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HYPEREXPONENTIAL WAITING TIME STRUCTURE IN HYPEREXPONENTIAL $H_K/H_L/1$ SYSTEMS

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Abstract Elementary congestion models sometimes require analysis of G/G/1 systems with hyperexponentially distributed interarrival time and service time distributions. It is shown that for such systems, the ergodic waiting time distribution is itself hyperexponentially distributed. A simple computational procedure is provided to find the parameters needed. Green's function methods are employed to motivate the factorization required. The relevance of these results to the delay in the overflow process of M/M/S is discussed.

1. Introduction. The Lindley Process

Let H_K be the class of hyperexponential variates of order K, i.e. of variates which are a mixture of K distinct exponential variates. In an $H_K/H_L/1$ queue, a sequence of customers C_K arrive at a single-server queue at epochs τ_k. The interarrival times $\Delta_k = \tau_{k+1} - \tau_k$ are i.i.d. with p.d.f. $f_\Delta(x) = \sum_{j=1}^{K} a_j \lambda_j e^{-\lambda_j x}$ $(\lambda_1 < \lambda_2 < \cdots < \lambda_K)$. The service times T_K also form an i.i.d. sequence with p.d.f. $f_T(x) = \sum_{i=1}^{L} b_i \mu_i e^{-\mu_i x}$ $(\mu_1 < \mu_2 < \cdots < \mu_L)$. If X_k is the time that the k-th customer must wait in queue for service, then $X_{k+1} = \max\,[0,\ X_k + \xi_{k+1}]$ where $\xi_{k+1} = T_k - \Delta_k$. The process X_k is called the Lindley waiting time process [6]. In this paper, the ergodic Lindley waiting time distribution is shown to be itself hyperexponential in density of order L apart from the mass point at the origin. To establish this result, Green's function methods are employed [2]. The hyperexponential structure is then obtained by complex plane arguments. Comparable results have been exhibited for $M/H_L/1$ systems previously [4]. The extension to $H_K/H_L/1$, however, is non-trivial and suggests that the exponential spectra obtained for $H_K/H_L/1$ might carry over for other system variates of interest such as the busy period.

It is known that for an *M/M/S* queue with finite storage, the overflow process has intervals between departures which are independent and of hyper-exponential distribution [7]. For a switching system in which the overflow stream is routed to a single server with hyperexponentially distributed service, the results exhibited may have practical value. The results may also be of interest elsewhere in congestion theory.

2. The Green's Function for the Underlying Homogeneous Process

It is assumed that the reader is familiar with Green's function methods and the idea of compensation. A simple short presentation may be found in [1] where other references are given. Other details may be found in [2].

It is shown in [3] that the ergodic Lindley waiting time distribution has a generalized density function $f_\infty(x)$ which may be expressed in terms of the ergodic Green density $g_\infty(x)$ of the underlying spatially homogeneous random walk and the compensation density $c_\infty(x)$ representing the influence of the boundary at $x=0$. The ergodic Green density is given by

$$(2.1)\qquad g_\infty(x) = \delta(x) + \sum_{n=1}^{\infty} a_{\xi_k}^{(n)}(x)$$

where $\delta(x)$ is the delta function for unit mass at 0 and $a_{\xi_k}^{(n)}(x)$ is the n-fold convolution of the p.d.f. of ξ_k. The generalized compensation density $c_\infty(x)$ has a delta function component of positive mass $c_{\infty 0}$ at $x=0$ and a negative density distributed on $(-\infty, 0)$ with mass equal to $-c_{\infty 0}$ so that the total compensation mass is zero. One then has as stated

$$(2.2)\qquad f_\infty(x) = \int_{-\infty}^{\infty} c_\infty(x')g_\infty(x-x')dx'.$$

The structure of $g_\infty(x)$ will be studied in this section through its Laplace transform

$$(2.3)\qquad \gamma_\infty(s) = L\{g_\infty(x)\} = \sum_{n=0}^{\infty} \alpha_{\xi_k}^{n}(s) = 1/(1-\alpha_{\xi_k}(s))$$

where $\alpha_{\xi_k}(s) = E[e^{-s\xi_k}]$.

Hereafter we write ξ instead of ξ_k for simplicity. From the definition of ξ, since $\xi = T_k-\Delta_k$, one has

(2.4) $$E[e^{-s\xi}] = \Big(\sum_{j=1}^{K} a_j \frac{\lambda_j}{\lambda_j - s}\Big)\Big(\sum_{i=1}^{L} b_i \frac{\mu_i}{\mu_i + s}\Big).$$

Since $\frac{\lambda_j}{\lambda_j-s}\frac{\mu_i}{\mu_i+s} = \frac{\lambda_j\mu_i}{\lambda_j+\mu_i}\Big(\frac{1}{\lambda_j-s}+\frac{1}{\mu_i+s}\Big)$,

$E[e^{-s\xi}]$ can be written as

(2.5) $$E[e^{-s\xi}] = \sum_{j=1}^{K} p_j \frac{\lambda_j}{\lambda_j - s} + \sum_{i=1}^{L} q_i \frac{\mu_i}{\mu_i + s}$$

where

(2.6) $$\sum_{j=1}^{K} p_j + \sum_{i=1}^{L} q_i = 1, \quad p_j,\ q_i > 0 \text{ for all } j=1, 2, \ldots, K \text{ and}$$

$=1, 2, \ldots, L$.

From (2.3), (2.5) and (2.6) one then finds

(2.7) $$L\{g_\infty(x)\} = \frac{1}{1-\sum_{j=1}^{K} p_j \frac{\lambda_j}{\lambda_j-s} - \sum_{i=1}^{L} q_i \frac{\mu_i}{\mu_i+s}} = \frac{1}{s\Big[\sum_{j=1}^{K} \frac{p_j}{s-\lambda_j} + \sum_{i=1}^{L} \frac{q_i}{\mu_i+s}\Big]}$$

Let

(2.8) $$X(s) = \sum_{j=1}^{K} \frac{p_j}{s-\lambda_j} + \sum_{i=1}^{L} \frac{q_i}{\mu_i+s}.$$

Then

$\frac{d}{ds}X(s) = -\sum_{j=1}^{K} \frac{p_j}{(s-\lambda_j)^2} - \sum_{i=1}^{L} \frac{q_i}{(\mu_i+s)^2} < 0$ for all real values of s which are not poles.

It follows that $X(s)$ has one simple zero between adjacent poles. If the ergodity condition $E[\Delta_k]-E[T_k]>0$ is satisfied, one has from (2.8)

(2.9) $$X(0) = (E[T_k]-E[\Delta_k])^{-1} < 0.$$

Equation (2.9) shows that $X(s)$ has one zero between the first negative pole and zero (See Figure 2.1). Hence, the number of negative zeros is equal to L and the number of positive zeros is equal to $K-1$. Let the negative and positive zeros respectively be denoted by $-\zeta_1^{(N)}, -\zeta_2^{(N)}, \ldots, -\zeta_L^{(N)}$ $(-\zeta_L^{(N)} < \ldots < -\zeta_2^{(N)} < -\zeta_1^{(N)})$ and $\zeta_1^{(P)}, \zeta_2^{(P)}, \ldots, \zeta_{K-1}^{(P)}$ $(\zeta_1^{(P)} < \zeta_2^{(P)} < \ldots < \zeta_{K-1}^{(P)})$. With this notation, (2.7) can be recast into the form

(2.10) $$L\{g_\infty(x)\} = \frac{\prod_{i=1}^{L}(s+\mu_i)\prod_{j=1}^{K}(s-\lambda_j)}{s\prod_{i=1}^{L}(s+\zeta_i^{(N)})\prod_{j=1}^{K-1}(s-\zeta_j^{(P)})} = \frac{\psi(s)}{\chi(s)}$$

where

(2.11) $$\psi(s) = \Big(\prod_{i=1}^{L}\frac{\zeta_i^{(N)}}{\mu_i}\Big)\Big(\prod_{i=1}^{L}\frac{s+\mu_i}{s+\zeta_i^{(N)}}\Big)$$

and

(2.12) $$\chi(s) = \Big(\prod_{i=1}^{L}\frac{\zeta_i^{(N)}}{\mu_i}\Big)\frac{s\prod_{j=1}^{K-1}(s-\zeta_j^{(P)})}{\prod_{j=1}^{K}(s-\lambda_j)} .$$

It should be noted that the following inequality is satisfied:

(2.13) $$-\mu_L < -\zeta_L^{(N)} < -\mu_{L-1} < -\zeta_{L-1}^{(N)} < \ldots . < -\mu_1 < -\zeta_1^{(N)} < 0$$
$$< \lambda_1 < \zeta_1^{(P)} < \lambda_2 < \zeta_2^{(P)} < \ldots . < \zeta_{K-1}^{(P)} < \lambda_K .$$

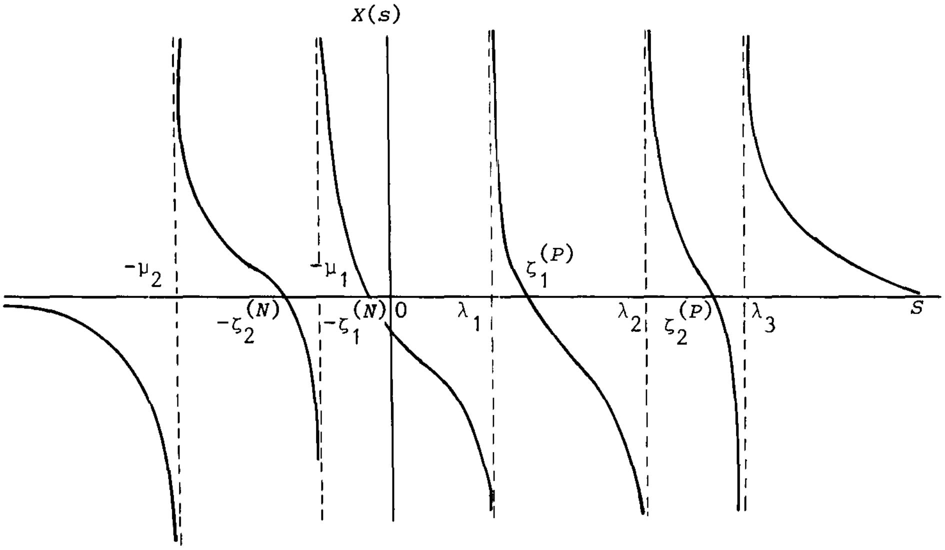

Figure 2.1 Zeros of $X(s)$ for $H_3/H_2/1$ queue

3. The Ergodic Waiting Time Distribution

The following factorization theorem for Green's function $g_\infty(x)$ may be employed:

Theorem. Let $\xi = T - \Delta$, where T and Δ are independent non-lattice random variables with finite first moments and $E(T)-E(\Delta)<0$. Let $R=\{s: Re s \geq 0\}$ and $L=\{s: Re s \leq 0\}$. Then $\{1-E[e^{-s\xi}]\}^{-1}$ may be written uniquely as a ratio

$$\{1-E[e^{-s\xi}]\}^{-1} = \frac{\psi_\infty(s)}{\chi_\infty(s)} ,$$

for which

a) $\psi_\infty(s)$ is regular inside R, and uniformly bounded and non-vanishing on R;

b) $\chi_\infty(s)$ is regular inside L and non-vanishing in the interior of L with a simple zero at $s=0$;

c) $\psi_\infty(0+)=1$.

Proof: This theorem is a variant of similar factorization theorems for Hilbert problems [8]. The non-vanishing of $\psi_\infty(s)$ is discussed in [5] in the context of the Spitzer identity. The non-vanishing of $\chi_\infty(s)$ arises from the structure of the compensation measure and is equivalent to the statement that for any non-positive, non-lattice variate X, $X(s)=E[e^{-sX}]$ has the value 1 on L only at $s=0$. The uniqueness is as usual obtained from Liouville's Theorem. Thus if we had distinct function pairs $\psi_1(s)$, $\chi_1(s)$ and $\psi_2(s)$, $\chi_2(s)$ with the properties stated, we would have $\{\psi_1(s)/\psi_2(s)\}=\{\chi_1(s)/\chi_2(s)\}$ on the imaginary axis. Each expression in curly brackets is regular and uniformly bounded in its half-plane, etc. Note that $E[T]<E[\Delta]$ assures that $\chi_1(s)$ and $\chi_2(s)$ have a simple zero at $s=0$ so that the ratio $\chi_1(s)/\chi_2(s)$ is bounded near $s=0$. Each expression must then be constant and the uniqueness follows.

By examination of (2.10) we verify that $\psi(s)$ and $\chi(s)$ given by (2.11) and (2.12) have the desired properties. Therefore, $\psi(s)$ is the Laplace transform of the ergodic Lindley waiting time density function $f_\infty(x)$. $\psi(s)$ can be written as

$$\psi(s) = \left(\prod_{i=1}^{L} \frac{\zeta_i^{(N)}}{\mu_i}\right)\left(1 + \sum_{j=1}^{L} w_j \frac{\zeta_j^{(N)}}{s+\zeta_j^{(N)}}\right) \tag{3.2}$$

where

$$w_1 = \frac{\prod_{k=1}^{L} (-\zeta_1^{(N)}+\mu_k)}{\zeta_1^{(N)} \prod_{k=2}^{L} (-\zeta_1^{(N)}+\zeta_k^{(N)})}$$

and

$$w_j = \frac{\prod_{k=1}^{j-1} (-\zeta_j^{(N)}+\mu_k)}{\zeta_j^{(N)} \prod_{k=1}^{j-1} (-\zeta_j^{(N)}+\zeta_k^{(N)})} \frac{\prod_{k=j}^{L} (-\zeta_j^{(N)}+\mu_k)}{\prod_{k=j+1}^{L} (-\zeta_j^{(N)}+\zeta_k^{(N)})}$$

It is clear from (2.13) that $w_1>0$. From $-\zeta_j^{(N)}+\mu_k<0$ and $-\zeta_j^{(N)}+\zeta_k^{(N)}<0$ for $k=1, 2, \ldots, j-1$, one has $\prod_{k=1}^{j-1} (-\zeta_j^{(N)}+\mu_k)/\prod_{k=1}^{j-1} (-\zeta_j^{(N)}+\zeta_k^{(N)})>0$. Noting that $-\zeta_j^{(N)}+\mu_k>0$ $k=j, \ldots, L$ and $-\zeta_j^{(N)}+\zeta_k^{(N)}>0$, $k=j+1, \ldots, L$, one has $w_j>0$. The inverse transform of (3.2) gives

$$(3.3) \qquad f_\infty(x) = f_{\infty 0}\delta(x) + \sum_{j=1}^{L} f_{\infty 0} w_j \zeta_j^{(N)} \exp(-\zeta_j^{(N)} x)$$

where $f_{\infty 0} = \prod_{i=1}^{L} \frac{\zeta_i^{(N)}}{\mu_i}$. From (3.3) we have

for the survival function $\bar{F}_\infty(x) = P[X_\infty>x]$ of the ergodic waiting time

$$(3.4) \qquad \bar{F}_\infty(x) = \sum_{j=1}^{L} f_{\infty 0} w_j \exp(-\zeta_j^{(N)} x), \quad x>0.$$

To obtain $f_\infty(x)$ and $\bar{F}_\infty(x)$ numerically we only need the zeros $\zeta_j^{(N)}$. These are easily obtained to great accuracy from (2.13) by standard bisection methods. It follows that $f_\infty(x)$ and $\bar{F}_\infty(x)$ are hyperexponential functions of order L apart from the mass point at the origin.

4. Structure of the Green's Function and Compensation Density

From (2.10) one then obtains the following explicit form for $g_\infty(x)$ away from the origin, at which point $g_\infty(x)$ has a delta function representing its mass point. One has

$$(4.1)\qquad g_\infty(x) = \begin{cases} \sum_{i=1}^{L} \rho_i \exp(-\zeta_i^{(N)} x), & x>0 \\ \frac{1}{E(\Delta_k)-E(T_k)} - \sum_{j=1}^{K-1} \theta_j \exp(\zeta_j^{(P)} x), & x<0, \end{cases}$$

where as may be seen from (2.7)

$$\rho_1 = \frac{\prod_{k=1}^{L}(-\zeta_1^{(N)}+\mu_k)}{\prod_{k=2}^{L}(-\zeta_1^{(N)}+\zeta_k^{(N)})} \frac{\prod_{j=1}^{K}(-\zeta_1^{(N)}-\lambda_j)}{(-\zeta_1^{(N)})\prod_{j=1}^{K-1}(-\zeta_1^{(N)}-\zeta_j^{(P)})},$$

$$\rho_j = \frac{\prod_{k=1}^{j-1}(-\zeta_j^{(N)}+\mu_k)}{\prod_{k=1}^{j-1}(-\zeta_j^{(N)}+\zeta_k^{(N)})} \frac{\prod_{k=j}^{L}(-\zeta_j^{(N)}+\mu_k)}{\prod_{k=j+1}^{L}(-\zeta_j^{(N)}+\zeta_k^{(N)})} \frac{\prod_{k=1}^{K}(-\zeta_j^{(N)}-\lambda_k)}{(-\zeta_j^{(N)})\prod_{k=1}^{K-1}(-\zeta_j^{(N)}-\zeta_k^{(P)})},$$

$$j = 2, 3, \ldots, L$$

$$-\theta_1 = \frac{\prod_{i=1}^{L}(\zeta_1^{(P)}+\mu_i)}{\prod_{i=1}^{L}(\zeta_1^{(P)}+\zeta_i^{(N)})} \frac{(\zeta_1^{(P)}-\lambda_1)\prod_{k=2}^{K}(\zeta_1^{(P)}-\lambda_k)}{\zeta_1^{(P)}\prod_{k=2}^{K-1}(\zeta_1^{(P)}-\zeta_k^{(P)})}$$

and

$$-\theta_j = \frac{\prod_{i=1}^{L}(\zeta_j^{(P)}+\mu_i)}{\prod_{i=1}^{L}(\zeta_j^{(P)}+\zeta_i^{(N)})} \frac{\prod_{k=1}^{j}(\zeta_j^{(P)}-\lambda_k)\prod_{k=j+1}^{K}(\zeta_j^{(P)}-\lambda_k)}{\zeta_j^{(P)}\prod_{k=1}^{j-1}(\zeta_j^{(P)}-\zeta_k^{(P)})\prod_{k=j+1}^{K-1}(\zeta_j^{(P)}-\zeta_k^{(P)})},$$

$$j = 2, 3, \ldots, K-1.$$

From (2.13) one can easily find $\rho_j>0$ for $j=1, 2, \ldots, L$ and $\theta_j>0$ for $j=1, 2, \ldots, K-1$.

From (2.2), (4.1) and properties of the compensation density,

$$f_\infty(x) = \int_{-\infty}^{0+} c_\infty(x')g_\infty(x-x')dx'$$

$$(4.2)\qquad = \int_{-\infty}^{0-} c_\infty(x')\sum_{i=1}^{L}\rho_i\exp(-\zeta_i^{(N)}(x-x'))dx' + c_{\infty 0}g_\infty(x)$$

$$= \sum_{i=1}^{L}\rho_i\exp(-\zeta_i^{(N)}x)\left[\int_{-\infty}^{0-} c_\infty(x')\exp(\zeta_i^{(N)}x')dx' + c_{\infty 0}\right], \qquad x>0 .$$

Noting that $\int_{-\infty}^{0-} c_\infty(x')dx'+c_{\infty 0} = 0$, $c(x')\leq 0$ for all $x'<0$ and $\exp(\zeta_i^{(N)}x')<1$ for all $x'<0$, one has

$$\int_{-\infty}^{0-} c_\infty(x')\exp(\zeta_i^{(N)}x')dx'+c_{\infty 0} > 0.$$

It then follows that $f_\infty(x)$ is a hyperexponential function of order L on $(0, \infty)$. This assures the result of Sec. 3. The hyperexponentiality of $\{E[\xi]\}^{-1}-g_\infty(x)$ of order $K-1$ on $(-\infty, 0)$ is obtained in the same way.

The compensation density $c_\infty(x)$ for negative x can be written as [2, pp.104-108].

$$c_\infty(x) = -\int_0^\infty a_\xi(x-y)f_\infty(y)dy, \qquad x<0 . \tag{4.3}$$

From (2.5) and (3.3), one has

$$\begin{aligned} c_\infty(x) &= -\int_0^\infty \{\sum_{j=1}^{K} p_j\lambda_j\exp(\lambda_j(x-x'))\}\{\sum_{i=1}^{L} f_{\infty 0}w_i\zeta_i^{(N)}\exp(-\zeta_i^{(N)}x')dx'\} \\ &= -f_{\infty 0}\sum_{j=1}^{K} p_j\lambda_j\exp(\lambda_j x)\left(\sum_{i=1}^{L}\frac{w_i\zeta_i^{(N)}}{\lambda_j+\zeta_i^{(N)}}\right), \qquad x<0 . \end{aligned} \tag{4.4}$$

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