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THE HORSE GAME

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Abstract N independent and identically distributed observations of some random variable with a continuous distribution F are sequentially presented to two players, White and Black. Each player is required to select exactly one observation without recall to rejected observations. As long as both players have not made a selection, White is always given the first option to accept or reject an observation. Both players are given the same information and are aware of the selection made by the other player. The player selecting the largest number wins the game. This problem is considered for the two cases, when either F is known or unknown. The probability of each player winning and the distribution of the location in the sample where selections are made is obtained. Also asymptotic results are derived. The problem is couched in terms of a horse bet.

1. Introduction

Two punters jointly own a horse called Jerome. Jerome is scheduled to run n races of a fixed length over the course of a racing season. Being gentlemen of both a gambling and sporting nature, they agree to make a game of their horse's official racing times. The winner will become sole owner of Jerome.

Over the course of the racing season, the punters will each select exactly one race time of their horse. After each has made a selection, the one with the smallest time, or equivalently the fastest track speed, wins. The selection process is as follows. The gamblers, call them Mr. White and Mr. Black adjourn to a bar after each race to discuss Jerome's past performances. If neither punter has made a previous selection, then White has the option to take the just completed race time as his selection. If White does not exercise his option, then and only then is Black given the same option.

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These options expire before the running of the next race. An expired option cannot be exercised at a later date. As soon as a selection is made by one of the players, future racing times are only available for selection, without recall, to the remaining player until such time as a selection is made. We assume the timing is fine enough to eliminate the possibility of a tie.

In summary, each player is allowed exactly one choice, without recall, in this sequential game. White is given the advantage as he raised the horse. Assuming that Jerome is a sought after reward, the optimal strategy will involve examining the probability that White wins. White will maximize and Black will minimize this probability.

The problem may be formulated in terms of searching for a larger number than your opponent in terms of track speed, or a smaller number, in terms of race times. These formulations are isomorphic and hence only one need be considered. We will consider the problem of players White and Black both searching for a number larger than that of their opponent.

The problem of a single player attempting to find the maximum of a sequentially presented sequence has an ample literature, see for example; Chow, et al (1964), Gilbert and Mosteller (1966) and Enns (1975). Two-person game variations on the secretary problem are given in Fushimi (1981) and Sakaguchi (1980). Sakaguchi (1978) also explores a two-person sequential decision problem where both players must accept an offer before it is finally taken. In our paper with two players, the object is not necessarily to pick the best race time, but merely to beat your opponent.

Let X_i, $i = 1, \ldots, n$ be the average track speed of the horse on the i^{th} race. Assume $X_1, \ldots, X_n$ are independent and identically distributed random variables with a continuous distribution $F(x) = P\{X_i < x\}$. We let $F(x)$ be continuous so that the n numbers generated are measurably different.

The problem has several variations, two of which are considered here. If White and Black have no previous racing knowledge of their horse, then the formulation involves an unknown distribution $F(x)$. On the other hand they may have an extensive past track record on their horse in which case $F(x)$ is known.

2. The Distribution $F(x)$ is Unknown

When the distribution $F(x)$ is unknown, it will be shown that the players observe a certain number of track speeds before one is seriously considered for selection.

Let r_{ik} = the k^{th} order statistic of $X_1, \ldots, X_i$ where r_{i1} is the largest. Let

$$Q_{ik} = P\{X_i > (X_{i+1}, X_{i+2}, \ldots, X_n) | X_i = r_{ik}\}$$

$$= \binom{i}{k}\Big/\binom{n}{k}, \text{ see DeGroot (1970).}$$

If W_i and B_i are the events that White or Black respectively select the i^{th} racing speed when it is presented to them, then the most general strategies of the players at each presentation can be represented by:

$$w_{ik} = P\{W_i | X_i = r_{ik}, W_1', \ldots, W_{i-1}', B_1', \ldots, B_{i-1}'\}$$

and $$b_{ik} = P\{B_i | X_i = r_{ik}, W_1', \ldots, W_i', B_1', \ldots, B_{i-1}'\},$$

where the primes denote complementary sets.

If $P\{\hat{W}_n\} = P\{\text{White wins} \mid n \text{ races}\}$, then the optimal strategy for the game involves finding the max-min value of $P\{\hat{W}_n\}$ with respect to the matrices w_{ik} and b_{ik} respectively where $k \leq i$. As we are dealing with a two person zero sum game, the above optimal strategy is equivalent to the Nash equilibrium solution, see J.C. Harsanyi (1967, 1968).

Now $P\{\hat{W}_n\}$ may be partitioned in terms of the location of the first track speed accepted. This is a reasonable partition, because when the first selection is made, the other player will merely wait until a number exceeding his opponent's choice occurs, if it exists.

$$(2.1) \qquad P\{\hat{W}_n\} = \sum_{i=1}^{n-1} \sum_{k=1}^{i} [Q_{ik}\, w_{ik} + (1-Q_{ik})(1-w_{ik})\, b_{ik}]/i$$

$$\times \prod_{r=0}^{i-1} h(r) \sum_{j=0}^{r} (1-w_{rj})(1-b_{rj}),$$

where $$h(r) = 1 \text{ if } r = 0$$

$$= \frac{1}{r} \text{ if } r > 0,$$

and where one defines $w_{r0} = b_{r0} = 1$ if $r > 0$ and $w_{00} = b_{00} = 0$.

For $l = 1, \ldots, n-1$, define:

$$S(l) = \sum_{i=1}^{l} \frac{1}{i!} \sum_{k=1}^{i} [Q_{ik}\, w_{ik} + (1-Q_{ik})(1-w_{ik})\, b_{ik}]$$

$$\times \prod_{r=0}^{i-1} \sum_{j=0}^{i} (1-w_{rj})(1-b_{rj})$$

and

$$(2.2)\qquad l\,\beta(l) = \sum_{k=1}^{l} b_{lk}\,(1-w_{lk})\{1-Q_{lk}-\beta(l+1)\}$$

$$+\; l\,\beta(l+1) + \sum_{k=1}^{l} w_{lk}\,\{Q_{lk}-\beta(l+1)\}$$

where $\beta(n) = 0$.

(2.1) can now be rewritten as:

$$(2.3)\qquad P\{\hat{W}_n\} = S(l) + \frac{\beta(l+1)}{l!}\prod_{r=0}^{l}\sum_{j=0}^{r}(1-w_{rj})(1-b_{rj})$$

for $l = 0, \ldots, n-1$ if $S(0) = 0$.

In (2.3), the remaining strategy for both White and Black commencing at the running of the $(l+1)^{\text{th}}$ race is contained in $\beta(l+1)$. Reverse sequential optimization beginning at the $l = (n-1)^{\text{th}}$ term yields the desired result. For example, when $l = n-1$, $b_{n-1,k} = 1$ for all k as each player is obliged to select one track speed. (2.2) becomes

$$(n-1)\,\beta\,(n-1) = \sum_{k=1}^{n-1}[w_{n-1,k}\,(2Q_{n-1,k}-1) + (1-Q_{n-1,k})].$$

Maximizing with respect to $w_{n-1,k}$ yields

$$\begin{aligned} w_{n-1,k} &= 1 \quad \text{if } Q_{n-1,k} \geq 0.5 \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

In general from (2.2), minimizing with respect to b_{lk}, one obtains

$$(2.4)\qquad b_{lk} = 1 \quad \text{if } Q_{lk} \geq 1 - \beta(l+1)\ .$$

Similarly one will find $w_{lk} = 1$ if $Q_{lk} \geq \alpha(l+1)$ for some $\alpha(l+1)$. There are now two possibilities, namely $\alpha(l+1) \geq 1 - \beta(l+1)$ or $\alpha(l+1) < 1 - \beta(l+1)$. Roughly speaking, the former implies that White is more fussy in his choice than Black while the latter implies the converse. Since White has an advantage, the former seems the most reasonable. One can assume the latter and show that it leads to a contradiction. This has been omitted for brevity. The first possibility is equivalent to saying that $w_{lk} = 1$ implies $b_{lk} = 1$. The consequence of this is that (2.2) can now be written as:

$$(2.5)\qquad l\,\beta(l) = l\,\beta(l+1) + \sum_{k=1}^{l} w_{lk}\,(2Q_{lk}-1)$$

$$+ \sum_{k=1}^{l} b_{lk}\{1-Q_{lk}-\beta(l+1)\} \ .$$

This yields the general optimal strategy for player White, namely:

$$w_{lk} = 1 \quad \text{if } Q_{lk} \geq 0.5 \qquad (2.6)$$
$$= 0 \quad \text{otherwise.}$$

For simplicity, let:

$$w_l = \max \{k;\ Q_{lk} \geq 0.5\}$$
$$= 0 \quad \text{otherwise.}$$
$$b_l = \max \{k;\ Q_{lk} \geq 1-\beta(l+1)\}$$
$$= 0 \quad \text{otherwise.}$$

Then $(w_l,\ b_l)$ represent the optimal game strategy when the l^{th} race result is considered. If w_l or b_l are zero, then the l^{th} race result is rejected by White or Black respectively. Otherwise:

Mr. White selects the l^{th} race result x_l if: $x_l \geq r_{l,w_l}$

Mr. Black selects the l^{th} race result if: $r_{l,b_l} \leq x_l < r_{l,w_l}$

When following the optimal strategy, (2.5) may be rewritten as:

$$l\ \beta(l) = b_l - w_l + (l-b_l)\ \beta(l+1) + G(l) \qquad (2.7)$$

where

$$G(l) = 2 \sum_{k=0}^{w_l} Q_{lk} - \sum_{k=0}^{b_l} Q_{lk}$$

and one defines $Q_{l0} = 0$.

The distribution $F(x)$ being unknown, both White and Black observe several initial race results without making a selection. Let j be the largest number such that $w_j = b_j = 0$. Then (2.7) yields $\beta(i) = \beta(j+1)$ for all $i \leq j+1$. Now from (2.3) when $l = 0$, one observes that:

$$P\{\hat{W}_n\} = \beta(1) = \beta(j+1) \ . \qquad (2.8)$$

The optimal strategies and winning probabilities may be computed from (2.4), (2.6), (2.7) and (2.8). Table 1 gives a short list of same.

It is worth noting that $P\{\hat{W}_n\}$ is not monotonic in n. This winning probability for white takes an upward jump whenever White's first choice strategy changes, namely at every second step. Hence $P\{\hat{W}_n\}$ is monotonic when n is odd, namely $n = 3, 5, 7, \ldots$ and also when n is even, namely $n = 4, 6, 8, \ldots$.

If $b_j = 0$, this implies $b_{j+1} \geq 1$. From (2.4), if $b_j = 0$, then $Q_{j1} = j/n < 1 - \beta(j+1)$. $b_{j+1} \geq 1$ in turn implies $Q_{j+1,1} = j+1/n \geq 1 - \beta(l+2)$.

Table 1

Optimal Strategies and Winning Probabilities when $F(x)$ is unknown.

Optimal strategy (w_k, b_k)

n	$P\{\hat{W}_n\}$	k										
		1	2	3	4	5	6	7	8	9	10	11
2	0.5000	any strategy will do.										
3	0.6667	(0,1)	(1,2)									
4	0.5833	(0,0)	(1,1)	(2,3)								
5	0.6333	(0,0)	(0,1)	(1,2)	(2,4)							
6	0.6111	(0,0)	(0,0)	(1,1)	(1,2)	(3,5)						
7	0.6238	(0,0)	(0,0)	(0,1)	(1,1)	(1,3)	(3,6)					
8	0.6205	(0,0)	(0,0)	(0,0)	(1,1)	(1,2)	(2,3)	(4,7)				
9	0.6234	(0,0)	(0,0)	(0,0)	(0,1)	(1,1)	(1,2)	(2,4)	(4,8)			
10	0.6190	(0,0)	(0,0)	(0,0)	(0,1)	(1,1)	(1,2)	(1,2)	(2,4)	(5,9)		
11	0.6204	(0,0)	(0,0)	(0,0)	(0,0)	(0,1)	(1,1)	(1,2)	(2,3)	(3,5)	(5,10)	
12	0.6175	(0,0)	(0,0)	(0,0)	(0,0)	(0,1)	(1,1)	(1,1)	(1,2)	(2,3)	(3,5)	(6,11)

When $l = j+1$ in (2.7), assuming $w_{j+1} = 0$ which is true for $n \geq 9$, one finds:

$$j(1 - \beta_{j+2}) = (j+1)\ \{1 - \beta_{j+1} - \frac{1}{n}\} \leq j(j+1)/n\ .$$

Incorporating (2.8), one obtains the inequality:

$$(2.9) \qquad j/n < 1 - P\{\hat{W}_n\} \leq (j+1)/n\ .$$

The inequality has been proved for $n \geq 9$. A similar inequality can be derived for $3 \leq n < 9$ which is equivalent to (2.9) in this range. Hence (2.9) is valid for $n \geq 3$.

The asymptotic probability of White winning was obtained empirically and found to be

$$\lim_{n\to\infty} P\{\hat{W}_n\} = 1 - \lim_{n\to\infty} j/n = 0.6086\ .$$

Hence White will observe $[n/2]$ races and for large n, Black will observe $j = 0.3914\, n$ races before considering a racing time for selection.

These pass periods are analogous to those in the classical secretary problem. The first race time that may quality for selection occurs after 50% of the races have been run. Since Black is at a disadvantage by always having second choice, he is less discriminating and optimally waits till only 39% of the races have been run before considering a race time for selection. This 39% also happens to be Black's probability of winning.

3. The Distribution $F(x)$ is known

When $F(x)$ is known, punters White or Black could choose any one of the racing times as no information useful to the game is available from past observations. As $F(x)$ is known, one need only consider $F(x)$ as the uniform distribution on $[0,1]$ as any continuous distribution can be transformed thereto. Hence let:

$$w_i(x) = P\{W_i | X_i = x,\ W_1',\ \ldots,\ W_{i-1}',\ B_1',\ \ldots,\ B_{i-1}'\}$$

$$b_i(x) = P\{B_i | X_i = x,\ W_1',\ \ldots,\ W_i,\ B_1',\ \ldots,\ B_{i-1}'\}\ .$$

For $n \geq 2$, the probability that player White wins is:

$$P\{\hat{W}_n\} = \int_0^1 [w_1(x)\ x^{n-1} + \{1 - w_1(x)\}\ b_1(x)(1 - x^{n-1})]\ dx$$
$$\qquad + P\{\hat{W}_{n-1}\} \int_0^1 \{1 - b_1(x)\}\{1 - w_1(x)\}\ dx \quad . \tag{3.1}$$

The optimal strategies in this finite extensive game with perfect recall represent the perfect equilibrium point as defined by R. Selten (1975). As we are dealing with only two players, the max-min concept applied to (3.1) is equivalent to finding the perfect equilibrium point.

As in section 2, it can be shown that $w_1(x) - 1$ implies $b_1(x) = 1$. This is intuitively reasonable as Black being at a disadvantage will not have a strategy as severe as White's. This allows (3.1) to read:

$$P\{\hat{W}_n\} = P\{\hat{W}_{n-1}\} + \int_0^1 w_1(x)\ \{2x^{n-1} - 1\}\ dx$$
$$\qquad + \int_0^1 b_1(x)\ [1 - P\{\hat{W}_{n-1}\} - x^{n-1}]\ dx\ . \tag{3.2}$$

The optimal strategies are now:

$$w_1^*(x) = 1 \quad \text{if} \quad x \geq (1/2)^{\frac{1}{n-1}}$$

$$= 0 \qquad \text{otherwise}$$

(3.3)

$$b_1^*(x) = 1 \quad \text{if} \quad x \geq \alpha_n = [1 - P\{\hat{W}_{n-1}\}]^{\frac{1}{n-1}}$$

$$= 0 \qquad \text{otherwise} \ .$$

Optimally, (3.2) becomes:

$$(3.4) \qquad P\{\hat{W}_n\} = \frac{1}{n} + \frac{n-1}{n} \left[\left(\frac{1}{2}\right)^{\frac{1}{n-1}} - (1 - P\{\hat{W}_{n-1}\})^{\frac{n}{n-1}}\right]$$

for $n \geq 2$ where $P\{\hat{W}_1\} = 1$.

Table 2 lists winning probabilities and strategies for small values on n.

The asymptotic winning probability can be found by allowing $u_n = 1 - P\{\hat{W}_n\}$. (3.4) now becomes:

$$u_n = \left(\frac{n-1}{n}\right) \left[1 - \left(\frac{1}{2}\right)^{\frac{1}{n-1}} + u_{n-1}^{\frac{n}{n-1}}\right]$$

or equivalently:

$$(3.5) \qquad n(u_n - u_{n-1}) = - u_{n-1} + (n-1) \left[1 - \left(\frac{1}{2}\right)^{\frac{1}{n-1}} - u_{n-1}\left(1 - u_{n-1}^{\frac{1}{n-1}}\right)\right] .$$

The sequence u_n is monotonic and bounded, hence

$$\underline{\lim}_{n\to\infty} n(u_n - u_n) = 0.$$

If $u = \lim\limits_{n\to\infty} u_n = 1 - \lim\limits_{n\to\infty} P\{\hat{W}_n\}$, then (3.5) becomes:

$$- u + \ln 2 + u \ln u = 0$$

or more succinctly:

$$(e/u)^u = 2.$$

Therefore $u = 0.327562414$ or

$$\lim_{n\to\infty} P\{\hat{W}_n\} = 0.6724.$$

Out of the n races run, it is also of interest to know in advance the likelihood of a selection being made at any particular race.

Table 2

Optimal Strategies and Winning Probabilities when $F(x)$ is known.

n,i	$P\{\hat{W}_n\}$	$w_{n-i+1}(x) = 1$ if $x \geq W$	$b_{n-i+1}(x) = 1$ if $x \geq B$
2	.7500	.5000	0
3	.7214	.7071	.5000
4	.7088	.7937	.6531
5	.7016	.8409	.7346
6	.6969	.8706	.7852
7	.6935	.8909	.8196
8	.6910	.9057	.8446
9	.6891	.9170	.8635
10	.6875	.9259	.8783
11	.6863	.9330	.8902
12	.6852	.9389	.9000
20	.6804	.9642	.9417
50	.6759	.9859	.9773
100	.6743	.9930	.9887
∞	.6724		

Note To determine strategies, select n and i from column and read corresponding strategy opposite choice of i. For example if $n = 5$. When $i = 5$, read $w_1(x) = 1$ if $x \geq .8409$; $b_1(x) = 1$ if $x \geq .7346$; when $i = 4$, read $w_2(x) = 1$ if $x \geq .7937$, $b_2(x) = 1$ if $x \geq .6531$ and so forth.

4. The Bivariate Distribution of the Races at Which Selections are Made.

Let K_n and L_n be random variables denoting the race numbers at which the first and second selections are made when following the optimal strategy.

The following form of the joint distribution is the easiest to write, namely:

$$P\{K_n=k,\ L_n \geq l\} = \int_0^1 \frac{dP}{dx}\{K_n=k,\ X_k < x\}\, x^{l-k-1}\, dx$$

(4.1)
$$= \alpha_n\, \alpha_{n-1} \cdots \alpha_{n-k+2} \int_{\alpha_{n-k+1}}^{1} x^{l-k-1}\, dx$$

$$= \alpha_n\, \alpha_{n-1} \cdots \alpha_{n-k+2}\, [1 - \alpha_{n-k+1}^{l-k}]/(l-k)$$

for $k = 1, \ldots, n-1$; $l = k+1, \ldots, n$ where α_n is defined by the optimal strategies in (3.3) and $\alpha_2 = 0$.

The first moments of K_n and L_n are now obtained

$$E(K_n) = 1 + \sum_{r=0}^{n-3} \prod_{i=0}^{r} \alpha_{n-i}$$

$$E(L_n) = 1 + E(K_n) + \sum_{r=0}^{n-3} \prod_{j=0}^{r} \alpha_{n-j+1} \sum_{i=2}^{n-1-r} (1 - \alpha_{n-i}^{i})/i$$

where $\alpha_{n+1} = 1$.

The asymptotic relations are more informative than those just derived.

5. Asymptotic Relations

Let $X_n = K_n/n$ and $Y_n = L_n/n$ and let $\overline{F}_n(x,y) = P\{X_n \geq x,\ Y_n \geq y\}$. Our aim is to find $\overline{F}(x,y) = \lim_{n\to\infty} \overline{F}_n(x,y)$. Since $\alpha_n = u_{n-1}^{\frac{1}{n-1}}$, a simple analysis of (4.1) yields:

(5.1)
$$G(x,y) = -\frac{d}{dx}\overline{F}(x,y) = (1-x)^{-ln\ u}\,(1-u^{\frac{y-x}{1-x}})/(y-x)\ .$$

The first limiting moment of X_n is:

$$\lim_{n\to\infty} E(X_n) = \int_0^1 xG(x,x)\, dx = (1 - ln\ u)^{-1}$$

$$= 0.4726\ .$$

It is interesting to note that the density of $Y = \lim_{n\to\infty} Y_n$ is discontinuous at $Y = 1$. This is due to the fact that after the first race selection has been made the other player often waits until the last race before a choice is forced on him. For example the second player to choose will select the last race time with probability

$$P\{Y = 1\} = \int_0^1 G(x,1)$$

$$= -\ (1-u)/\ln u$$

$$= 0.6025$$

The first moment of Y can now be written as:

$$E(Y) = \int_0^1 xG(x,x)\ dx + \int_0^1 dx \int_x^1 G(x,y)\ dy$$

$$= (1+v)/(1-\ln u),$$

where

$$v = \int_0^1 \frac{1-u^y}{y}\ dy$$

$$= -\sum_{k=1}^{\infty} \frac{(\ln u)^k}{k \cdot k!}$$

$$= 0.868235\ .$$

Hence

$$E(Y) = 0.8829.$$

In summary, if n is large the punters will make their first and second selections on average after 47% and 88% of the races have been run. Mr. White will win Jerome with a probability of 0.67.

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