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QUALITATIVE PROPERTIES OF SYSTEMS OF LINEAR CONSTRAINTS

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Abstract Systems of linear constraints are examined from a qualitative point of view. A property is said to be qualitative if it holds for all possible parameter values of certain prefixed sign patterns. In economics, such an approach has been traditionally taken within the perspective of comparative static analysis. However, the results obtained in this area appear somewhat limited in practical applicability.

This paper extends qualitative approach to the system of linear constraints in general, and presents necessary and sufficient conditions for qualitative feasibility/infeasibility, qualitative boundedness/unboundedness of feasible regions, and qualitative boundedness/unboundedness of objective functions. Duality theorem for linear programming problems is reconsidered from the qualitative point of view. Possible application areas of such an approach are also discussed.

1. Introduction

Systems of linear constraints are fairly common in many areas of sciences, including mathematics, operations research, economics and engineering, just to mention a few. Theory of linear algebra provides a strong background for theoretical analysis of such a system, and recent rapid progress of computer technology makes it possible to treat bigger problems numerically. However, there are many problems, especially in the area of economics and social sciences, which can at best be formulated vaguely in qualitative terms. Indeed, theoretical models in economics are often described in terms of the structure, with no or only limited amount of information being given.

A question that may arise in such a situation is to ask whether there exist properties that always hold irrespective to the specific realization of parameter values. Such a qualitative approach seems to have its root in comparative static analysis [17] of economics, where the central issue is to find the direction of changes in endogenous variables induced by a small changes in exogenous variables. Stated mathematically, this problem reduces to the following question: Given a system of linear equation

(1) $A\,x + b = 0$

with (A,b) being an $m\times(n+1)$ sign matrix, is it possible to determine the signs of each component of the n dimentional solution vector x uniquely? Here, a matrix (vector, resp.) is said to be a *sign matrix* (*vector*) [7], or alternatively a *qualitative matrix* (*vector*) [1], [9], if each entry of that matrix (vector) is specified only up to its sign, i.e., +, −, or 0.

For the limited case of $m = n$ with non-vanishing $\det(A)$, it is known that (1) is sign-solvable, i.e., the sign of x is uniquely determined from those of (A,b), if and only if (A,b) is of the following *standard form* [7]

(2) $$(A,b) = \begin{bmatrix} - & + & + & \cdots & + & + \\ & - & + & \cdots & + & + \\ & & - & \ddots & & \vdots \\ & 0 & & \ddots & + & + \\ & & & & - & + \end{bmatrix}$$

or its slight modifications. (See also [3],[6],[8],[12]) Since such a specific pattern of signs seldom occurs in practical applications, the above result is of limited significance in practice.[1)]

Another topic of qualitative nature is the problem of qualitative stability [14], [15], where the problem is to find the sign patterns of A that guarantees the stability of

$$\dot{x} = A\,x$$

irrespective to the specific values of parameters in A. Although the complete characterization of sign stability has been obtained by Jefferies *et al.* [4], the sign pattern that satisfies the required condition is extremely restrictive, and most matrices of practical significance do not seem to pass this requirement.

Thus, although the qualitative approach appears quite attractive in the sense that it requires neither statistical data nor numerical calculations, the practical applicability of the results obtained up to date seems rather

1) An important exception of practical significance is the *Leontief substitution system* [10] whose sign pattern is exactly that of (2). In many I/O-models of national economy, the Leontief matrices have been observed to be (approximately) triangulable, in which case the result will be the Leontief substitution system.

limited. However, there exist problems where qualitative information is actually helpful. For example, consider a set of large number of linear equations. Such a system might be feasible or infeasible, and the feasible region might be bounded or unbounded, depending on the values of parameters in the coefficient matrix. Since such a system of linear constraints arises rather frequently in operations research as well as in many other areas of sciences and engineering, it is hoped that investigating the qualitative properties of such a system might provide a new viewpoint and new techniques of analyzing these systems.

Therefore, the purpose of this paper is to investigate qualitative properties of systems of linear constraints in its most general form, but before stating the problem in mathematical terms, we need to prepare some notations.

Notation:

$A, B, \ldots$ $(a, b, \ldots)$: sign matrix (vector)

$\bar{A}, \bar{B}, \ldots$ $(\bar{a}, \bar{b}, \ldots)$: numerical matrix (vector) of sign pattern $A, B, \ldots$ $(a, b, \ldots)$

A^T, $\bar{b}^T$: transpose of A, $\bar{b}$

$\mathrm{sgn}(\cdot)$: sign operator; e.g., $\mathrm{sgn}(\bar{A}) = A$, $\mathrm{sgn}(\bar{a}) = a$

$x \geqq (>)\ 0$: each component of x is nonnegative (strictly positive)

$x \geq 0$: $x \geqq 0$ with some components being strictly positive

$\underline{n} = \{1, 2, \ldots, n\}$

$x_S = (x_i, x_j, \ldots, x_r)^T$ for $S = \{i, j, \ldots, r\} \subseteq \underline{n}$, $x = (x_1, x_2, \ldots, x_n)^T$

$\|x\|$: Euclidian norm $(= \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2})$

$R \dot{\cup} S = \underline{n}$: $R, S \subseteq \underline{n}$, $R \cup S = \underline{n}$, $R \cap S = \phi$

iff : if and only if

2. Description of the Problem

Consider the set of linear constraints of the form

$$(3) \qquad A\,x = 0 \qquad x_R \geq 0 \qquad x_S \geqq 0$$

where A is an $m \times n$ sign matrix and $R \dot{\cup} S = \underline{n}$. Such a system might be redundant, feasible, or infeasible, depending on the specific numerical matrix $\bar{A}$ of sign pattern A. (Throughout this paper, the bar attached to a sign matrix (vector) should be interpreted to implicitly represent a numerical matrix (vector) of the sign pattern of that sign matrix (vector). Thus, $\bar{A}$ implies $\mathrm{sgn}(\bar{A}) = A$.)

However, for some sign matrices, these properties might be true qualitatively, i.e., they may hold for any matrix $\bar{A}$. This paper is concerned with these qualitative properties, such as qualitative feasibility/infeasibility, qualitative boundedness/unboundedness of feasible regions, and qualitative boundedness/unboundedness of objective functions, as well as the conditions, or the structure of the sign matrix A, under which these qualitative properties hold. A few remarks should be made concerning the form of (3) before we go on to investigate the conditions, however.

Remark 1: Nonnegativeness of x may be replaced by any other sign pattern requirement on x. Indeed, if $x_i \leqq 0$ is the original requirement, this can be transformed to the condition $x_i \geqq 0$ by inverting the sign of the i-th column of A.

Remark 2: Inequality conditions and inhomogeneous equations can be treated within the framework of (3). If, for example, the original system is

$$A\,x + b \geqq 0, \quad C\,x + d = 0, \quad x \geqq 0$$

we can rewrite the above system as

$$\begin{bmatrix} A & -E \\ C & 0 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} b \\ d \end{bmatrix} y = 0 \tag{4}$$

by introducing the slack variable z and a strictly positive scalar variable y. Here, E is a diagonal sign matrix, with all of its diagonal elements being strictly positive. With $R = \{y\}$ and $S = \{x, z\}$, (4) is precisely of form (3).

Qualitative information concerning linear systems of constraints might be helpful in the following areas of applications:

(i) Some typical linear programming (LP) problems have significant sign patterns, although parameter values may change from one problem to the other. See, e.g., [2], [11].

(ii) In large-scale economic models, the profit maximizing behavior of the component economic agents is often modeled in the form of LP. Such a LP problem must be solved repeatedly with different set of parameter values, since the overall equilibrium solution is usually obtained by some iterative algorithms.

(iii) Some problems in the area of combinatorial mathematics can be formulated in the form of integer programming (LP) problems if the feasible region associated with the set of linear constraints is bounded [5].

In all the above situations, if those properties are known to hold qualitatively, i.e., irrespective to the specific values of parameters, we are freed from the task of ascertaining those properties for each system.

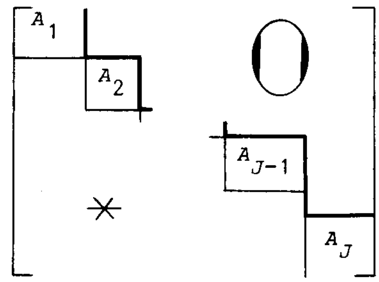
3. Qualitative Infeasibility

Let $A = (a_1, a_2, \ldots, a_m)^T$ be a sign matrix. A has a non-positive (non-negative, resp.) row if $a_i \leqq 0$ ($\geqq 0$) for some $i (1 \leqq i \leqq m)$. If the i-th row of A is either non-negative or non-positive, the system (3) implies that $x_j \equiv 0$ for all j satisfying $a_{ij} \neq 0$. This observation motivates us to introduce the following canonical form of a sign matrix.

Canonical Form of a Sign Matrix:

Sign matrix A can be transformed, by appropriate permutation matrices P and Q, into the following *canonical form:*

$A^* \triangleq PAQ =$



Here, $A_1, A_2, \ldots, A_{J-1}$ are (not necessarily square) sign matrices consisting either non-positive or non-negative rows only, while A_J containes neither of these rows. Furthermore, none of $A_1, A_2, \ldots, A_J$ containes zero columns, and A_J does not contain zero rows. This transformation of a sign matrix into the canonical form can be accomplished by rearranging the columns and rows of that matrix in such a way as to move the non-positive and non-negative rows to its top left corner.

Correspondingly, x is rearranged into $(x_1^T, x_2^T, \ldots, x_J^T)^T$ by

$$x = Q(x_1^T, x_2^T, \ldots, x_J^T)^T$$

Let I_J denote the set of indices of x_J represented in terms of x. That is, if $x_J = (x_p, x_q, \ldots, x_s)^T$, $I_J \triangleq \{p, q, \ldots, s\} \subseteq \underline{n}$. Furthermore, let $R_J \triangleq R \cap I_J$ and $S_J \triangleq S \cap I_J$. Then, (3) clearly implies $x_1 = 0$, $x_2 = 0, \ldots, x_{J-1} = 0$, and

$$\text{(5)} \qquad A_J x_J = 0, \quad x_{R_J} \geq 0, \quad x_{S_J} \geq 0$$

Thus, the system (3) is reduced to (5). A is said to be *irreducible* if $A = A_J$. Put another way, A is irreducible *iff* there exist neither non-positive nor non-negative rows in A.

Qualitative Infeasibility:

The following lemma is needed in the proof of the subsequent infeasibility theorem.

Lemma 1. Let A be an irreducible sign matrix. Then, there exist $\bar{A}$ and $\bar{x}$ such that $\bar{A}\,\bar{x} = 0$, $\bar{x} > 0$.

Proof: Since the i-th row of A contains at least one + and one − elements, we can assign $1+\alpha_i$ to the + element, −1 to the − element, and $+(-)\varepsilon$ to all other +(−) elements of that row. Let α_i be determined by $\alpha_i = \Delta_i\varepsilon$, where $\Delta_i \triangleq$ number of −'s − number of +'s in the i-th row. For sufficiently small ε, $1+\alpha_i > 0$. $\bar{A}$ is obtained by doing the same for all rows. Then, clearly $\bar{A}$ and $\bar{x} = (1,1,..,1)^T$ satisfy the requirement of this Lemma. Q.E.D.

Now we can state the following *qualitative infeasibility theorem*.

Theorem 1. System (3) is qualitatively infeasible (*QIF*) *iff* $R_J = \phi$.

Proof: If $R_J = \phi$, by definition we have $R \subseteq \underline{n}\backslash I_J$. However, since $x_j \equiv 0$ for $j \in \underline{n}\backslash I_J$, $x_R = 0$, which violates the condition $x_R \geq 0$.

Conversely, if $R_J \neq \phi$, the system (3) can be reduced to (5). By Lemma 1, there exists $\bar{A}_J$ to which $\bar{x}_J = (1,1,..,1)$ is a solution. This and $\bar{x}_{\underline{n}\backslash I_J} = 0$ satisfies all requirements of (3). Therefore, (3) is not *QIF*. Q.E.D.

4. Qualitative Feasibility

Let the sign matrix A be denoted column-wisely as $A = (c_1, c_2, \ldots, c_n)$. We introduce *sign convex cones* associated with each column of A as follows:

$$C_j \triangleq \{\bar{y} \in R^m \mid \operatorname{sgn}(\bar{y}) = c_j\}$$
$$C_j^- \triangleq \{\bar{y} \in R^m \mid \bar{y}^T\bar{c}_j \leq 0 \text{ for } \textit{some } \bar{c}_j \in C_j\}$$
$$\mathring{C}_j^- \triangleq \{\bar{y} \in R^m \mid \bar{y}^T\bar{c}_j < 0 \text{ for } \textit{some } \bar{c}_j \in C_j\}$$
$$C_j^+ \triangleq \{\bar{y} \in R^m \mid \bar{y}^T\bar{c}_j \geq 0 \text{ for } \textit{all } \bar{c}_j \in C_j\}$$
$$\mathring{C}_j^+ \triangleq \{\bar{y} \in R^m \mid \bar{y}^T\bar{c}_j > 0 \text{ for } \textit{all } \bar{c}_j \in C_j\}$$

Clearly, by definition we have:

Lemma 2. $C_j^+ = R^m \backslash \mathring{C}_j^-$, $\quad \mathring{C}_j^+ = R^m \backslash C_j^-$

Proof: See Lanchaster [9]. Q.E.D.

We are now ready to state the *qualitative feasibility theorem*.

Theorem 2. System (3) is qualitatively feasible (*QF*) *iff*

(6) $$(\bigcup_{j\in R} C_j^{+}) \cup (\bigcup_{j\in S} \mathring{C}_j^{+}) = R^m$$

Proof: Note first that, by Tucker's theorem [13], [20], (3) is QF iff $\bar{y}^T\bar{c}_j < 0$ $(j \in R)$, $\bar{y}^T\bar{c}_j \leqq 0$ $(j \in S)$ is infeasible for any choice of $\bar{A} = (\bar{c}_1, \bar{c}_2, \ldots, \bar{c}_n)$. The latter condition can be written as

$$(\bigcap_{j\in R} \mathring{C}_j^{-}) \cap (\bigcap_{j\in S} C_i^{-}) = \phi$$

which is equivalent to the condition of this Theorem. Q.E.D.

Remark 3. This theorem is a slight generalization of the Lanchaster's result (Theorem 5.3 of [9]), which gives the same result for the limited case of $S = \phi$.

Structures of Sign Convex Cones:

Consider the following *extended sign vector* $e = (e_1, e_2, \ldots, e_m)^T$, where e_i $(e=1,2,..,m)$ are non-empty subsets of $\{+,0,-\}$. Associated with e, we define its *convex cone representation*, Co(e), by

$$\text{Co}(e) \triangleq \{\bar{y} \in R^m \mid \bar{y} = (y_1, y_2, .., y_m),\ \text{sgn}(y_i) \in e_i\ \ i=1,2,..,m\}$$

Another way of seeing extended sign vectors is to consider e as the set of allowable sign patterns; namely,

$$e \triangleq \{(\varepsilon_1, \varepsilon_2, .., \varepsilon_m) \mid \varepsilon_i \in e_i\ \ i=1,2,..,m\}$$

For a sign matrix $A = (c_1, c_2, \ldots, c_n)$ with $c_j = (a_{1j}, a_{2j}, \ldots, a_{mj})^T$, we define the extended sign vectors c_j^{+} and $c_j^{\dagger}$ by

$$c_j^{+} \triangleq (a_{1j}^{+}, a_{2j}^{+}, \ldots, a_{mj}^{+}), \quad c_j^{\dagger} \triangleq (a_{1j}^{\dagger}, a_{2j}^{\dagger}, \ldots, a_{mj}^{\dagger})$$

Here, superfixes + and † are both point-to-set mappings from $\{+,0,-\}$ to itself which is defined by

$$\{+\}^{+} = \oplus \triangleq \{+,0\},\ \{0\}^{+} = * = \{+,0,-\},\ \{-\}^{+} = \ominus \triangleq \{0,-\}$$

$$\{+\}^{\dagger} - \{0\},\ \{-\}^{\dagger} = \{0\},\ \{0\}^{\dagger} = *$$

When we consider c_j^{+} and $c_j^{\dagger}$ as the sets of allowable sign patterns, we can define their substraction. That is,

$$\mathring{c}_j^{+} \triangleq c_j^{+} \setminus c_j^{\dagger}$$

Note that c_j^{+} and $\mathring{c}_j^{+}$ are proper representation of C_j^{+} and $\mathring{C}_j^{+}$, in the sense that any vector of the sign pattern c_j^{+} ($\mathring{c}_j^{+}$, resp.) lies in C_j^{+} ($\mathring{C}_j^{+}$) and vice versa.

Therefore, by Theorem 2, qualitative feasibility of (3) can be tested by first constructing the vectors of sign patterns c_j^+ $(j \in R)$ and $\overset{\circ}{c}{}_j^+$ $(j \in S)$, and then checking whether all 3^m possible sign patterns are included in $(\bigcup_{j\in R} c_j^+) \cup (\bigcup_{j\in S} \overset{\circ}{c}{}_j^+)$.

Example 1. Let

$$A = (c_1, c_2) = \begin{bmatrix} + & - \\ 0 & + \\ + & - \end{bmatrix} \quad S = \{1\}, \quad R = \{2\}.$$

Then, we have $c_1^+ = (\oplus, *, \oplus)^T$, $c_1^\dagger = (0, *, 0)^T$, and $c_2^+ = (\ominus, \oplus, \ominus)^T$. This system is not *QF*, since, e.g.,

$$(-,-,+) \notin \overset{\circ}{c}{}_1^+ \cup c_2^+ .$$

For inequality constrained systems, we can obtain more straightforward criterion:

Theorem 3. Let A be an $m \times n$ sign matrix. $A\,x \geqq 0$, $x \geq 0$ is *QF iff* there exists a non-negative column in A.

Proof: Since sufficiency is obvious, we only need to prove necessity. Suppose the above condition is not true. Then, each solumn of A containes at least one - element. Assign -1 to such an element, and $+(-)\varepsilon$ to all other +(-) elements. Doing this for all columns defines $\bar{A}$. For sufficiently small ε, we have $1^T\bar{A} \risingdotseq -1^T$.

If the inequality constraint is *QF*, there exists $\bar{x}$ such that $\bar{A}\,\bar{x} \geqq 0$, $\bar{x} \geq 0$. This is a contradiction, since

$$\begin{aligned} 1^T\bar{A}\bar{x} &= 1^T(\bar{A}\bar{x}) \geqq 0 \\ &= (1^T\bar{A})\bar{x} < 0. \end{aligned}$$

Q.E.D.

5. Qualitative Boundedness/Unboundedness of Feasible Regions

In this section, we consider the special case of (3); namely the system of the form

$$(7) \qquad A\,x + b = 0, \quad x \geqq 0$$

where (A,b) is an $m \times (n+1)$ sign matrix. Throughout this section, we maintain the assumption that (7) is *QF*. However, the feasible region of (7) may be bounded or unbounded, depending on the parameter values of (A,b). If the region is always bounded (unbounded, resp.) irrespective to the specific parameter values, the system (7) is said to be *qualitatively bounded (unbounded)*, and denoted *QB* (*QUB*).

The homogeneous system

(8) $A\ x = 0, \quad x \geq 0$

associated with (7) plays an essential role in developing the criteria for *QB* and *QUB*. Before stating the theorem, however, we consider the numerical versions of (7) and (8). Namely,

(9) $\bar{A}\ \bar{x} + \bar{b} = 0, \quad \bar{x} \geqq 0$

(10) $\bar{A}\ \bar{x} = 0, \quad \bar{x} \geq 0$

Then, we have the following:

Lemma 3. Suppose that (9) is feasible. Then, the feasible region of (9) is unbounded *iff* (10) is feasible.

Proof: Assume that (10) is feasible. Then, there exists $\bar{x}$ such that $\bar{A}\ \bar{x} = 0$, $\bar{x} \geq 0$. Also, since (9) is feasible, there exists $\bar{\bar{x}}$ such that $\bar{A}\ \bar{\bar{x}} + \bar{b} = 0$, $\bar{\bar{x}} \geqq 0$. Let $\bar{x}_\theta = \bar{\bar{x}} + \theta\bar{x}$ $(\theta > 0)$. Then, we have

$$\bar{A}\ \bar{x}_\theta + \bar{b} = 0, \quad \bar{x}_\theta \geq 0, \quad \|\bar{x}_\theta\| \to \infty\ (\theta \to \infty)$$

Thus, $\{\bar{x}_\theta\}_{\theta=1,2,\ldots}$ is an unbounded sequence of feasible solutions to (9).

Conversely, suppose that (10) is infeasible. If the feasible region of (9) is unbounded, there exists a sequence $\bar{x}_1, \bar{x}_2, \ldots,$ such that

$$\bar{A}\ \bar{x}_k + \bar{b} = 0, \quad \bar{x}_k \geqq 0, \quad \|\bar{x}_k\| \to \infty\ (k \to \infty)$$

Let $\theta_k = \|\bar{x}_k\|$, and $\bar{z}_k = \bar{x}_k/\theta_k$. Then, we have $\bar{A}\ \bar{z}_k + \bar{b}/\theta_k = 0$, $\bar{z}_k \geqq 0$, $\|\bar{z}_k\| = 1$, and $\theta_k \to \infty$ $(k \to \infty)$.

Let $\bar{z}^*$ be an accumulating point of $\{\bar{z}_k\}$ (which does exist, since this sequence is bounded on a compact set $\{\bar{z} \in R^m \mid \bar{z} \geqq 0, \|\bar{z}\| = 1\}$), and let $k \to \infty$. Then, we obtain $\bar{A}\ \bar{z}^* = 0$, $\bar{z}^* \geq 0$. This contradict the assumption that (10) is infeasible. Therefore, the feasible region of (9) is bounded. Q.E.D.

Now we are ready to state the *qualitative boundedness/unboundedness theorem*.

Theorem 4. Assume that (7) is *QF*. Then, (7) is:

(i) *QB iff* (8) is *QIF*, and (ii) *QUB iff* (8) is *QF*.

Proof: (i) Assume that (8) is not *QIF*. Then, there exists $\bar{A}$ such that (10) is feasible. Since (7) is *QF*, for an arbitrary $\bar{b}$, the feasible region of (9) is unbounded, by virtue of Lemma 3. Therefore, (7) is not *QB*.

Conversely, if (7) is not *QB*, there exists $(\bar{A}, \bar{b})$ such that (9) has an unbounded feasible region. By Lemma 3, this implies that (10) is feasible, which further implies that (8) is not *QIF*.

(ii) Assume that (7) is *QUB*. Then, for *any* $(\bar{A},\bar{b})$, (9) has an unbounded feasible region. By Lemma 3, this implies that (10) is feasible for *any* choice of $\bar{A}$. Therefore, (8) is *QF*.

Conversely, if (8) is *QF*, (10) is feasible for any choice of $\bar{A}$. By Lemma 3, this implies that (9) has an unbounded feasible region for any choice of $(\bar{A},\bar{b})$. Thus (7) is *QUB*. Q.E.D.

6. Qualitative Boundedness/Unboundedness of Objective Functions

In this section, we consider the qualitative LP problem

$$\text{Maximize: } z = c^T x$$
$$\text{subject to} \qquad\qquad (\text{P})$$

(11) $\quad A\,x \leqq b, \quad x \geqq 0$

where $\begin{bmatrix} A & b \\ c^T & 0 \end{bmatrix}$ is an $(m+1) \times (n+1)$ sign matrix. Throughout this section, (11) is assumed to be *QF*. It is clear that z is bounded if the feasible region of (11) is bounded. However, it may or may not be bounded if the feasible region is unbounded. We call z *qualitatively bounded* (*unbounded*, resp.), if for any parameter values of $\bar{A}$, $\bar{b}$, and $\bar{c}$, $z_{\max} < \infty$ ($z_{\max} = \infty$), and denote this as $Q\{z_{\max} < \infty\}$ ($Q\{z_{\max} = \infty\}$).

Again, before going on, we prepare a lemma for the numerical LP problem

$$\text{Maximize: } \bar{z} = \bar{c}^T \bar{x}$$
$$\text{subject to} \qquad\qquad (\bar{\text{P}})$$

(12) $\quad \bar{A}\,\bar{x} \leqq \bar{b}, \quad \bar{x} \geqq 0$

Associated with this problem, we consider the system

(13) $\quad \bar{A}\,\bar{x} \leqq \bar{0}, \quad \bar{x} \geq 0, \quad \bar{c}^T\bar{x} > 0$

Then, we have the following well known lemma (See [18] for proof).

Lemma 4. Assume that (12) is feasible. Then, $\{\bar{z}_{\max} < \infty\}$ *iff* (13) is infeasible.

We are now in a position to state the following *qualitative boundedness/unboundedness theorem of objective functions*.

Theorem 5. Assume (11) is *QF*. Then, the following three conditions are all equivalent:

(i) $Q\{z_{\max} < \infty\}$

(ii) $A\,x \leqq 0,\quad x \geq 0,\quad c\,x > 0$ is *QIF*

(iii) $A^T\lambda \geqq c,\quad \lambda \geqq 0$ is *QF*

Proof: (i) $\rightleftarrows$ (ii) Obvious from Lemma 4.
(ii) $\rightleftarrows$ (iii) This is a straightforward consequence of the Slater's theorem [13], [19]. Q.E.D.

Similarly, we have:

Theorem 6. Assume (11) is *QF*. Then, the following three conditions are all equivalent:

(i) $Q\{z_{\max} = \infty\}$

(ii) $A\,x \leqq 0,\quad x \geq 0,\quad c^T x > 0$ is *QF*

(iii) $A^T\lambda \geqq c,\quad \lambda \geqq 0$ is *QIF*

Proof: (i) $\rightleftarrows$ (ii) Obvious from Lemma 4.
(ii) $\rightleftarrows$ (iii) Also straightforward from Slater's theorem. Q.E.D.

Qualitative Duality Theorem:

Combining the above theorems, we can derive a qualitative version of the *duality theorem* for LP problems. To this end, let us consider the dual to (P); namely,

Minimize: $w = b^T\lambda$

Subject to (D)

(14) $A^T\lambda \geqq c,\quad \lambda \geqq 0$

Then, we have:

Theorem 7. Consider the qualitative LP problem (P) and its dual (D). Then,

(i) (11) is *QF* and $Q\{z_{\max} < \infty\}$ *iff* (14) is *QF* and $Q\{w_{\min} > -\infty\}$

(ii) Assume (11) is *QF*. Then, $Q\{z_{\max} = \infty\}$ *iff* (14) is *QIF*.

(iii) Assume (14) is *QF*. Then, $Q\{w_{\min} = -\infty\}$ *iff* (11) is *QIF*.

Proof: Restate Theorems 5 and 6 with respect to the dual LP problem (D), and let the resulting theorems be Theorem 5' and Theorem 6'.
(i) From Theorem 5, $Q\{z_{\max} < \infty\}$ implies that (14) is *QF*. Then, by Theorem 5'. we obtain $Q\{w_{\min} > -\infty\}$, since (11) is *QF*. The converse is also true by the symmetry of (P) and (D).

(ii) and (iii) are obvious from Theorem 6 and 6'. Q.E.D.

Remark 4. Theorem 7 can be readily derived from the duality theorem for LP problems [2], [11], but the above alternative derivation serves to strengthen the validity of our earlier results.

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