 \leqq i \leqq 2N-1 \\ \eta_t^i M_t \{ \eta_t^{-M} exp(-\lambda a + \lambda a \eta_t) - 1 \} & : i \geqq 2N. \end{cases}$$

We will show the existence of numbers $(1=)\eta_0 < \eta_1 < \cdots < \eta_d$ satisfying the inequalities

$$(4.4) \qquad \eta_t^{-N} exp(-\lambda a + \lambda a \eta_t) < 1 \qquad (t = 1, \cdots, d)$$

and

$$(4.5) \qquad \sum_{j=1}^{\infty} p_{ij} \eta_t^j < \eta_t^i \qquad (i = 1, \cdots, 2N-1).$$

Then we can obtain M_t recursively. That is, for given $M_0, \cdots, M_{t-1}$ the inequality (4.3) is satisfied if M_t is sufficiently large.

Since the function $x^{-N} exp(-\lambda a + \lambda a x)$ is smaller than 1 on a certain interval $(1, \delta_1)$ of x, it suffices to prove the existence of the value δ_2 such that

$$(4.6) \qquad \sum_{j=1}^{\infty} p_{ij} x^j < x^i \qquad (i = 1, \cdots, 2N-1)$$

for any x such as $1 < x < \delta_2$. When $p_{io} > 0$, from Lemma 4.1 the inequalities of (4.6) hold in the neighborhood of $x = 1$. In the case $P_{io} = 0$, i.e., the case $N < i < 2N$, the inequalities of (4.6) hold if we get the inequality

$$\sum_{j=1}^{\infty} j\beta_{tj} \leqq N\lambda b + t, \tag{4.7}$$

for

$$\begin{aligned}(\frac{d}{dx}\sum_{j=1}^{\infty} P_{ij}x^{j})_{x=1} &= \sum_{t=i=N}^{\infty} \alpha_{it}\sum_{j=t}^{\infty} j\beta_{tj} \\ &\leqq \sum_{t=i-N}^{\infty} \alpha_{it}(N\lambda b+t) \\ &= \lambda a + i - N < i.\end{aligned}$$

In the case $t \geqq N$, since

$$\sum_{j=t}^{\infty} (j-t)\beta_{tj} = \sum_{j=t}^{\infty} (j-t)\,\frac{(N\lambda b)^{j-t}}{(j-t)!}\,exp(-N\lambda b) = N\lambda b,$$

(4.7) holds.

In the case $t < N$, we will prove first the inequality

$$\sum_{j=t}^{s} \beta_{tj} \geqq \sum_{j=t}^{s} \beta_{t+1,j+1} \qquad \text{for all } s. \tag{4.8}$$

Let $(na <)\ x_1 < x_2 < \cdots$ be the arrival instants of passengers after the time instant na. Let $\zeta(L_n, x_1, x_2, \cdots)$ be the state number of (s_n, r_n) when the nth streetcar leaves the station. Since the arrival instants of passengers follow the Poisson process, the probabilistic behavior of $\{x_i\}$ does not depend on the value L_n. Hence $\sum_{j=t}^{s} \beta_{t+p,j+p}$ $(p = 0$ or $1)$ is the probability of the set

$$A_p = \{(x_1, x_2, \cdots):\ t + p \leqq \zeta(t+p, x_1, x_2, \cdots) \leqq s + p\}.$$

Let the sequence $\{x_i\}$ be given. If the nth streetcar with $L_n = t$ leaves at $na + cb$ and if the r_n passengers remain there at $na + cb$, then there are the $r_n + 1$ passengers at the platform just before $na + cb$ in the case $L_n = t + 1$. This means $\zeta(t+1, x_1, x_2, \cdots) \geqq \zeta(t, x_1, x_2, \cdots) + 1$ and so $A_1 \subset A_0$, from which we obtain (4.8). Hence

$$\sum_{j=t}^{\infty} (j-t)\beta_{tj} = \sum_{k=t}^{\infty} (1-\sum_{j=t}^{k} \beta_{tj}) \leqq \sum_{k=t}^{\infty} (1-\sum_{j=t}^{k} \beta_{t+1,j+1}) = \sum_{j=t}^{\infty} (j-t)\beta_{t+1,j+1}.$$

Repeating this procedure, we obtain

$$\sum_{j=t}^{\infty} (j-t)\beta_{tj} \leqq \sum_{j=N}^{\infty} (j-N)\beta_{Nj} = N\lambda b. \qquad \text{Q.E.D.}$$

Theorem 4.3. If $\lambda a < N$, the spectral distribution F of the stationary S_n process is absolutely continuous and its spectral density function is an infinitely differentiable function.

Proof: The autocovariance γ_k is given as

$$\gamma_k = E(S_n - ES_n)(S_{n+k} - ES_{n+k})$$

$$= \sum_{i,j=0}^{N} ij[Pr\{S_n=i,\ S_{n+k}=j\} - Pr\{S_n=i\}Pr\{S_{n+k}=j\}].$$

And

$$Pr\{S_n=i,\ S_{n+k}=j\} - Pr\{S_n=i\}Pr\{S_{n+k}=j\}$$

$$= \begin{cases} u_i(P_{ij}^{(k)} - u_j) & : i < N,\ j < N \\ u_i \sum_{t=0}^{N-1} (u_t - P_{it}^{(k)}) & : i < N,\ j = N \\ \sum_{t=0}^{N-1} (u_j - P_{tj}^{(k)}) u_t & : i = N,\ j < N \\ \sum_{h=0}^{N-1} u_h \sum_{t=0}^{N-1} (P_{ht}^{(k)} - u_t) & : i = j = N \end{cases}$$

where $P^{(k)} = (P_{ij}^{(k)})$ is the k-step transition matrix of (S_n, r_n). Hence we obtain $\lim_{k\to\infty} \gamma_k = 0$. Therefore F is continuous (see [3] p.495 Theorem 7.2 or [8] p.246 Theorem 8.4) and it is represented as

$$F(x) = \frac{\gamma_o}{\pi} x + 2\lim_{n\to\infty} \sum_{s=1}^{n} \gamma_s \frac{\sin sx}{\pi s}$$

(see [3] p.482 or [8] p.36). Since the finiteness of $\sum_{s=0}^{\infty} |\gamma_s|$ is easily derived from Lemma 4.2, $\frac{d}{dx}(\frac{\gamma_o}{\pi} x + 2\sum_{s=1}^{n} \gamma_s \frac{\sin sx}{\pi s})$ converges uniformly as $n \to \infty$. Hence $F(x)$ has a derivative $F'(x) = \pi^{-1}(\gamma_o + 2\lim_{n\to\infty} \sum_{s=1}^{n} \gamma_s \cos sx)$ and by the Lebesgue's bounded convergence theorem F is absolutely continuous. Similarly the infinitely differentiability of the spectral density function follows from the finiteness of $\sum_{s=0}^{\infty} s^k |\gamma_s|$ for the arbitrary positive integer k.

Q.E.D.

By this theorem the spectral density $f(x)$ on $[0, \pi]$ is represented as

$$f(x) = \frac{\gamma_o}{\pi} + \frac{2}{\pi} \sum_{t=1}^{\infty} \gamma_t \cos tx.$$

Especially $f(0)$ is given by the next theorem.

Theorem 4.4.

$$f(0) = \frac{\lambda a}{\pi}$$

Proof: By [8] p.274 or [4] p.52,

$$(4.9)\qquad \begin{aligned} f(0) &= \lim_{n\to\infty} \frac{1}{n\pi} Var\left(\sum_{i=1}^{n} s_i\right) \\ &= \lim_{n\to\infty} \frac{1}{n\pi} E(L_o + f_{n+1} - L_{n+1} - n\lambda a)^2. \end{aligned}$$

Let X_i be the number of passengers coming from $(i-1)a$ to ia. If $Var(L_o) = Var(L_{n+1}) < \infty$, then (4.9) is

$$\begin{aligned} f(0) &= \lim_{n\to\infty} \frac{1}{n\pi} E\{\sum_{i=1}^{n} (X_i - \lambda a)\}^2 \\ &= \frac{1}{\pi} Var(X_i) \end{aligned}$$

We will prove the finiteness of $Var(L_n)$. Let $\tilde{L}_n$, $\tilde{S}_n$ and $\tilde{r}_n$ be the L_n, S_n and r_n respectively in the model of $b = 0$. In this particular case

$$\tilde{r}_{n+1} = \max(\tilde{r}_n + X_{n+1} - N,\ 0).$$

This is the well known form of waiting time of the queueing model $GI/G/1$, so that by the Kiefer-Wolfowitz' theorem (see [5] or [6]) $\tilde{r}_n$ has the finite moment for any order. Hence $\tilde{L}_n$ has also the finite moment for any order, because of the relation $\tilde{L}_n = \tilde{r}_{n-1} + X_n$. For the L_n in the model with the positive b, the inequality; $\tilde{L}_n \geqq L_n$ holds if $\tilde{L}_0 = L_0$. Moreover the number ϕ_{jn} of elements of the set $\{i : i \leqq n,\ L_0 = 0,\ L_i = j\}$ satisfies $\lim_{n\to\infty} \phi_{jn}/n = Pr\{L_n = j\}$ and similar argument holds for $\tilde{L}_n$. Therefore L_n has the finite moment for any order.

Q.E.D.

5. Case for $N = 1$

We now consider the special case $N = 1$. In this case the exact value of u_i will be obtained. Each transition probability p_{ij} is easily derived such as

$$p_{oo} = e^{-\lambda a}$$

$$p_{oj} = \frac{\lambda^j}{j!} e^{-\lambda(a+b)}\{(a+b)^j - b^j\}, \qquad (j \geqq 1)$$

$$P_{1o} = e^{-\lambda(a-b)}$$

$$P_{1j} = \frac{\lambda^j}{j!} e^{-\lambda a}(a^j - b^j), \qquad (j \geqq 1)$$

and for $i \geqq 2$,

$$P_{ij} = \begin{cases} 0 & : i > j+1 \\ \dfrac{(\lambda a)^{j-i+1}}{(j-i+1)!} e^{-\lambda a} & : i \leqq j+1. \end{cases}$$

To derive the stationary distribution u_j, we let $U(z) = \sum_{j=0}^{\infty} u_j z^j$. Then

$$\begin{aligned} U(z) &= \sum_{j=0}^{\infty} \Big(\sum_{i=0}^{j+1} u_i P_{ij} \Big) z^j \\ &= \sum_{j=0}^{\infty} (u_o P_{oj} + u_1 P_{1j}) z^j + \sum_{i=2}^{\infty} \sum_{j=i-1}^{\infty} u_i P_{ij} z^j \\ &= u_o \{ e^{-\lambda a} + e^{-\lambda(a+b)(1-z)} - e^{-\lambda(a+b)+\lambda bz} \} + u_1 \{ e^{-\lambda(a-b)} - e^{-\lambda a+\lambda bz} \} \\ &\quad + z^{-1} e^{-\lambda a+\lambda az} \{ U(z) - u_o \} \end{aligned}$$

Therefore

$$U(z) = \frac{u_o [z\{ e^{-\lambda a} + e^{-\lambda(a+b)(1-z)} - e^{-\lambda(a+b)+\lambda bz} \} - e^{-\lambda a+\lambda az}] + z u_1 [e^{-\lambda(a-b)} - e^{-\lambda a+\lambda bz}]}{z - e^{-\lambda a+\lambda az}}$$

By L'Hospital's theorem and $U(1) = 1$, we can derive

$$(5.1) \qquad (1+\lambda b-\lambda b e^{-\lambda a}) u_o - \lambda b e^{-\lambda a+\lambda b} u_1 = 1 - \lambda a.$$

From $u_o = \sum_{i=0}^{\infty} u_i P_{io}$,

$$(5.2) \qquad (1 - e^{-\lambda a}) u_o - e^{-\lambda a+\lambda b} u_1 = 0.$$

By solving (5.1) and (5.2), we obtain

$$u_o = 1 - \lambda a$$

$$u_1 = (e^{\lambda a-\lambda b} - e^{-\lambda b})(1 - \lambda a).$$

Hence, although the calculation is cumbersome, the other u_i's are found such as

$$u_i = e^{-\lambda b}(1-\lambda a) \Big\{ \sum_{k=1}^{i} \frac{(\lambda b-k\lambda a)^{i-k}}{(i-k)!} e^{k\lambda a} - \sum_{k=0}^{i-1} \frac{(\lambda b-k\lambda a)^{i-k-1}}{(i-k-1)!} e^{k\lambda a} \Big\}, \quad (i=2,3,\cdots).$$

Next we will consider the properties of S_n in the steady state.

Theorem 5.1. In the steady state with $N = 1$, $Var(S_n) = \lambda a(1-\lambda a)$ and $Cov(S_n,\ S_{n+k}) = (1-\lambda a)[-1+\lambda a+e^{-k\lambda a}\sum_{j=0}^{k-1} \frac{k^{j-1}}{j!} (k-j)(\lambda a)^j]$, $(k=1,2,\cdots)$.

Proof:

$$
\begin{aligned}
Cov(S_n,\ S_{n+k}) &= ES_nS_{n+k} - (ES_n)^2 \\
&= Pr\{S_{n+k}=S_n=1\} - (Pr\{S_n=1\})^2 \qquad (5.3) \\
&= 1 - 2u_o + Pr\{S_{n+k}=0|S_n=0\}u_o - (\lambda a)^2
\end{aligned}
$$

Hence $Var(S_n) = \lambda a(1-\lambda a)$. We will obtain $Pr\{S_{n+k}=0|S_n=0\}$ for a positive integer k. Let $X_{k,j}$ be the probability that j passengers arrive in the interval $(na,\ na+ka]$ and that less than $k - p$ passengers arrive in the interval $(na+pa,\ na+ka]$ for $p=0,\cdots,k-1$. Since $X_{1,0} = e^{-\lambda a}$ and

$$
X_{k,j} = \begin{cases} \sum_{i=0}^{j} \frac{(\lambda a)^i}{i!} e^{-\lambda a} X_{k-1,j-i} & : j < k \\ 0 & : j \geqq k, \end{cases}
$$

we find by the mathematical induction method that

$$
X_{k,j} = \frac{k^{j-1}}{j!} (k-j)(\lambda a)^j e^{-k\lambda a}, \qquad (j < k).
$$

Hence we obtain theorem by substituting $Pr\{S_{n+k}=0|S_n=0\} = \sum_{j=0}^{k-1} X_{k,j}$ into (5.3).

Q.E.D.

Corollary. The autocovariances $\gamma_k = Cov(S_n,\ S_{n+k})$'s are positive for all k and decreasing monotonically.

Proof: We define $h_k(x) = e^{-kx}\sum_{j=0}^{k-1} \frac{k^{j-1}}{j!} (k-j)x^j$. Then positivity follows from $h_k(0) = 1$ and from

$$
\frac{d}{dx} \{-1+x+h_k(x)\} = 1 - e^{-kx}\sum_{j=0}^{k-1} \frac{1}{j!} (kx)^j > 0.
$$

For all k

$$
\begin{aligned}
h_k'(1) - h_{k+1}'(1) &= \int_0^1 \{h_k''(x) - h_{k+1}''(x)\}dx \\
&= \frac{1}{k!} \int_0^1 x^{k-1} e^{-kx}\{k^{k+1} - (k+1)^{k+1} x e^{-x}\}dx \\
&= \frac{k}{k!} \int_0^k y^{k-1} e^{-y} dy - \frac{1}{k!}\int_0^{k+1} y^k e^{-y} dy
\end{aligned}
$$

$$= \frac{1}{k!}\{k^k e^{-k} - \int_k^{k+1} y^k e^{-y} dy\}$$

$$> 0$$

and $h_k'(0) - h_{k+1}'(0) = 0$. Moreover, $h_k''(x) - h_{k+1}''(x)$ is positive if $0 < x < x^*$ and negative if $x^* < x < 1$, where x^* is the unique root on $(0,1)$ of $h_k''(x) - h_{k+1}''(x) = 0$. Therefore $h_k'(x) - h_{k+1}'(x) > 0$ for all x on $(0,1)$. Hence

$$Cov(S_n,\ S_{n+k}) - Cov(S_n,\ S_{n+k+1}) = (1-\lambda a)\int_0^{\lambda a} \{h_k'(x) - h_{k+1}'(x)\}dx$$

$$> 0.$$

Q.E.D.

6. Numerical Examples

The spectral density functions and autocorrelations of S_n discussed above were calculated for $N = 1,2,4$ and 20 by the computer. These functions with N larger than 1 depend on the three parameters λa, λb and N and these parameters must satisfy the relation $N\lambda b \leqq \lambda a < N$. In the case $N = 1$ it depends only on λa.

When N is larger than 1, it seems difficult to obtain the stationary distributions by the analytical method. Here we used the approximate method. The 400×400 north west corner truncation $\tilde{P}$ of P was substituted for P and the stationary distributions were obtained by multiplying the vector $(1,0,0,\cdots)$ by $\tilde{P}$ repeatedly. Next, the $400 \times N$ left half of $\tilde{P}^s$ was computed and the autocovariances γ_s's were derived by the equations in Theorem 4.3. Lastly, the spectral density was obtained by the following approximation:

$$f(x) = \frac{\gamma_0}{\pi} + \frac{2}{\pi}\sum_{s=1}^{p} \gamma_s \cos sx,$$

were $p = 100 \sim 300$. Figure 2 $\sim$ 14 show these spectral densities. Fortunately, since the values of the mean ES_n and $f(0)$ are known, we can check the goodness of approximation. The differences between the means of the calculated distributions and the true mean λa are less than 10^{-5}, even if $\lambda a = 0.9N$.

These figures show the existence of three types of the spectral density on $[0, \pi]$, i.e., the monotone decreasing functions, the monotone increasing functions and the functions which decrease till a certain point and increase thereafter. In particular, in the case $N = 1$, the relation $f(0) > f(\pi)$ holds from Corollary of Theorem 5.1. This relation and Figure 2 suggest that there

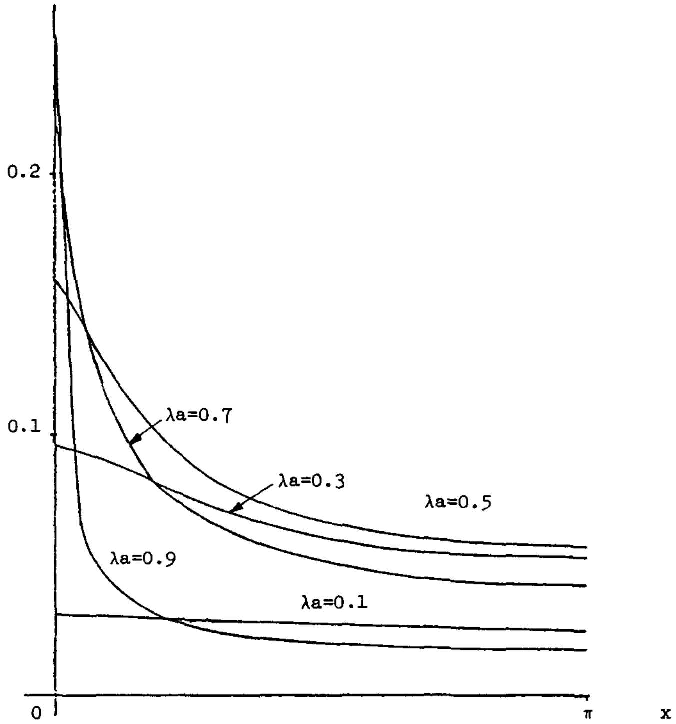


Figure 2. Spectral density: N = 1.

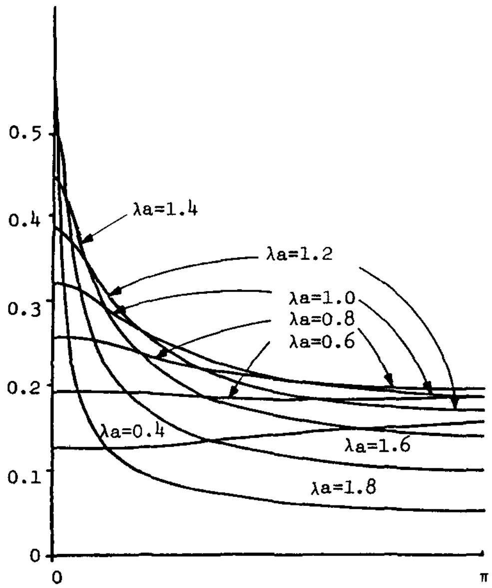


Figure 3. Spectral density: N=2, λb=0.2.

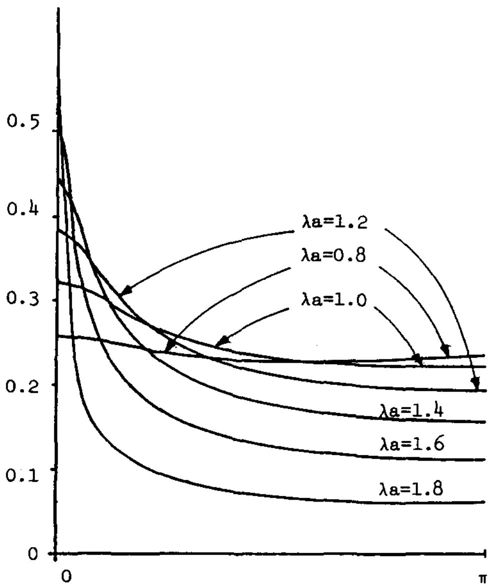


Figure 4. Spectral density: N=2, λb=0.4.

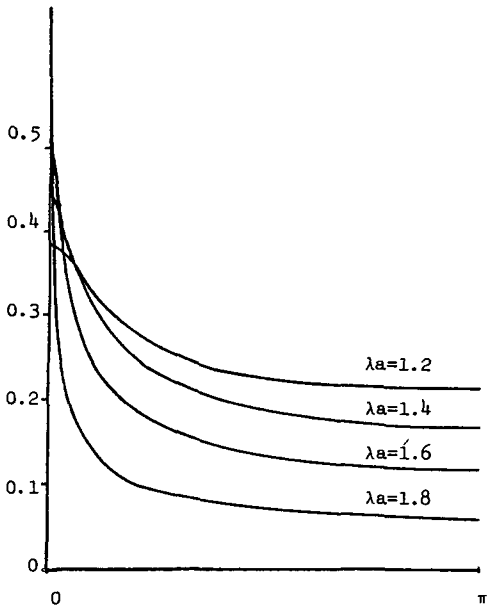


Figure 5. Spectral density: N=2, λb=0.6.

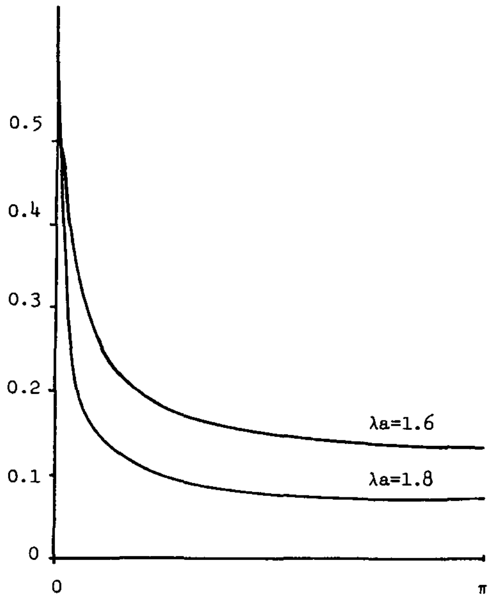


Figure 6. Spectral density: N=2, λb=0.8.

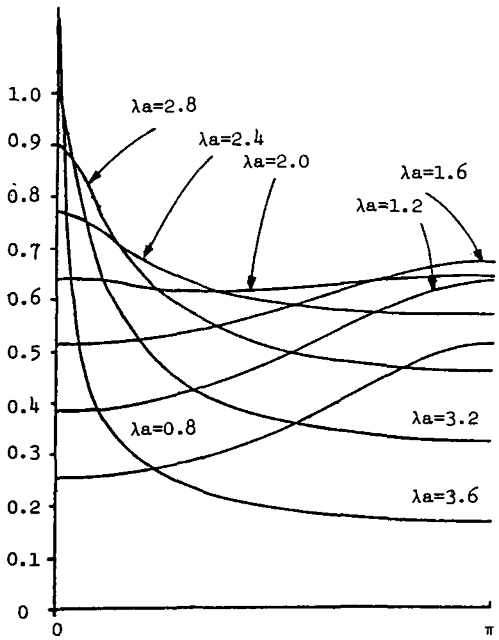


Figure 7. Spectral density: N=4, λb=0.2.

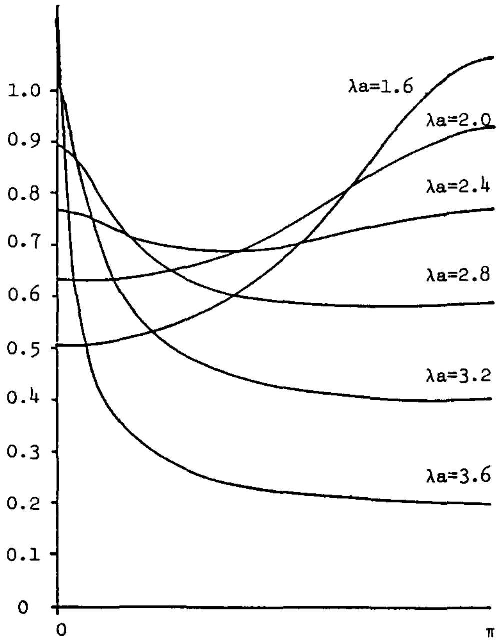


Figure 8. Spectral density: N=4, λb=0.4

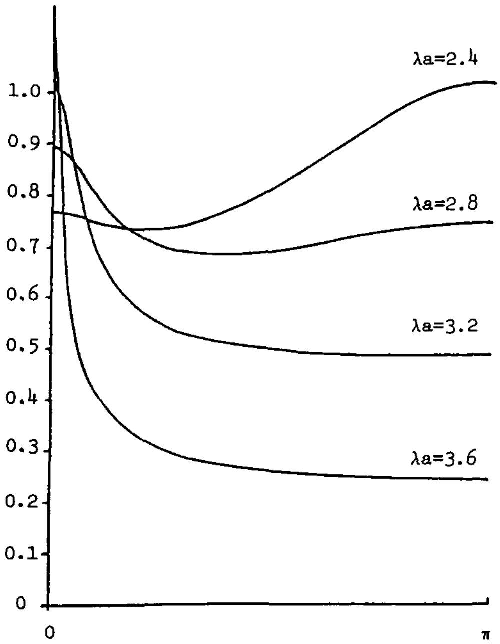


Figure 9. Spectral density: N=4, λb=0.6.

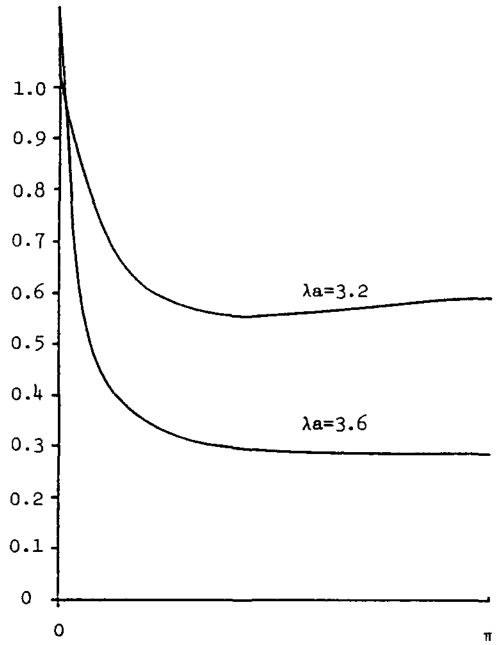


Figure 10. Spectral density: N=4, λb=0.8.

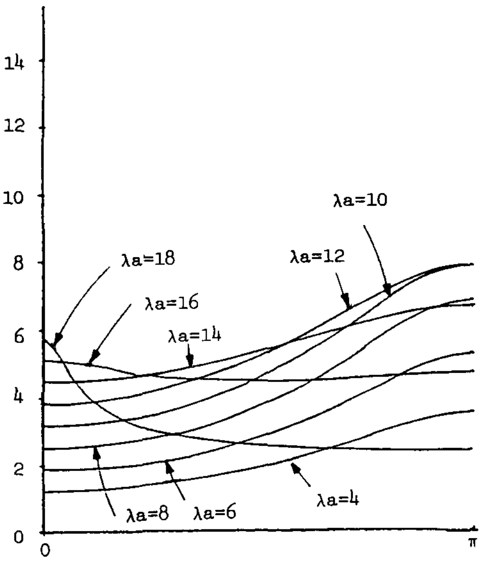


Figure 11. Spectral density: N=20, λb=0.2.

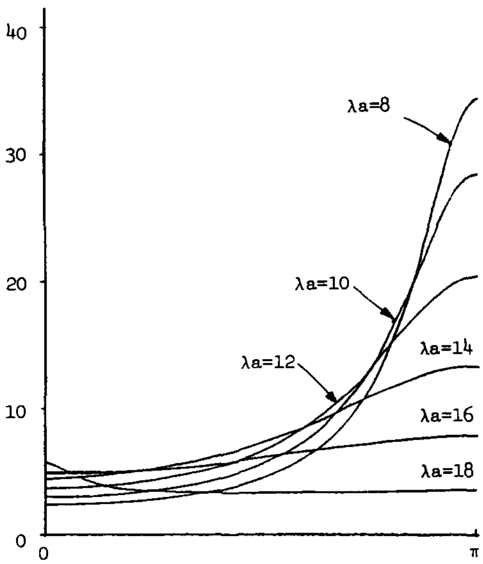


Figure 12. Spectral density: N=20,λb=0.4.

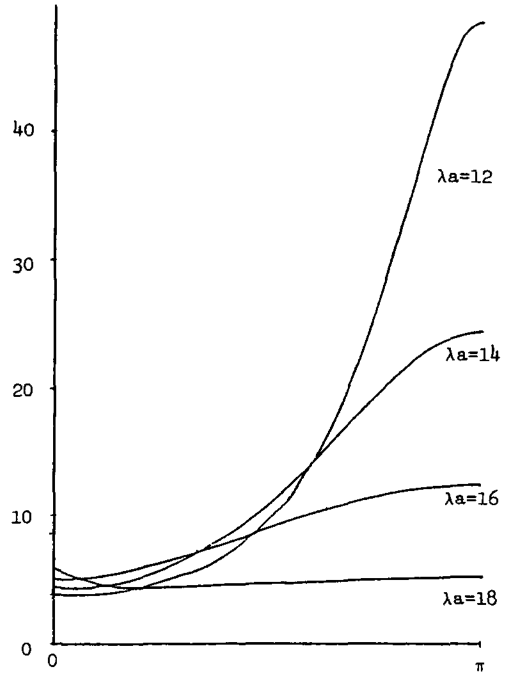


Figure 13. Spectral density: N=20, λb=0.6.

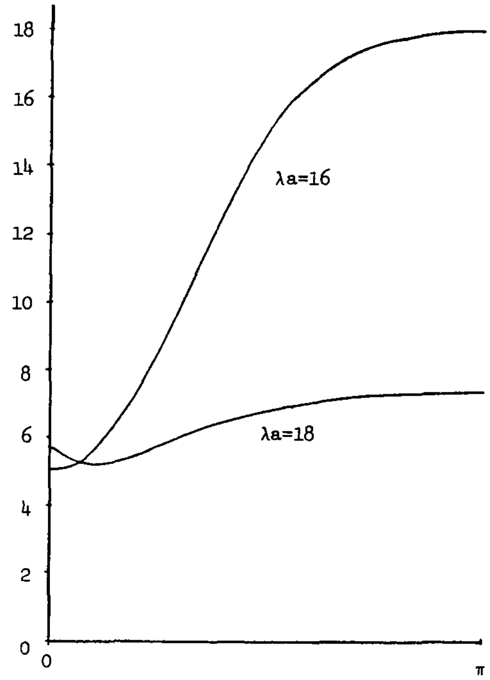


Figure 14. Spectral density: N=20, λb=0.8.

is only the first type in this case.

It is noted that these spectral density functions are not convex on $[0, \pi]$, which is shown as follows. Since $\sum_{s=1}^{\infty} |s\gamma_s| < \infty$ from Lemma 4.2, the derivative of $f(x)$ is given by $f'(x) = -\frac{2}{\pi}\sum_{s=1}^{\infty} s\gamma_s \sin sx$. Hence $f'(0) = f'(\pi) = 0$ and thus $f(x)$ is not convex except for the uniform function.

The monotone decreasing spectral density in these figures, for example the case for $N = 4$ and $\lambda a = 3.6$, decreases rapidly for small x and eventually becomes flat. To understand the stationary process with the spectral density of this type, it is useful to partition $f(x)$ into the sum of two spectral densities $f_1(x)$ and $f_2(x)$, where $f_1(x)$ has the mass mostly in the neighborhood of $x = 0$ and $f_2(x)$ is the uniform function. In the stationary process y_t with $f_1(x)$ the effect of each variable y_t on other variables is positive and remains for a long time. In the stationary process z_t with $f_2(x)$ each variable z_t is mutually uncorrelated. Since our process S_n is regarded as $y_t + z_t$, it has no periodic characteristics, the effect of congestion of each streetcar on that of others is positive and small effect remains for a long time. The last one in these three characteristics is more clearly found in the column

Table 1. Variance and autocorrelations, N=4, λb=0.4.

	λa	1.6	2.4	3.6
	variance	2.27635	2.27940	0.87577
order of autocorrelation	1	-0.18254	-1.1707×10^{-2}	0.22398
	2	4.0442×10^{-2}	2.1346×10^{-2}	0.15143
	3	-7.8976×10^{-3}	8.1240×10^{-3}	0.11541
	4	1.7076×10^{-3}	4.1143×10^{-3}	9.3332×10^{-2}
	5	-3.3451×10^{-4}	2.1014×10^{-3}	7.8170×10^{-2}
	6	7.2683×10^{-5}	1.1111×10^{-3}	6.7019×10^{-2}
	7	-1.4090×10^{-5}	6.0091×10^{-4}	5.8429×10^{-2}
	8	3.1085×10^{-6}	3.3098×10^{-4}	5.1589×10^{-2}
	9	-5.9012×10^{-7}	1.8499×10^{-4}	4.6004×10^{-2}
	10	1.3375×10^{-7}	1.0465×10^{-4}	4.1354×10^{-2}
	11	-2.4506×10^{-8}	5.9802×10^{-5}	3.4048×10^{-2}
	12	5.8063×10^{-9}	3.4468×10^{-5}	3.1127×10^{-2}
	13	-1.0034×10^{-9}	2.0013×10^{-5}	2.8574×10^{-2}
	14	2.5558×10^{-10}	1.1695×10^{-5}	2.6323×10^{-2}
	15	-4.0067×10^{-11}	6.8732×10^{-6}	2.4326×10^{-2}

of λa = 3.6 in Table 1, where the decreasing of the sequence $\{\gamma_t\}$ is very slow for $t \geqq 5$. However, it may be difficult to ascertain these effects from the real behavior of s_n, because the variance of s_n is small in the model of this type and many of s_n take the value N. In other words, the heap in the neighborhood of x = 0 in the spectral density function means that the overflowed customers cause the congestion of the subsequent streetcars.

When the parameter λa is near $N\lambda b$, the spectral density function increases monotonically. In this case, the streetcar following the crowded (vacant) one is comparatively vacant (crowded). This is clearly shown in the column of λa = 1.6 in Table 1, where the sign of autocorrelation coefficients varies alternately. This case is what Terada pointed out.

Interesting enough, there is the third type of the spectral density function (e.g. the case λa = 2.4 in Figure 8 or Table 1). The function of this type changes smoothly. Particularly, if λb is not large, we can select λa with which the spectral density function is like the uniform function on $[0, \pi]$. This suggests that if we suitably select the interarrival time of the streetcars, we can eliminate both the periodical congestion made by the boarding time and the successive congestion made by the capacity of the streetcar.

To state the other property, these figures show that the spectral density function is monotonically increasing with respect to λb for fixed λa. From this fact, when λb becomes large, the variance of S_n is increasing. Moreover, in the spectral density of the first type the weight of the factor z_t stated above becomes large and in the second type the periodical property becomes more remarkable.

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