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EIGENVALUES OF THE TRANSITION RATE MATRICES IN A GI/E_k/m QUEUEING SYSTEM

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Abstract In this paper, the eigenvalues of the transition rate matrices in a GI/E_k/m queueing system are analytically obtained for any k and m. First, it is supposed that each channel is distinguishable from others, as a semi-homogeneous queueing system. Here, a transition rate matrix $S_m(\theta)$ and the eigenvalues of it are easily found by the mathematical induction on m, for any fixed k, where θ is a complex parameter. It can be shown that the matrix $S_m(\theta)$ is similar to a diagonal matrix, and that an eigenvalue of $S_m(\theta)$ takes the form of a m-sum of d(j)'s, where d(j) is the eigenvalue of $S_1(\theta)$. On the other hand, the transition rate matrix $T_m(\theta)$ in a homogeneous queueing system is different from $S_m(\theta)$ in appearance. But $T_m(\theta)$ can be made from $S_m(\theta)$ by using an equivalence relation. Then it can be shown that the matrix $T_m(\theta)$ is similar to a diagonal matrix, and the matrices $T_m(\theta)$ and $S_m(\theta)$ have the same eigenvalues except the multiplicity. Finally, to clarify the description, an example (k = 3 and m = 3) is shown.

1. Introduction

The GI/E_k/m system has arbitrarily distributed inter-arrival times as $A(t)$ and an infinite single queue served by m-service channels. The service times in each channel have a k-stage Erlangian distribution with mean rate μ (homogeneous service system). That is, each service-channel is divided into k-phases. The first (or enter) phase is called by 1, the second phase is called by 2,..., and the last (or exit) phase is called by k. The phase-states in the system are lexicographically arranged in accordance with a certain rule which is based on the total number of customers n, so the probability densities $P_{n;h}(t)$ in the steady state can be put as follows;

$$P_n(t) = [P_{n;1}(t), P_{n;2}(t), \ldots, P_{n;h}(t), \ldots, P_{n;M(n)}(t)]' \quad (n=0,1,2,\ldots)$$

where the components of a vector $P_n(t)$ are arranged in the same rule as the phase-state order and t denotes an elapsed time since the last arrival time,

at this time.
Here $M(n)$ is correspondingly determined when the each channel is distinguishable or not (refer to Section 2 and 4).

Then the balance equations for $P_n(t)$ are written as

$$(1.1)\quad \begin{aligned} &[\tfrac{d}{dt} + \lambda(t) + kn\mu]P_n(t) = k\mu\{G_nP_n(t) + H_nP_{n+1}(t)\} \\ &P_{n+1}(0) = \int_0^\infty P_n(t)\lambda(t)dt \quad \text{or } 0 \qquad (n=0,1,2,\dots,m-1) \end{aligned}$$

$$(1.2)\quad \begin{aligned} &[\tfrac{d}{dt} + \lambda(t) + km\mu]P_n(t) = k\mu\{G_mP_n(t) + H_mP_{n+1}(t)\} \\ &P_{n+1}(0) = \int_0^\infty P_n(t)\lambda(t)dt \qquad (n=m,m+1,\dots.) \end{aligned}$$

and $$\sum_n P_n = 1$$

where the coefficient matrix G_n is of order $M(n)\times M(n)$ and H_n is of order $M(n)\times M(n+1)$, provided that $M(n)=M(m)$ for $n\geq m+1$, and

$$P_n = \int_0^\infty \sum_h P_{n;h}(t)dt \quad \text{and} \quad \lambda(t) = \frac{1}{1-A(t)}\frac{d}{dt}A(t).$$

In accordance with a technique for solving differential difference equations, we seek a solution of the form

$$\begin{aligned} &P_n(t) = [1-A(t)]exp\{-km\mu t\}Q_n(t) \\ &Q_{n+1}(t) = \theta Q_n(t) \end{aligned} \qquad (n=m,m+1,\dots\ )$$

where the complex parameters θ's are independent of n and t, and are concerned with the inter-arrival distribution function $A(t)$ (for further details, see [3] and [4]).

In this work, the parameters can be assumed to be known from the beginning because we pay attention only to the structure of the eigenvalues, and we treat θ as a fixed parameter. Thus (1.2) is rewritten as

$$(1.3)\quad \frac{d}{dt}Q_n(t) = k\mu R(\theta)Q_n(t) \qquad (n=m,m+1,\dots\ ),$$

where $R(\theta) = G_m + \theta H_m$

For any k and m, if all eigenvalues of $R(\theta)$ are known, we easily find $Q_n(t) = exp\{k\mu R(\theta)t\}Q_n(0)$ for n=m, m+1, ..., and work out $P_n(t)$ in (1.1) for n=m-1, m-2, ..., 1, 0. That is, the matrix $R(\theta)$ is the key to the balance equations. So we discuss the matrix $R(\theta)$ in the $GI/E_k/m$ system with the following methods:

(1) Each channel is distinguishable from others. The queueing system is, so to speak, a semi-homogeneous.

(2) All channels are indistinguishable, as usual. This system is a homogeneous.

In this paper, the matrix $R(\theta)$ is denoted by $S_m(\theta)$ in case (1) and by $T_m(\theta)$ in case (2). We take out the connection between $S_m(\theta)$ and $T_m(\theta)$. Here the transpose of $R(1)/m$ is usually called a transition rate matrix, so we name $R(\theta)$ the transition rate matrix after $R(1)/m$. The eigenvalues of the matrix $R(\theta)$ are depend on $M(n)$ but independ of the arrangement in $\mathbf{P}_n(t)$, that is, the above arrangement rule is not a unique for the eigenvalues of it.

In the $GI/E_k/m$ queueing system, almost all the researchers in this field make use of $T_m(\theta)$, yet it is difficult to directly analyse $T_m(\theta)$. Although the structure of $S_m(\theta)$ is a simple, it is never used directly. So we deduce some properties of $T_m(\theta)$ from $S_m(\theta)$. $T_m(\theta)$ plays not only the key to $\mathbf{P}_n(t)$, but an important role in the waiting time distribution (see [4]).

In regard to the matrix $T_m(\theta)$, Shapiro[7] has presented first the eigenvalues of $T_m(1)$ in case of $k=2(m=2,3,\ldots)$. After that, the characteristic polynomial of $T_m(1)$ has been discussed in case of individual k and m: Mayhugh & McCormik[5] have treated it in case of $k=3(m=3)$. Heffer[2] has shown the eigenvalues in case of $k=2(m=2,3,\ldots)$, $k=3(m=2,3,4,5)$, $k=4(m=2,3)$ and $k=5,6$ $(m=2)$. Poyntz & Jackson[6] have dealt it in case of $k=3(m=3)$. On the other hand, Yu[8] has considered a heterogeneous $GI/E_k^{(m)}/m$ system and has discussed the matrix $S_2(1)$ in case of $k=3(m=2)$.

Our result in this paper does not contradict the above results, also includes them.

2. Semi-homogeneous System

In Section 2 and 3, suppose a semi-homogeneous queueing system where each channel is distinguishable from others. Namely, each channel is numbered as $1,2,\ldots$, or m. For $n\geqq m+1$, let $(n;[h_1,h_2,\ldots,h_m])$ denote a system-state, where n indicates a total number of customers in the system, the notation $[h_1,h_2,\ldots,h_m]$ means a phase-state, and then each h_j indicates an occupied phase-position in the j-th channel ($h_j=1,2,\ldots,k$ and $j=1,2,\ldots,m$). The phase-state $[h_1,h_2,\ldots,h_m]$ is abbreviated to a phase-state notation E_h. Assuming E_h's are arranged in the above rule and E_h is also called the h-th phase-state (see Section 5 as a concrete example). Since each E_h has one to one corresponding to *a repeated arrangement*, there are $K = k^m$ different arrangements of size m with replacement from k objects (that is, h=1,2,...,K). Then $M(n) = K (n \geq m)$ and $M(n) = \binom{m}{n} k^n$ $(0 \leq n \leq m-1)$.

Seeing that $S_m(1)/m$ is formed as a transition rate matrix, the element in the g-th row and h-th column of $S_m(\theta)$ means the situation of the one-stage transition from a state E_h to a state E_g. Here, let $\{E_h(i)\}$ denote the destinations of one-stage transition from E_h. Then the elements of $S_m(\theta)$ are given by

$$[S_m(\theta)]_{g,h} = \begin{cases} 1 & (E_g = E_h(i) \text{ and } h_i \neq k) \\ \theta & (E_g = E_h(i) \text{ and } h_i = k) \\ 0 & (\text{otherwise}), \end{cases}$$

where $E_g = [g_1, g_2, \ldots, g_m]$, $E_h(i) = [h_1, h_2, \ldots, h_i+1, \ldots, h_m]$ (if $h_i \neq k$) or $[h_1, h_2, \ldots, 1, \ldots, h_m]$ (if $h_i = k$), for g and h = 1,2,...,K, and i = 1,2,...,m.

2.1. Essential eigenvalues

In the especial case of m=1, for any fixed k, we have $E_h = [h]$ (h=1,2,..., k). The destination of one-stage transition from E_h is shown as $E_h(1) = [h+1]$ (h=1,2,...,k-1) or $E_k(1) = [1]$. The matrix $S_1(\theta)$ forms

$$S_1(\theta) = \begin{bmatrix} 0 & 0 & . & . & . & \theta \\ 1 & 0 & . & . & . & 0 \\ 0 & 1 & . & . & . & 0 \\ 0 & 0 & . & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & 1 & 0 \end{bmatrix}$$

The characteristic equation of $S_1(\theta)$ is given by $|dI_{(1)} - S_1(\theta)| = d^k - \theta = 0$ where $I_{(1)}$ is the identity matrix of order k.
Let d_j denote the j-th eigenvalue of $S_1(\theta)$ and let a(j) denote the associated eigenvector of d_j, then d_j and a(j) are given as

(2.1) $\quad d_j = \eta\zeta^{k-j} \qquad (j=1,2,\ldots,k),$

(2.2) $\quad a(j) = N\,[d_j^{k-1}, d_j^{k-2}, \ldots, d_j, 1]'$

where $\eta^k = \theta$, $\zeta^k = 1$, $N^2\{1+(k-1)\eta\bar{\eta}\} = 1$ (normalization condition) and $\bar{\eta}$ is a conjugate of η.

Because $d_1, d_2, \ldots, d_k$ are distinct, the matrix $S_1(\theta)$ is similar to a diagonal matrix D (so-called semisimple), as follows;

$$S_1(\theta) = ADA^{-1}$$

where $A = [a(1),a(2),\ldots,a(k)]$ and $D = diag\ \{d_1,d_2,\ldots,d_k\}$.
The above eigenvalues d_j's have an essential role in this work, the details will be described later on.

2.2. The connection between $S_m(\theta)$ and $S_{m+1}(\theta)$

In case of a (m+1)-channel system $GI/E_k/m+1$, let $G_g=[g_1,g_2,\ldots,g_m,g_{m+1}]$ denote the g-th phase-state for $n \geqq m+2$ $(g=1,2,\ldots,k^{m+1})$, as well as the m-channel system. Then, for any G_g, a certain E_h exists such that $h_j=g_j$ $(j=1,2,\ldots,m)$ and $G_g=[E_h,g_{m+1}]$. The destinations of one-stage transition from G_g are shown as $\{[E_h(i),g_{m+1}]\}$, and $[E_h,(g_{m+1}+1)]$ (if $g_{m+1} \neq k$) or $[E_h,1]$ (if $g_{m+1}=k$). Using the notations of the Kronecker product and the Kronecker sum (see Appendix), we have

$$S_{m+1}(\theta) = S_1(\theta) \otimes I_{(m)} + I_{(1)} \otimes S_m(\theta)$$

$$= S_1(\theta) \oplus S_m(\theta) \qquad (m=1,2,\ldots),$$

where $I_{(n)}$ is the identity matrix of order k^n $(n=1,2,\ldots)$.
Namely, the matrix $S_1(\theta)\otimes I_{(m)}$ implies a change of phase in the only (m+1)-th channel, one hand $I_{(1)}\otimes S_m(\theta)$ implies a change of phase in some other channel when the (m+1)-th channel is invariant.

From the mathematical induction on m, $S_m(\theta)$ is given by

$$S_m(\theta) = \oplus_m S_1(\theta) \qquad (m=2,3,\ldots). \tag{2.3}$$

Although $S_m(\theta)$ has the multiple eigenvalues in case of $m \geqq 2$, $S_m(\theta)$ always becomes semisimple.

Theorem 2.1. The matrix $S_m(\theta)$ is similar to a diagonal matrix as

$$S_m(\theta) = UXU^{-1}$$

where $U = \otimes_m A$ and $X = \oplus_m D$.

Proof: It is clear that $X=\oplus_m D$ becomes a diagonal matrix, because D is a diagonal.

Next, we proceed with induction on m. We know $S_m(\theta)=UXU^{-1}$ when m=1. From (A.3), (A.4), (2.3) and the inductive hypothesis, we have

$$S_{m+1}(\theta) = S_1(\theta) \otimes I_{(m)} + I_{(1)} \otimes Sm(\theta)$$

$$= (ADA^{-1}) \otimes (UI_{(m)}U^{-1}) + (AI_{(1)}A^{-1}) \otimes (UXU^{-1})$$

$$= (A \otimes U)(D \otimes I_{(m)} + I_{(1)} \otimes X)(A^{-1} \otimes U^{-1})$$

$$= (\otimes_{m+1}A)(\oplus_{m+1})(\otimes_{m+1}A^{-1}).$$

This completes the induction proof.

Therefore, let $x(h)$ denote the h-th eigenvalue of $S_m(\theta)$ and let $u(h)$ denote associated eigenvector of $x(h)$, that is, $X=diag\{x(1), x(2), \ldots, x(K)\}$, and $U = [u(1), u(2), \ldots, u(k)]$, then we obtain

$$x(h) = d(h_m) + d(h_{m-1}) + \ldots + d(h_1)$$

(2.4) $\qquad$ (h=1,2,...,K),

$$u(h) = a(h_m) \otimes a(h_{m-1}) \otimes \ldots \otimes a(h_1)$$

where $E_h = [h_1,h_2,\ldots,h_m]$ and $d(i) = d_i$.

3. Classification of Phase-states

For any fixed m and k, a phase-state E_h has one to one correspondence to *an arrangement* of size m, so we shall regard a phase-state as an arrangement. Hence, we define the following equivalence relation **R**.

Definition 3.1. Two phase-state $E_h=[h_1,h_2,\ldots,h_m]$ and $E_g=[g_1,g_2,\ldots,g_m]$ are said to be in the relation **R**, if there exists a permutation mapping σ such that $\sigma(E_h)=\sigma([h_1,h_2,\ldots,h_m])=[g_1,g_2,\ldots,g_m]=E_g$.

It is trivial that the relation **R** is an equivalence relation. Therefore, a set of $\{E_h\}$ is classified into equivalence classes $C_1,C_2,\ldots,C_L$ by the relation **R**. If E_h and E_g are in the relation **R**, then E_h and E_g belong to the same class C_α, and then E_h and E_g are congruent in the sense of *the repeated combination*. From (2.4) and the commutative law of addition, we have the following result.

Theorem 3.1. If E_h and E_g are in the relation R, then the h-th eigenvalue and the g-th eigenvalue have the same value, that is, if $\sigma(E_h)=E_g$ then $x(h) = x(g)$. But the converse is not true.

For a phase-state $E_r = [r_1,r_2,\ldots,r_m] \in C_\alpha$, if $r_1 \geqq r_2 \geqq \ldots \geqq r_m$, then we shall call E_r the representative state of C_α. And the representative state E_r is also expressed as the notation e_α. Since a representative state $e_\alpha(=E_r)$ has one to one corresponding to a certain *repeated combination*, there are $L = \binom{k+m-1}{m}$ different repeated combinations of size m from k objects. If E_h and E_g belong to C_α, then for any $E_h(i)\in C_\beta$, a certain $E_g(j)$ which belongs to C_β, exists. Here $E_g(j)$ is a destination of one-stage transition from E_g (j=1,2,...,k). In short, if $\sigma(E_h)=E_g$, then for any i, a certain j exists such that $\sigma(E_h(i))=E_g(j)$. So we have the following result.

Theorem 3.2. If E_h and E_g belong to the same class, then each destination of one-stage transition from E_h consists with a certain destination of one-stage transition from E_g.

Now, using the relation R, we define the matrices B and C:

$B = [b_{\alpha,h}]$ where $b_{\alpha,h} = 1$ ($E_h \in C_\alpha$) or $b_{\alpha,h} = 0$ (otherwise);
$C = [c_{h,\alpha}]$ where $c_{h,\alpha} = 1$ ($E_h = e_\alpha$) or $c_{h,\alpha} = 0$ (otherwise);
for $h=1,2,...,K$ and $\alpha=1,2,...,L$.

These matrices can be expressed by the fundamental vectors as follows:

$$B = [b_1, b_2, ..., b_K] = [B_1', B_2', ..., B_L']'$$

where

$$(3.1) \qquad b_h = \phi(\alpha) \qquad (E_h \in C_\alpha)$$
$$B_\alpha' = \sum_{E_r \in C_\alpha} f'(r).$$

And $C = [c_1, c_2, ..., c_L] = [C_1', C_2', ..., C_K']'$

where

$$(3.2) \qquad c_\beta = f(g) \qquad (e_\beta = E_g)$$
$$C_h' = \phi'(\alpha) \qquad (E_h = e_\alpha)$$
$$o' \qquad \text{(otherwise)}.$$

Here $\phi(\alpha)$ is the α-th fundamental vector in the L-dimensional vector space ($\alpha=1,2,...,L$), f(g) is the g-th fundamental vector in the K-dimensional vector space (g=1,2,...,K) and o is the zero vector in the L-dimensional vector space.

From (3.1) and (3.2), the element of BC becomes

$$[BC]_{\alpha,\beta} = B_\alpha' c_\beta$$
$$= \sum_{E_r \in C_\alpha} f(r) f(g) \qquad (e_\beta = E_g)$$
$$= \sum_{E_r \in C_\alpha} \delta_{rg} \qquad \text{(where } \delta_{rg} \text{ is the Kronecker delta).}$$

If $\alpha=\beta$, then $E_g \in C_\beta = C_\alpha$ and $[BC]_{\alpha,\beta} = 1$, else ($\alpha \neq \beta$) $E_g \notin C_\alpha$ and $[BC]_{\alpha,\beta} = 0$. That is, $[BC]_{\alpha,\beta} = \delta_{\alpha\beta}$, we have

$$(3.3) \qquad BC = I_L \qquad \text{(the identity matrix of order } L\text{).}$$

On the other hand, let $Z = [z_1, z_2, \ldots, z_K]$ denote CB, then

$$(3.4)\quad \begin{aligned} z_h &= Cb_h \\ &= C\phi(\alpha) && (E_h \in C_\alpha) \\ &= c_\alpha \\ &= f(g) && (C_\alpha \ni e_\alpha = E_g). \end{aligned}$$

Here

$$(3.5)\qquad f(g) = e(g_m) \otimes e(g_{m-1}) \otimes \ldots \otimes e(g_1)$$

where $e(i)$ is the i-th fundamental vector in the k-dimensional vector space $(i=1,2,\ldots,k)$, and the g-th phase-state E_g is described as $[g_1, g_2, \ldots, g_m]$.

So we have the following Theorem 3.3, 3.4 and 3.5 concerning with the matrix B.

Theorem 3.3. If E_h and E_g are in the relation R; $\sigma(E_h)=E_g$, then $Bu(h) = Bu(g)$.

The proof is led from the next lemma, because any permutation mapping σ can be made of the product of the interchange (transposition) mappings σ_i's; $\sigma = \sigma_1 \cdot \sigma_2 \cdot \cdots \cdot \sigma_s$.

Lemma. Let a interchange mapping σ_1 define (i,j). If $\sigma_1(E_h) = E_r$, then $Bu(h) = Bu(r)$.

Proof of lemma: Let E_h denote $[h_1, h_2, \ldots h_i, \ldots, h_j, \ldots, h_m]$. By the assumptions, we have $E_r = \sigma_1(E_h) = [h_1, h_2, \ldots, h_j \ldots, h_i, \ldots, h_m]$.
From (3.1), the β-th component of Bu(h) becomes

$$[Bu(h)]_\beta = B'_\beta u(h) = \sum_{E_p \in C_\beta} f'(p)u(h) \qquad (\beta=1,2,\ldots,L).$$

From (2.4) and (3.5), we have

$$\begin{aligned} f'(p)u(h) &= (e(p_m)\otimes e(p_{m-1})\otimes\ldots\otimes e(p_1))'(a(h_m)\otimes a(h_{m-1})\otimes\ldots\otimes a(h_1)) \\ &= \prod_{v=1}^{m} a(p_v, h_v) \\ &= a(p_1,h_1)\cdot a(p_2,h_2)\cdots a(p_i,h_i)\cdots a(p_j,h_j)\cdots a(p_m,h_m) \end{aligned}$$

and $f'(p)u(r) = a(p_1,h_1)\cdot a(p_2,h_2)\cdots a(p_i,h_j)\cdots a(p_j,h_i)\cdots a(p_m,h_m)$

where $a(x,y) = e'(x)a(y) = [A]_{x,y}$.

Here, let $C_\beta(1) = C_\beta(1;i,j)$ and $C_\beta(2) = C_\beta(2;i,j)$ denote disjoint subsets of C_β depend on the i-th channel and j-th channel, as follows;

$$C_\beta(1) = \{E_p | E_p \in C_\beta,\ p_i = p_j\}$$

and $$C_\beta(2) = \{E_p | E_p \in C_\beta,\ p_i \neq p_j\}$$

where $$E_p = [p_1, p_2, \ldots, p_i, \ldots, p_j, \ldots, p_m].$$

So if $E_p \in C_\beta(1)$, then $f'(p)u(h) = f'(p)u(r)$. If $E_p \in C_\beta(2)$, then E_q exists such that $E_q = [p_1, p_2, \ldots, p_j, \ldots, p_i, \ldots, p_m] = \sigma_1(E_p) \in C_\beta(2)$, and then we have

$$f'(p)u(h) = f'(q)u(r) \quad \text{and} \quad f'(q)u(h) = f'(p)u(r).$$

As a result, we see that

$$\begin{aligned}[Bu(h) - Bu(r)]_\beta &= \sum_{E_p \in C_\beta} f'(p)\{u(h)-u(r)\} \\ &= \sum_{E_p \in C_\beta(1)} f'(p)\{u(h)-u(r)\} + \sum_{E_p, E_q \in C_\beta(2)} \{f'(p)+f'(q)\}\{u(h)-u(r)\} \\ &= 0 \qquad (\beta=1,2,\ldots,L),\end{aligned}$$

which proves this lemma.

Theorem 3.4. If $\sigma(E_h) = E_g$, then $Bs(h) = Bs(g)$, where $s(h)$ is the h-th column vector of $S_m(\theta)$.

Proof: From (3.1), the β-th component of $Bs(h)$ becomes

$$\begin{aligned}B'_\beta[S_m(\theta)]_h &= \sum_{E_p \in C_\beta} f'(p)\ [S_m(\theta)]_h \\ &= \sum_{E_p \in C_\beta} [S_m(\theta)]_{p,h} \qquad (\beta=1,2,\ldots,L).\end{aligned}$$

Thus the β-th component of $Bs(h)$ means the situation of the one-stage transitions from a state E_h to a class C_β. Similarly, the β-th component of $Bs(g)$ means the situation of the one-stage transitions from E_g to C_β ($=1,2,\ldots,L$). Since E_h and E_g belong to the same class, it is clear that the situation of the transitions from E_h and from E_g agree with each other, by the use of Theorem 3.2. Thus we get $Bs(h) = Bs(g)$.

Theorem 3.5.

(3.6) $$BUZ = BU$$

(3.7) $$BS_m(\theta)Z = BS_m(\theta)$$

Proof: From (3.4), the h-th column vector of BUZ becomes

$$[BUZ]_h = BU\mathbf{z}_h$$
$$= BUf(g) \qquad (E_h \in C_\alpha \text{ and } E_g = e_\alpha \in C_\alpha)$$
$$= Bu(g)$$

Namely, we have $[BUZ]_h = Bu(g) = Bu(h) = [BU]_h$ by using Theorem 3.3, where $E_h \in C_\alpha$ and $C_\alpha \ni e_\alpha = E_g$. Therefore, (3.6) is led.

By the use of Theorem 3.4, the similar proof holds for (3.7).

4. Homogeneous System

In this section, we assume the usual homogeneous queueing system where all channels are indistinguishable. In this case, we have $M(n) = L\,(n \geqq m)$ and $M(n) = \binom{n+k-1}{n}$ $(0 \leqq n \leqq m-1)$. Any phase-state E_h which belongs to C_α is regarded as the representative state e_α. Thus the element in the α-th row and β-th column of the transition rate matrix $T_m(\theta)$ is given by the sum of one-stage transitions from a representative state $e_\beta (= E_g \in C_\beta)$ to a class C_α, after the manner of $S_m(\theta)$. That is,

$$[T_m(\theta)]_{\alpha,\beta} = \sum_{E_h \in C_\alpha} [S_m(\theta)]_{h,g} \qquad (\alpha \text{ and } \beta = 1,2,\ldots,L).$$

Then we have the following Theorem 4.1.

Theorem 4.1. $T_m(\theta) = BS_m(\theta)C$.

Proof: From (3.1) and (3.2), the element in the α-th row and β-th column of $BS_m(\theta)C$ becomes

$$[BS_m(\theta)C]_{\alpha,\beta} = \mathbf{B}'_\beta S_m(\theta)\mathbf{c}_\beta$$
$$= \{\sum_{E_h \in C_\alpha} \mathbf{f}'(h)\} S_m(\theta)\mathbf{f}(g) \qquad (e_\beta = E_g)$$
$$= \sum_{E_h \in C_\alpha} [S_m(\theta)]_{h,g} \qquad (\alpha \text{ and } \beta = 1,2,\ldots,L),$$

which proves our assertion.

Let the matrices Y and V denote BXC and BUC respectively, then we obtain the following Theorem 4.2.

Theorem 4.2.

(4.1) $\quad Y = diag\{y(1), y(2), \ldots, y(L)\}$

where $y(\alpha) = x(\mathrm{h}) = d(\mathrm{h}_m) + d(\mathrm{h}_{m-1}) + \ldots + d(\mathrm{h}_1)$

(on the condition of $e_\alpha = E_\mathrm{h} = [\mathrm{h}_1,\mathrm{h}_2,\ldots,\mathrm{h}_m]$, $\alpha=1,2,\ldots,L$).

(4.2) $$V^{-1} = BU^{-1}C$$

and

(4.3) $$T_m(\theta) = VYV^{-1}.$$

Proof: We shall prove in order. The element of Y becomes

$$[Y]_{\alpha,\beta} = [BXC]_{\alpha,\beta}$$

$$= \sum_{E_\mathrm{h}\in C_\alpha} [X]_{\mathrm{h},\mathrm{g}} \qquad (e_\beta = E_\mathrm{g}),$$

as well as $BS_m(\theta)C$. In the same manner as $BC = I_L$, we have $[Y]_{\alpha,\beta} = x(\mathrm{h})$ (if $\alpha=\beta$ and $e_\alpha=E_\mathrm{h}$) and $[Y]_{\alpha,\beta} = 0$ (otherwise), so (4.1) holds. From (3.3) and (3.6), $V(BU^{-1}C)$ becomes

$$(BUC)(BU^{-1}C) = BU(CB)U^{-1}C = (BUZ)U^{-1}C = (BU)U^{-1}C = BC = I_L,$$

which proves (4.2). From (3.7) and $S_m(\theta)U = UX$, we have

$$T_m(\theta)V = (BS_m(\theta)C)(BUC) = (BS_m(\theta)Z)UC = (BS_m(\theta))UC = B(S_m(\theta)U)C$$

$$= B(UX)C = (BU)XC = BUZXC = (BUC)(BXC) = VY.$$

Therefore, $T_m(\theta) = VYV^{-1}$ holds.

In other words, (2.4), (4.1) and (4.3) can be rewritten as follows:

Theorem 4.3. An arbitrary m-sum of $d(j)$'s, say $d(j_m)+d(j_{m-1})+\ldots+d(j_1)$, is an eigenvalue of $S_m(\theta)$ and also $T_m(\theta)$.

Theorem 4.4. The matrices $S_m(\theta)$ and $T_m(\theta)$ have the same eigenvalues except the multiplicity. Furthermore, the matrix $T_m(\theta)$ is similar to a diagonal matrix, even if it has multiple eigenvalues.

Theorem 4.5. Let $f_m(\lambda)$ and $F_m(\lambda)$ denote the characteristic polynomial of $T_m(\theta)$ and of $S_m(\theta)$, respectively. Then the polynomials are given by

$$f_m(\lambda) = \prod_{\alpha=1}^{L} (\lambda - y(\alpha)) \quad \text{and} \quad F_m(\lambda) = \prod_{\alpha=1}^{L} (\lambda - y(\alpha))^{\gamma_\alpha}$$

where $\gamma_1+\gamma_2+\ldots+\gamma_L=K$ and γ_α is the cardinal number of a class C_α (the number of elements of a class C_α in a wider sense), $\alpha=1,2,\ldots,L$.

On the other hand, any eigenvalue $y(= d(j_m)+d(j_{m-1})+ \ldots +d(j_1))$ of $T_m(\theta)$

always becomes an eigenvalue of $T_{m+k}(\theta)$ and also $S_{m+k}(\theta)$. Because $y+d(1)+d(2)+\ldots+d(k)$ is formed as a (m+k)-sum of $d(j)$'s and $d(1)+d(2)+\ldots+d(k) = \eta(1+\zeta+\zeta^2+\ldots+\zeta^{k-1}) = 0$. As an immediate consequence of the above results, we have the following Theorem.

Theorem 4.6. In a $GI/E_k/(m+k)$ queueing system, the characteristic polynomial $f_{m+k}(\lambda)$ of $T_{m+k}(\theta)$ is divisible by $f_m(\lambda)$, and the characteristic polynomial $F_{m+k}(\lambda)$ of $S_{m+k}(\theta)$ is divisible by $F_m(\lambda)$. That is,

$$f_{m+k}(\lambda) = g_{k,m+k}(\lambda)f_m(\lambda) \quad \text{and} \quad F_{m+k}(\lambda) = G_{k,m+k}(\lambda)F_m(\lambda)$$

where the polynomial $g_{k,m+k}(\lambda)$ is of degree $\binom{m+2k-1}{m+k} - \binom{m+k-1}{m}$ and $G_{k,m+k}(\lambda)$ is of degree $k^{m+k} - k^m$.

5. Example

To clarify the description, we shall discuss the case of k=3 and m=3, as an example. First, $S_1(\theta)$, d_j and $a(j)$ are set as,

$$S_1(\theta) = \begin{pmatrix} 0 & 0 & \theta \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad d_1 = \eta\,\zeta^2, \quad d_2 = \eta\zeta, \quad d_3 = \eta,$$

$$a(1) = N\begin{pmatrix} \eta^2 & \zeta \\ \eta & \zeta^2 \\ & 1 \end{pmatrix}, \quad a(2) = N\begin{pmatrix} \eta^2 & \zeta^2 \\ \eta & \zeta \\ & 1 \end{pmatrix}, \quad a(3) = N\begin{pmatrix} \eta^2 \\ \eta \\ 1 \end{pmatrix},$$

where $\eta^3 = \theta$, $\zeta^3 = 1$ and $N^2\{1+2\eta\bar{\eta}\} = 1$.

Then the matrices $S_2(\theta)$, $S_3(\theta)$, B, C, and $T_3(\theta)$ are obtained as follows:

$$S_2(\theta) = S_1(\theta)\otimes I_{(1)}+I_{(1)}\otimes S_1(\theta) = S_1(\theta)\oplus S_1(\theta)$$

$$= \begin{pmatrix} S_1(\theta) & 0 & \theta I_{(1)} \\ I_{(1)} & S_1(\theta) & 0 \\ 0 & I_{(1)} & S_1(\theta) \end{pmatrix}$$

$$= \begin{pmatrix} 0&0&\theta&0&0&0&\theta&0&0 \\ 1&0&0&0&0&0&0&\theta&0 \\ 0&1&0&0&0&0&0&0&\theta \\ 1&0&0&0&0&\theta&0&0&0 \\ 0&1&0&1&0&0&0&0&0 \\ 0&0&1&0&1&0&0&0&0 \\ 0&0&0&1&0&0&0&0&\theta \\ 0&0&0&0&1&0&1&0&0 \\ 0&0&0&0&0&1&0&1&0 \end{pmatrix},$$

$$S_3(\theta) = S_1(\theta)\otimes I_{(2)} + I_{(1)}\otimes S_2(\theta) = S_1(\theta)\oplus S_2(\theta)$$

$$= \begin{bmatrix} S_2(\theta) & 0 & \theta I_{(2)} \\ I_{(2)} & S_2(\theta) & 0 \\ 0 & I_{(2)} & S_2(\theta) \end{bmatrix} = \left(\begin{array}{ccccccccccccccccccccccccccc}
&&\theta&&&&\theta&&&&&&&&&&&&\theta \\
1&&&&&&&\theta&&&&&&&&&&&&\theta \\
&1&&&&&&&\theta&&&&&&&&&&&&\theta \\
1&&&&&\theta&&&&&&&&&&&&&&&&\theta \\
&1&&1&&&&&&&&&&&&&&&&&&&\theta \\
&&1&&1&&&&&&&&&&&&&&&&&&&\theta \\
&&&1&&&&&\theta&&&&&&&&&&&&&&&&\theta \\
&&&&1&&1&&&&&&&&&&&&&&&&&&&\theta \\
&&&&&1&&1&&&&&&&&&&&&&&&&&&&\theta \\
1&&&&&&&&&&&\theta&&&&\theta \\
&1&&&&&&&&1&&&&&&&\theta \\
&&1&&&&&&&&1&&&&&&&\theta \\
&&&1&&&&&&1&&&&&\theta \\
&&&&1&&&&&&1&&1 \\
&&&&&1&&&&&&1&&1 \\
&&&&&&1&&&&&&1&&&&&\theta \\
&&&&&&&1&&&&&&1&&1 \\
&&&&&&&&1&&&&&&1&&1 \\
&&&&&&&&&1&&&&&&&&&&&\theta&&&&\theta \\
&&&&&&&&&&1&&&&&&&&1&&&&&&&\theta \\
&&&&&&&&&&&1&&&&&&&&1&&&&&&&\theta \\
&&&&&&&&&&&&1&&&&&&1&&&&&\theta \\
&&&&&&&&&&&&&1&&&&&&1&&1 \\
&&&&&&&&&&&&&&1&&&&&&1&&1 \\
&&&&&&&&&&&&&&&1&&&&&&1&&&&&\theta \\
&&&&&&&&&&&&&&&&1&&&&&&1&&1 \\
&&&&&&&&&&&&&&&&&1&&&&&&1&&1
\end{array}\right)$$

$$B = \left(\begin{array}{ccccccccccccccccccccccccccc}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&1&0&1&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&1&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0 \\
0&0&0&0&1&0&0&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&1&0&1&0&0&0&1&0&0&0&1&0&0&0&1&0&1&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&1&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0&1&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&1&0&1&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1
\end{array}\right)$$

$$C = \left(\begin{array}{cccccccccc}
1&0&0&0&0&0&0&0&0&0 \\
0&1&0&0&0&0&0&0&0&0 \\
0&0&1&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0 \\
0&0&0&1&0&0&0&0&0&0 \\
0&0&0&0&1&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&1&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&1&0&0&0 \\
0&0&0&0&0&0&0&1&0&0 \\
0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&1&0 \\
0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&1
\end{array}\right)$$

$$T_3(\theta) = BS_3(\theta)C = \left(\begin{array}{cccccccccc}
0&0&\theta&0&0&0&0&0&0&0 \\
3&0&0&0&\theta&0&0&0&0&0 \\
0&1&0&0&0&2\theta&0&0&0&0 \\
0&2&0&0&0&0&0&\theta&0&0 \\
0&0&2&2&0&0&0&0&2\theta&0 \\
0&0&0&0&1&0&0&0&0&3\theta \\
0&0&0&1&0&0&0&0&0&0 \\
0&0&0&0&1&0&3&0&0&0 \\
0&0&0&0&0&1&0&2&0&0 \\
0&0&0&0&0&0&0&0&1&0
\end{array}\right)$$

Here, E_h, C_α, e_α, $x(h)$ and u(h) are shown by

E_h	$[h_1,h_2,h_3]$	C_α	e_α	$x(h)$	u(h)
1	[1,1,1]	1	1	$d(1)+d(1)+d(1)=3\eta\zeta^2$	$a(1)\otimes a(1)\otimes a(1)$
2	[2,1,1]	2	2	$d(1)+d(1)+d(2)=\eta\zeta\{2\zeta+1\}$	$a(1)\otimes a(1)\otimes a(2)$
3	[3,1,1]	3	3	$d(1)+d(1)+d(3)=\eta\{2\zeta^2+1\}$	$a(1)\otimes a(1)\otimes a(3)$
4	[1,2,1]	2		$d(1)+d(2)+d(1)=x(2)$	$a(1)\otimes a(2)\otimes a(1)$
5	[2,2,1]	4	4	$d(1)+d(2)+d(2)=\eta\zeta\{\zeta+2\}$	$a(1)\otimes a(2)\otimes a(2)$
6	[3,2,1]	5	5	$d(1)+d(2)+d(3)=0$	$a(1)\otimes a(2)\otimes a(3)$
7	[1,3,1]	3		$d(1)+d(3)+d(1)=x(3)$	$a(1)\otimes a(3)\otimes a(1)$
8	[2,3,1]	5		$d(1)+d(3)+d(2)=x(6)$	$a(1)\otimes a(3)\otimes a(2)$
9	[3,3,1]	6	6	$d(1)+d(3)+d(3)=\eta\{\zeta^2+2\}$	$a(1)\otimes a(3)\otimes a(3)$
10	[1,1,2]	2		$d(2)+d(1)+d(1)=x(2)$	$a(2)\otimes a(1)\otimes a(1)$
11	[2,1,2]	4		$d(2)+d(1)+d(2)=x(5)$	$a(2)\otimes a(1)\otimes a(2)$
12	[3,1,2]	5		$d(2)+d(1)+d(3)=x(6)$	$a(2)\otimes a(1)\otimes a(3)$
13	[1,2,2]	4		$d(2)+d(2)+d(1)=x(5)$	$a(2)\otimes a(2)\otimes a(1)$
14	[2,2,2]	7	7	$d(2)+d(2)+d(2)=3\eta\zeta$	$a(2)\otimes a(2)\otimes a(2)$
15	[3,2,2]	8	8	$d(2)+d(2)+d(3)=\eta\{2\zeta+1\}$	$a(2)\otimes a(2)\otimes a(3)$
16	[1,3,2]	5		$d(2)+d(3)+d(1)=x(6)$	$a(2)\otimes a(3)\otimes a(1)$
17	[2,3,2]	8		$d(2)+d(3)+d(2)=x(15)$	$a(2)\otimes a(3)\otimes a(2)$
18	[3,3,2]	9	9	$d(2)+d(3)+d(3)=\eta\{\zeta+2\}$	$a(2)\otimes a(3)\otimes a(3)$
19	[1,1,3]	3		$d(3)+d(1)+d(1)=x(3)$	$a(3)\otimes a(1)\otimes a(1)$
20	[2,1,3]	5		$d(3)+d(1)+d(2)=x(6)$	$a(3)\otimes a(1)\otimes a(2)$
21	[3,1,3]	6		$d(3)+d(1)+d(3)=x(9)$	$a(3)\otimes a(1)\otimes a(3)$
22	[1,2,3]	5		$d(3)+d(2)+d(1)=x(6)$	$a(3)\otimes a(2)\otimes a(1)$
23	[2,2,3]	8		$d(3)+d(2)+d(2)=x(15)$	$a(3)\otimes a(2)\otimes a(2)$
24	[3,2,3]	9		$d(3)+d(2)+d(3)=x(18)$	$a(3)\otimes a(2)\otimes a(3)$
25	[1,3,3]	6		$d(3)+d(3)+d(1)=x(9)$	$a(3)\otimes a(3)\otimes a(1)$
26	[2,3,3]	9		$d(3)+d(3)+d(2)=x(18)$	$a(3)\otimes a(3)\otimes a(2)$
27	[3,3,3]	10	10	$d(3)+d(3)+d(3)=3\eta$	$a(3)\otimes a(3)\otimes a(3)$

Consequently the characteristic polynomials $F_3(\lambda)$ and $f_3(\lambda)$ are given by

$$F_3(\lambda) = \lambda^6(\lambda - 3\eta)(\lambda - 3\eta\zeta)(\lambda - 3\eta\zeta^2)(\lambda - \eta[2\zeta^2+\zeta])^3(\lambda - \eta[2\zeta^2+1])^3$$
$$(\lambda - \eta[\zeta^2+2\zeta])^3(\lambda - \eta[\zeta^2+2])^3(\lambda - \eta[2\zeta+1])^3(\lambda - \eta[\zeta+2])^3$$
$$= \lambda^6(\lambda^3 - 27\theta)(\lambda^6 + 27\theta^2)^3$$

$$f_3(\lambda) = \lambda(\lambda - 3\eta)(\lambda - 3\eta\zeta)(\lambda - 3\eta\zeta^2)(\lambda - \eta[2\zeta^2+\zeta])(\lambda - \eta[2\zeta^2+1])$$
$$(\lambda - \eta[\zeta^2+2\zeta])(\lambda - \eta[\zeta^2+2])(\lambda - \eta[2\zeta+1])(\lambda - \eta[\zeta+2])$$
$$= \lambda(\lambda^3 - 27\theta)(\lambda^6 + 27\theta^2).$$

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Appendix

In this paper, the following terms on the Kronecker product are applied (see, for example, Bellman[1] 235-239).

Definitions:

(A.1) $A \otimes B = [a_{ij}B]$ (the Kronecker product).

(A2) $A \oplus B = A \otimes I_n + I_m \otimes B$ (the Kronecker sum),

where A is an m×m matrix and B is an n×n matrix.

Properties:

(A.3) $(A\otimes B)(C\otimes D) = (AC) \otimes (BD)$ (where AC and BD are defined).

(A.4) Let (λ,x) and (μ,y) be the eigenvalues and associated eigenvectors of A and of B, respectively. Then $A\oplus B(x\otimes y) = (\lambda+\mu)(x\otimes y)$.

Notations:

(A.5) $\otimes_n A = A\otimes A\otimes \ldots \otimes A$ (the n-time Kronecker product of A to itself).

(A.6) $\oplus_n A = A\oplus A\oplus \ldots \oplus A$ (the n-time Kronecker sum of A to itself).

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