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THE TRANSIENT BEHAVIOR OF THE PH/M/S/K QUEUE

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Abstract We analyze the transient behavior of the PH/M/S/K queue. Emphasis is placed on the overflow process, busy period distribution, idle period distribution, accepted arrival process, departure process and the transition probabilities of the system states. All results are represented by the addition and multiplication of the matrices whose dimensions are at most equal to the number of phases for the inter-arrival time distribution.

1. Introduction

The GI/M/S/K queue with S servers and K-S waiting positions is one of the basic models in Telephony, because many queueing systems in telephone network have non-Poisson arrivals such as overflows. In this paper, the PH/M/S/K queue is considered and inter-overflow time density function, all-busy period density function, idle period density function, inter-accepted-arrival time density function, inter-departure time density function, transition probabilities of the system states and so on are analyzed using Laplace transforms. All the formulae are given by simple recurrences and have a great advantage in numerical computation. Moreover, these formulae have much applicability. For example, we can obtain the moments of overflow traffic by the Laplace transform of the inter-overflow time density function, using the theory of GI/M/∞ queue.(5)(15) The Laplace transforms of the all-busy period density function and the inter-departure time density function are important in the analyses of priority queues and tandem queues, respectively. The unsolved model $PH^{[X]}$+PH/M/S/K queue with mixed group PH-renewal arrivals and simple PH-renewal arrivals become possible to be analyzed by applying the Laplace transforms of the transition probabilities of the system states to the theory of piecewise Markov process.(6)(9)(11) Another paper will discuss these applications in detail.

There are a number of papers in which the transient behavior of the GI/M/S/K queue have been studied. In regard to the overflow process, Çinlar-Disney [4] and Matsumoto-Watanabe [12] dealt with the GI/M/1/K queue and H_2/M/S/K queue, respectively. We can extend their results and also simplify their computation methods.

The busy period for the GI/M/S/K queue have been studied by Raju-Bhat [14], but their computation algorithm is complicated except for Poisson arrival case. Our results simplify their algorithm and the moments of the busy period can be easily computed.

The accepted arrival process for the loss system GI/M/S/S queue have been studied by Heffes-Holtzman. [8] They characterize this process as a semi-Markov process. We also characterize the accepted arrival process for the PH/M/S/K queue as a semi-Markov process which depends on the number of customers in the system and derive a semi-Markov matrix. [13]

Departure processes have been analyzed by Laslett [10] for the GI/M/1/K queue, and have been analyzed by Heffes [7] for the H_2/M/S/K queue. In this paper, for the more general model PH/M/S/K queue, the departure process is characterized as a semi-Markov process which depends on the number of customers in the system and the arrival phase states and a semi-Markov matrix [13] is derived.

The transition probabilities for the system states have been analyzed by Bhat. [1] However, the computational algorithm is complicated. We propose simpler computational algorithm.

2. Model and Notation

We consider the finite queueing system PH/M/S/K. Suppose that the interarrivals of customers are mutually independent and identically distributed with a phase type distribution whose density function is $f(t)$ and the service times of customers are independent and exponentially distributed with the mean μ^{-1}.

Let us assume that the total number of phases of $f(t)$ is m and assign a number to each phase. Let $A = \{1,2, \ldots, m\}$ denote the space of the numbered phases and consider a stochastic process $\{W(t) \in A;\ t \geq 0\}$. Additionally, let us consider another stochastic process $\{Y(t);\ t \geq 0\}$ where $Y(t)$ signifies the number of customers in the system. It is clear that the bivariate stochastic process $\{Y(t),\ W(t);\ t \geq 0$ is Markovian.

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3. First Passage Time Density Functions and Their Application

3.1. First Passage Time Density Functions

Define the density function of the first passage times (first entrance times[2]) in the bivariate stochastic process $\{Y(t),\ W(t);\ t\geq 0,\ 0\leq Y(t)\leq K,\ 1\leq W(t)\leq m\}$ as follows:

(1) $g^{+}_{i.}(t;n)dt \triangleq P\{Y(x)=n+1$ for some $x\varepsilon(t,\ t+dt),\ Y(y)\leq n$ for all $y\varepsilon[0,t]$ $\mid Y(0)=n,\ W(0)=i\},\qquad 0\leq n\leq K-1,\quad 1\leq i\leq m,$

$g^{+}_{i.}(t;K)dt \triangleq P\{$an overflow occurs at some $x\varepsilon(t,\ t+dt)$ for the first time after time 0 $\mid Y(0)=K,\ W(0)=i\},\qquad 1\leq i\leq m,$

(3) $g^{-}_{ij}(t;n)dt \triangleq P\{Y(x)=n-1,\ W(x)=j$ for some $x\varepsilon(t,\ t+dt),\ x\varepsilon(t,\ t+dt),$ $Y(y)\geq n$ for all $y\varepsilon[0,t]$, no overflows occur in $[0,t]$ $\mid Y(0)=n,\ W(0)=i\},\qquad 1\leq n\leq K,\quad 1\leq i,\ j\leq m,$

and

(4) $g^{\pm}_{ij}(t;n)dt \triangleq P\{Y(x)=n-1,\ W(x)=j$ for some $x\varepsilon(t,\ t+dt),\ Y(y)\geq n$ for all $y\varepsilon[0,t]$ $\mid Y(0)=n,\ W(0)=i\},\qquad 1\leq n\leq K,\quad 1\leq i,\ j\leq m.$

We will introduce the definitions (5)∿(10) in order to analyze (1)∿(4).

First, let us define the first passage time density functions for transitions of the number of customers in the system.

(5) $f^{+}_{i.}(t;n)dt \triangleq P\{Y(x)=n+1$ for some $x\varepsilon(t,\ t+dt),\ Y(y)=n$ for all $y\varepsilon[0,t]$ $\mid Y(0)=n,\ W(0)=i\},\qquad 0\leq n\leq K-1,\quad 1\leq i\leq m.$

(6) $f^{+}_{i.}(t;K)dt \triangleq P\{$an overflow occurs at $x\varepsilon(t,\ t+dt)$ for the first time after time 0, $Y(y)=K$ for all $y\varepsilon[0,t]$ $\mid Y(0)=K,\ W(0)=i\},$ $1\leq i\leq m.$

(7) $f^{-}_{ij}(t;n)dt \triangleq P\{Y(x)=n-1,\ W(x)=j$ for some $x\varepsilon(t,\ t+dt),\ Y(y)=n$ for all $y\varepsilon[0,t]$ $\mid Y(0)=n,\ W(0)=i\},\qquad 1\leq n\leq K-1,\quad 1\leq i,\ j\leq m.$

(8) $f^{-}_{ij}(t;K)dt \triangleq P\{Y(x)=K-1,\ W(x)=j$ for some $x\varepsilon(t,\ t+dt),\ Y(y)=K$ for all $y\varepsilon[0,t]$, no customers arrive in $[0,t]$ $\mid Y(0)=K,\ W(0)=i\},$ $1\leq i,\ j\leq m.$

$f^{\pm}_{ij}(t;K)dt \triangleq P\{Y(x)=K-1,\ W(x)=j$ for some $x\varepsilon(t,\ t+dt),\ Y(y)=K$ for all

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(9) $y\varepsilon[0,t] \mid Y(0)=n,\ W(0)=i\}$, $1\leq i,\ j\leq m$.

Next, let us consider the following transition probabilities of phase states.

(10) $h^+_{\cdot j}(n) \triangleq P\{W(t_k+0)=j,\ Y(t_k+0)=n+1 \mid Y(t_k)=n\}$, $0\leq n\leq K$, $1\leq j\leq m$.

where t_k and t_k+0 signify the arrival epochs of k-th customers and the instant immediately after this epoch, respectively. Since $h^+_{\cdot j}(n)$ $(0\leq n\leq K)$ does not depend on n, we simply write $h^+_{\cdot j}$ instead of $h^+_{\cdot j}(n)$.

Let us consider the following matrices:

(11) $G^+_n(t) = (g^+_{i\cdot}(t;n))$ $1\leq i\leq m$ ($m\times 1$ matrix),

(12) $G^-_n(t) = (g^-_{ij}(t;n))$ $1\leq i,\ j\leq m$ ($m\times m$ matrix),

(13) $G^\pm_n(t) = (g^\pm_{ij}(t;n))$ $1\leq i,\ j\leq m$ ($m\times m$ matrix),

(14) $F^+_n(t) = (f^+_{i\cdot}(t;n))$ $1\leq i\leq m$ ($m\times 1$ matrix),

(15) $F^-_n(t) = (f^-_{ij}(t;n))$ $1\leq i,\ j\leq m$ ($m\times m$ matrix),

(16) $F^\pm_n(t) = (f^\pm_{ij}(t;n))$ $1\leq i,\ j\leq m$ ($m\times m$ matrix),

and

(17) $H^+ = (h^+_{\cdot j})$ $1\leq j\leq m$ ($1\times m$ matrix).

Using (14)∿(17), we obtain both the semi-Markov matrix[13] for the accepted arrival process and that for the departure process. Refer to these in the Appendix for more detail. The semi-markov matrix is called the semi-markov kernel in Çinlar.[3]

Now we have the following recurrences, i.e.,

(18) $G^+_0(t) = F^+_0(t)$,

(19) $G^+_n(t) = \sum_{l=0}^{\infty} (F^-_n(t)*G^+_{n-1}(t)H^+)^{l*}*F^+_n(t)$, $1\leq n\leq K$,

where $A(t)*B(t)$ signifies the convolution of $A(t)$ and $B(t)$, and $A(t)^{l*}$ signifies the l-th convolution of itself.

Proof of (19): Let us define the following first passage time density matrices

$$G^+_n(t;l) = (g^+_{i\cdot}(t;n;l)) \qquad 1\leq i\leq m,\quad 1\leq n\leq K,$$

where

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$g^+_{i.}(t;n;l)dt \triangleq P\{Y(x)=n+1$ for some $x\varepsilon(t, t+dt)$, $Y(y)\leq n$ for all $y\varepsilon[0,t]$, the upward transitions from $Y(\cdot)=n-1$ to $Y(\cdot)=n$ occur l times in $[0,t]$ $|$ $Y(0)=n$, $W(0)=i\}$, $\quad 0\leq n\leq K$,

and

$g^+_{i.}(t;K;l)dt \triangleq P\{$an overflow occurs at some $x\varepsilon(t, t+dt)$ for the first time after time 0, the upward transitions from $Y(\cdot)=K-1$ to $Y(\cdot)=K$ occur l times in $[0,t]$ $|$ $Y(0)=n$, $W(0)=i\}$.

When we divide the interval $[0,t]$ into its subintervals $[0,s_1]$, $(s_1,t_1]$, $(t_1,s_2]$, $(s_2,t_2]$, ..., $(t_{l-1},s_l]$, $(s_l,t_l]$, $(t_l,t]$ where epochs s_i and t_i signify the epoch of the transition from $Y(\cdot)=n$ to $Y(\cdot)=n-1$ and that from $Y(\cdot)=n-1$ to $Y(\cdot)=n$, respectively, the transition probability density matrices of the intervals $[0,s_1]$, $[s_j+0,t_j](j=1, \ldots, l)$, $[t_j+0, s_{j+1}]$ $(j=1, \ldots, l-1)$, $[t_l+0,t]$ can be represented by $F^-_n(t)$, $G^+_{n-1}(t)$, $F^-_n(t)$ and $F^+_n(t)$, respectively. Since the transition matrices of $[s_j, s_j+0]$ and $[t_j, t_j+0]$ are $m\times m$ identity matrix and matrix H^+, respectively, we obtain

$$(20) \qquad G^+_n(t,l) = (F^-_n(t)*G^+_{n-1}(t)H^+)^{l*}*F^+_n(t).$$

Since $F^-_n(t)*G^+_{n-1}(t)H^+(1,1, \ldots, 1)^T<(1,1, \ldots, 1)^T$ (elementwise), the right hand of (19) converges. Then we obtain (19).

Similarly, we obtain

$$(21) \qquad G^-_K(t) = F^-_K(t),$$

$$(22) \qquad G^-_n(t) = \sum_{l=0}^{\infty} (F^+_n(t)H^+*G^-_{n+1}(t))^{l*}*F^-_n(t),$$

$$K-1\geq n\geq 1 \quad (n=K-1,\ K-2,\ \ldots,\ 1),$$

and

$$(23) \qquad G^{\pm}_K(t) = F^{\pm}_K(t) = F^-_K(t) + \sum_{m=0}^{\infty} F^+_K(t)*(\exp(-K\mu t)f(t))^{m*}*H^+F^-_K(t),$$

$$(24) \qquad G^{\pm}_n(t) = \sum_{l=0}^{\infty} (F^+_n(t)H^+*G^{\pm}_{n+1}(t))^{l*}F^-_n(t), \qquad K-1\geq n\geq 1.$$

Let $\tilde{x}(s)$ denote the Laplace transform of $x(t)$ and let us consider the following matrices, i.e.,

$$(25) \qquad \tilde{G}^+_n(s) \triangleq (\tilde{g}^+_{i.}(s;n)) \qquad 1\leq i\leq m\ ,$$

$$(26) \qquad \tilde{G}^-_n(s) \triangleq (\tilde{g}^-_{ij}(s;n)) \qquad 1\leq i,\ j\leq m\ ,$$

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(27) $\tilde{G}_n^{\pm}(s) \triangleq (\tilde{g}_{ij}^{\pm}(s;n)) \qquad 1 \leqq i,\ j \leqq m$,

(28) $\tilde{F}_n^{+}(s) \triangleq (\tilde{f}_{i.}^{+}(s;n)) \qquad 1 \leqq i \leqq m$,

(29) $\tilde{F}_n^{-}(s) \triangleq (\tilde{f}_{ij}^{-}(s;n)) \qquad 1 \leqq i,\ j \leqq m$,

and

(30) $\tilde{F}_n^{\pm}(s) = (\tilde{f}_{ij}^{\pm}(s;n)) \qquad 1 \leqq i,\ j \leqq m$.

From (19), we obtain the following equations.

(31) $\tilde{G}_0^{+}(s) = \tilde{F}_0^{+}(s)$,

(32)
$$\tilde{G}_n^{+}(s) = \sum_{l=0}^{\infty} (\tilde{F}_n^{-}(s)\tilde{G}_{n-1}^{+}(s)H^{+})^{l}\tilde{F}_n^{+}(s)$$
$$= \tilde{F}_n^{+}(s)+\tilde{F}_n^{-}(s)\tilde{G}_{n-1}^{+}(s)(1-H^{+}\tilde{F}_n^{-}(s)\tilde{G}_{n-1}^{+}(s))^{-1}H^{+}\tilde{F}_n^{+}(s),$$
$1 \leqq n \leqq K.$

In a similar fashion, from (21)∿(24), we obtain the following equations.

(33) $\tilde{G}_n^{-}(s) = \tilde{F}_K^{-}(s)$,

(34)
$$\tilde{G}_n^{-}(s) = \tilde{F}_n^{-}(s)+\tilde{F}_n^{+}(s)(1-H^{+}\tilde{G}_{n+1}^{-}(s)\tilde{F}_n^{+}(s))^{-1}H^{+}\tilde{G}_{n+1}^{-}(s)\tilde{F}_n^{-}(s),$$
$K-1 \leqq n \leqq 1$,

and

(35) $\tilde{G}_K^{\pm}(s) = \tilde{F}_K^{\pm}(s) = \tilde{F}_K^{-}(s)+(1-f(s+K\mu))^{-1}\tilde{F}_K^{+}(s)H^{+}\tilde{F}_K^{-}(s)$,

(36)
$$\tilde{G}_n^{\pm}(s) = \tilde{F}_n^{-}(s)+\tilde{F}_n^{+}(s)(1-H^{+}\tilde{G}_{n+1}^{\pm}(s)\tilde{F}_n^{+}(s))^{-1}H^{+}\tilde{G}_{n+1}^{\pm}(s)\tilde{F}_n^{-}(s),$$
$K-1 \leqq n \leqq 1.$

Since H^{+}, $\tilde{F}_n^{-}(s)$ and $\tilde{G}_{n-1}^{+}(s)$ are $1\times m$ matrix, $m\times m$ matrix and $m\times 1$ matrix, respectively, $H^{+}\tilde{F}_n^{-}(s)\tilde{G}_{n-1}^{+}(s)$ in (32) is scalar. Hence, $\tilde{G}_n^{+}(s)$ can be easily computed by the addition and multiplication of matrices whose dimensions are m at most. Complex procedures such as the computations of matrix inverses are not necessary at all. Since $H^{+}G_{n+1}^{-}(s)\tilde{F}_n^{+}(s)$ in (34) and $H^{+}G_{n+1}^{\pm}(s)F_n^{+}(s)$ in (36) are scalar, the same conclusion follows for $\tilde{G}_n^{-}(s)$ and $\tilde{G}_n^{\pm}(s)$, respectively. Our algorithmic procedures to compute $\tilde{G}_n^{+}(s)$, $\tilde{G}_n^{-}(s)$ and $\tilde{G}_n^{\pm}(s)$ are much simpler than the known procedures such as Matsumoto-Watanabe[12] and Raju-Bhat[14] in which the computations of $m\times m$ matrix inverses are required.

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3.2. Inter-Overflow Time Distribution and Overflow Traffic

The Laplace-Stieltjes transform $\psi(s)$ of the inter-overflow time distribution can be represented using (17) and (25) as follows:

$$\psi(s) = H^{+}\tilde{G}^{+}_{K}(s) \tag{37}$$

Since the overflow-process from the PH/M/S/K queue is renewal, we can quantify overflow traffic using the theory of the GI/M/∞ queue.(15)(6)

Example 1. Overflows from the H_m/M/S/K queue

In this example, we assume the inter-arrival time density function $f(t)$ to be $\sum_{j=1}^{m} k_j r_j \exp(-r_j t)$ $(k_j>0,\ r_j>0,\ \sum_{i=1}^{m} k_j=1)$. Next, we assign a number to each phase as shown in Fig. 1. From the definitions of $\tilde{f}^{+}_{i.}(s;n)$, $\tilde{f}^{-}_{ij}(s;n)$ and $h^{+}_{.j}$, we obtain

$$\tilde{f}^{+}_{i.}(s;n) = \frac{r_i}{s+r_i+\mu_n}, \tag{38}$$

$$\tilde{f}^{-}_{ij}(s;n) = \begin{cases} \dfrac{\mu_n}{s+r_i+\mu_n} & (i=j) \\ 0 & (i\neq j), \end{cases} \tag{39}$$

and

$$h^{+}_{.j} = k_j, \tag{40}$$

where $\mu_n = \min(n,S)\cdot\mu$.

Therefore, from (31) and (32), we obtain

$$\tilde{g}^{+}_{i.}(s;0) = \frac{r_i}{s+r_i}, \tag{41}$$

$$\tilde{g}^{+}_{i.}(s;n) = \frac{r_i}{s+r_i+\mu_n} + \frac{\mu_n}{s+r_i+\mu_n}\tilde{g}^{+}_{i.}(s;n-1)\sum_{j=1}^{m}\frac{k_j r_j}{s+r_j+\mu_n} \Big/ \left(1-\sum_{j=1}^{m}\frac{k_j\mu_n}{s+r_j+\mu_n}\tilde{g}_{j.}(s;n-1)\right), \tag{42}$$

$$1\leq n\leq K,$$

and then

$$\psi(s) = a_K + a_K b_K/(1-b_K), \tag{43}$$

where

$$a_K = \sum_{j=1}^{m}\frac{k_j r_j}{s+r_j+\mu_K}, \quad b_K = \sum_{j=1}^{m}\frac{k_j\mu_K}{s+r_j+\mu_K}\tilde{g}^{+}_{j.}(s;K-1). \tag{43}$$

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Example 2. Overflows from the $E_m/M/S/K$ queue

In this example, we assume the inter-arrival time density function $f(t)$ to be the Erlang type $(m\lambda)^m t^{m-1} t^{-\lambda mt}/(m-1)!$ If we assign a number to each phase as shown in Fig. 2, we obtain

(44) $$\tilde{f}^{+}_{i.}(s;n) = \left(\frac{m\lambda}{s+m\lambda+\mu_n}\right)^{i},$$

(45) $$\tilde{f}^{-}_{ij}(s;n) = \begin{cases} \frac{\mu_n}{m\lambda}\left(\frac{m\lambda}{s+m\lambda+\mu_n}\right)^{i-j+1}, & i \geqq j, \\ 0, & i<j, \end{cases}$$

and

(46) $$h^{+}_{.j} = \begin{cases} 1, & j=m, \\ 0, & j \neq m. \end{cases}$$

From (31) and (32), we obtain

(47) $$\tilde{g}^{+}_{i.}(s;0) = \left(\frac{m\lambda}{s+m\lambda}\right)^{i},$$

(48) $$\tilde{g}^{+}_{i.}(s;n) = \left(\frac{m\lambda}{s+m\lambda+\mu_n}\right)^{i} + \left(\sum_{j=1}^{i} \frac{\mu_n}{m\lambda}\left(\frac{m\lambda}{s+m\lambda+\mu_n}\right)^{i-j+1} \tilde{g}^{+}_{j.}(s;n-1)\right)\left(\frac{m\lambda}{s+m\lambda+\mu_n}\right)$$
$$/\left(1 - \sum_{j=1}^{m} \frac{\mu_m}{m\lambda}\left(\frac{m\lambda}{s+m\lambda+\mu_n}\right)^{m-j+1} \tilde{g}^{+}_{j.}(s;n-1)\right),$$

$$1 \leqq n \leqq K,$$

and then

(49) $$\psi(s) = \tilde{g}^{+}_{m.}(s;k).$$

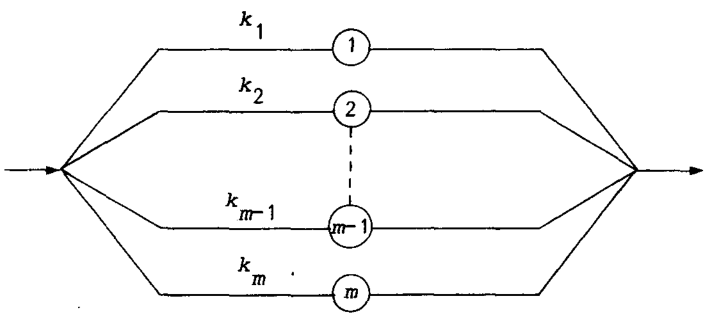

Fig. 1 Phase of Hyper-exponential Distribution

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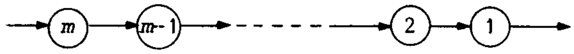

Fig. 2 Phase of Erlang Distribution

3.3. Joint distribution of all-busy period and number of overflows

Since the all-busy period is defined as the first passage time from the epoch at which all servers become busy due to a certain arrival until the epoch at which a server becomes idle as a result of a certain departure, the Laplace-Stieltjes transform, vector $\tilde{B}(s)$, of this can be written as follows:

(50) $$\tilde{B}(s) = H^{+}\tilde{G}^{\pm}_{S}(s).$$

Let T denote the all-busy period and let $O(T)$ denote the number of overflows in this period. Here, we consider the following joint probability density function.

(51) $$B(t,j)dt \triangleq P\{t<T\leq t+dt,\ O(T)=j\}.$$

Let us define the following first passage time density function with taboo states $Y(\cdot) < S$.

(52)
$$g^{+}_{i\cdot}(t;n,\ Y\geq S)dt \triangleq P\{Y(x)=n+1 \text{ for some } x\varepsilon(t,\ t+dt),$$
$$S\leq Y(y)\leq n \text{ for all } y\varepsilon[0,t], \text{ no overflows occur in } [0,t]$$
$$\mid Y(0)=n,\ W(0)=i\}, \qquad S\leq n\leq K,\quad 1\leq i\leq m.$$

If we consider the column vector

$$\tilde{G}^{+}_{n}(s;Y\geq S) \triangleq (\tilde{g}^{+}_{i\cdot}(s;n,Y\geq S)) \qquad 1\leq i\leq m,$$

we obtain the following equations in a similar fashion to (31) and (32), i.e.,

(53) $$\tilde{G}^{+}_{S}(s;Y\geq S) = \tilde{F}^{+}_{S}(s),$$

(54) $$\tilde{G}^{+}_{n}(s;Y\geq S) = \tilde{F}^{+}_{n}(s)+\tilde{F}^{-}_{n}(s)\tilde{G}^{+}_{n-1}(s;Y\geq S)(1-H^{+}\tilde{F}^{-}_{n}(s)\tilde{G}^{+}_{n-1}(s;Y\geq S))^{-1}\ H^{+}\tilde{F}^{+}_{n}(s),$$
$$n\geq S+1.$$

The Laplace transform $\tilde{B}(s,j)$ of $\tilde{B}(t,j)$, i.e.,

$$\tilde{B}(s,j) = \int_{0}^{\infty} e^{-st}B(t,j)dt$$

can be written from the definitions of $\tilde{G}^{+}_{n}(s:Y\geq S)$, $\tilde{G}^{-}_{n}(s)$, and H^{+} as follows:

(55) $$\tilde{B}(s,j) = H^{+}\tilde{G}^{+}_{S}(s;Y\geq S)H^{+}\tilde{G}^{+}_{S+1}(s;Y\geq S)\ \dots\ H^{+}\tilde{G}^{+}_{K-1}(s;Y\geq S)$$

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$$\cdot(H^{+}\tilde{G}_{K}^{+}(s;Y\geq S))^{j}H^{+}\tilde{G}_{K}^{-}(s)\tilde{G}_{K-1}^{-}(s)\ \ldots\ \tilde{G}_{S}^{-}(s),$$

$j\geq 0$.

Specifically, it is clear that

$$\tilde{B}(s,0)=H^{+}G_{S}^{-}(s). \tag{56}$$

Therefore, defining

$$\tilde{B}^{*}(s,z)\ \triangleq\ \int_{0}^{\infty}e^{-st}\sum_{j=0}^{\infty}z^{j}B(t,j)dt,$$

we obtain from (55) and (56)

$$\tilde{B}^{*}(s,z)=H^{+}\tilde{G}_{S}^{-}(s)+z(1-zH^{+}\tilde{G}_{K}^{+}(s;Y\geq S))^{-1}(\prod_{j=0}^{K-S}H^{+}\tilde{G}_{S+j}^{+}(s;Y\geq S))H^{+}(\prod_{j=0}^{K-S}\tilde{G}_{K-j}^{-}(s)). \tag{58}$$

Using (58), we can easily compute the covariance $Cov(T,O)$ of the all-busy period and the number of overflows during this period, i.e.,

$$\begin{aligned}Cov(T,O)&=-\frac{\partial}{\partial s}\frac{\partial}{\partial z}\tilde{B}^{*}(s,z)\Big|_{z=1,\ s=0}\cdot\tilde{B}^{T}(0)-E(T)E(O)\\&=[-\frac{\partial}{\partial s}\{(1-H^{+}\tilde{G}_{K}^{+}(s;Y\geq S))^{-1}[1+H^{+}\tilde{G}_{K}^{+}(s;Y\geq S)(1-H^{+}\tilde{G}_{K}^{+}(s;Y\geq S))^{-1}]\\&\quad\cdot(\prod_{j=0}^{K-S}H^{+}\tilde{G}_{S+j}^{+}(s;Y\geq S))H^{+}(\prod_{j=0}^{K-S}\tilde{G}_{K-j}^{-}(s))\}\Big|_{s=0}]\cdot\tilde{B}^{T}(0)-E(T)E(O),\end{aligned} \tag{59}$$

where $\tilde{B}^{T}(s)$ is a transposed vector of $\tilde{B}(s)$,

$$E(T)=-\frac{\partial}{\partial s}\tilde{B}(s)\Big|_{s=0}\cdot\tilde{B}^{T}(0)\qquad\text{(from (50))},$$

and

$$E(O)=(1-H^{+}\tilde{G}_{K}^{+}(0;Y\geq S))^{-1}[1+H^{+}\tilde{G}_{K}^{+}(0;Y\geq S)(1-H^{+}\tilde{G}_{K}^{+}(0;Y\geq S))^{-1}]\cdot(\prod_{j=0}^{K-S}H^{+}\tilde{G}_{S+j}^{+}(0;Y\geq S))H^{+}(\prod_{j=0}^{K-S}\tilde{G}_{K-j}^{-}(0)).$$

4. Transition Probability

In this section, we consider the transition probability

$$p_{ij}(t;k,l)\ \triangleq\ P\{Y(t)=j,\ W(t)=l\,|\,Y(0)=i,\ W(0)=k\},\qquad 0\leq i,\ j\leq K,\ 1\leq k,\ l\leq m. \tag{60}$$

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Let us introduce

(61) $$p_{jj}(t;k,l;n) \triangleq P\{Y(t)=j,\ W(t)=l,\ \text{the upward transitions from } Y(\cdot)=j-1 \text{ to } Y(\cdot)=j \text{ occur } n \text{ times in } [0,t]\ |\ Y(0)=j,\ W(0)=k\},$$

and

(62) $$q_{jj}(t;k,l) \triangleq P\{Y(u)=j \text{ for all } u\in[0,t],\ W(t)=l\,|\,Y(0)=j,\ W(0)=k\}.$$

From (60) and (61),

(63) $$p_{jj}(t;k,l) = \sum_{n=0}^{\infty} p_{jj}(t;k,l;n).$$

Let $P_{jj}(t;n)$ and $Q_{jj}(t)$ denote the matrices which have the elements $p_{jj}(t;k,l;n)(1\leqq k,\ l\leqq m)$ and $q_{jj}(t;k,l)(1\leqq k,\ l\leqq m)$, respectively, i.e.,

(64) $$P_{jj}(t;n) = (p_{jj}(t;k,l;n)) \qquad 1\leqq k,\quad l\leqq m\ ,$$

and

(65) $$Q_{jj}(t) = (q_{jj}(t;k,l)) \qquad 1\leqq k,\quad l\leqq m\ .$$

(i) In the case of $n = 0$

Considering the number of downward transitions from $Y(\cdot)=j+1$ to $Y(\cdot)=j$, we obtain

(66) $$P_{jj}(t;0) = \sum_{i=0}^{\infty} (F_j^{+}(t)H^{+} * G_{j+1}^{\pm}(t))^{i*} * Q_{jj}(t)\ , \qquad 0\leqq j<K\ ,$$

and

(67) $$P_{KK}(t;0) = Q_{KK}(t)\ .$$

(ii) In the case of $n \geqq 1$

From (61), we obtain

(68) $$P_{00}(t;n) = 0 \qquad \text{(zero matrix)},$$

and

(69) $$P_{jj}(t;n) = (G_j^{\pm}(t) * G_{j-1}^{+}(t)H^{+})^{n*} * P_{jj}(t;0)\ , \qquad 1\leqq j\leqq K,\quad n\geqq 1\ .$$

Therefore, it follows from (63), (66), (68) and (69) that

(70) $$P_{00}(t) = \sum_{n=0}^{\infty} (F_0^{+}(t)H^{+} * G_1^{\pm}(t))^{n*} * Q_{00}(t)\ ,$$

and

(71) $$P_{jj}(t) = \sum_{n=0}^{\infty} (G_j^{\pm}(t) * G_{j-1}^{+}(t)H^{+})^{n*} * P_{jj}(t;0)\ , \qquad 1\leqq j\leqq K\ .$$

Taking the Laplace transform of both sides of (70), we obtain

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(72) $\tilde{P}_{00}(s) = \tilde{Q}_{00}(s)+\tilde{F}_0^+(s)(1-H^+\tilde{G}_1^{\pm}(s)\tilde{F}_0^+(s))^{-1}H^+\tilde{G}_1^{\pm}(s)\tilde{Q}_{00}(s)$.

It should be noted that $H^+\tilde{G}_1^{\pm}(s)\tilde{F}_0^+(s)$ is scalar.

Taking the Laplace transform of both sides of (71), we obtain

(73) $\tilde{P}_{jj}(s) = \tilde{P}_{jj}(s;0)+\tilde{G}_j^{\pm}(s)\tilde{G}_{j-1}^+(s)(1-H^+\tilde{G}_j^{\pm}(s)\tilde{G}_{j-1}^+(s))^{-1}H^+\tilde{P}_{jj}(s;0)$, $1 \leqq j \leqq K$,

where from (66)

(74) $\tilde{P}_{jj}(s;0) = \tilde{Q}_{jj}(s)+\tilde{F}_j^+(s)(1-H^+\tilde{G}_{j+1}^{\pm}(s)\tilde{F}_j^+(s))^{-1}H^+\tilde{G}_{j+1}^{\pm}(s)\tilde{Q}_{jj}(s)$.

It should be noted that $H^+\tilde{G}_j^{\pm}(s)\tilde{G}_{j-1}^+(s)$ and $H^+\tilde{G}_{j+1}^{\pm}(s)\tilde{F}_j^+(s)$ are scalar.

We can easily obtain $\tilde{Q}_{jj}(s)$ as follows:

(75) $\tilde{Q}_{jj}(s) = \frac{1}{\mu_j} \tilde{F}_j^-(s)$, $1 \leqq j \leqq K$,

(76) $\tilde{Q}_{KK}(s) = \frac{1}{\mu_K} \tilde{F}_K^{\pm}(s)$.

We can also obtain $\tilde{Q}_{00}(s)$ in a similar fashion to $\tilde{Q}_{jj}(s)$ $(1 \leqq j \leqq K)$.

Next, let us consider $P_{ij}(t)$ which is defined as

(77) $P_{ij}(t) \triangleq (p_{ij}(t;k,l))$ $1 \leqq k,\ l \leqq m,\ i \neq j$.

Since $P_{ij}(t)$ is represented as the convolution of the first passage time from $Y(\cdot)=i$ to $Y(\cdot)=j$ and $P_{jj}(t)$, the Laplace transform $\tilde{P}_{ij}(s)$ of $P_{ij}(t)$ can be obtained as follows:

(78)
$$\tilde{P}_{ij}(s) = \begin{cases} \tilde{G}_i^+(s)H^+\tilde{G}_{i+1}^+(s)H^+ \cdots\cdots \tilde{G}_{j-1}^+(s)H^+\tilde{P}_{jj}(s) , & i<j , \\ \tilde{G}_i^{\pm}(s)\tilde{G}_{i-1}^{\pm}(s) \cdots\cdots \tilde{G}_{j+1}^{\pm}(s)\tilde{P}_{jj}(s) & i>j . \end{cases}$$

In this way, we have obtained the Laplace transform $\tilde{P}_{ij}(s;k,l)$ of the transition of the transition probability $P_{ij}(t;k,l)$ for each i, j, k, l.

5. Conclusion

We have analyzed the following types of transient behavior of the PH/M/S/K queue. All results are represented by the addition and multiplication of the matrices whose dimensions are at most equal to the total number of

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phases for the inter-arrival time density function.

(i) Overflow Process

We have shown that overflows form a renewal process and have derived the Laplace transform of the inter-overflow time density function.

(ii) Busy Period and Idle Period

We have derived the Laplace transform of the busy (idle) period density function.

(iii) Busy Period and Number of Overflows

We have derived the joint distribution of the busy period and the number of overflows during this period.

(iv) Accepted Arrival Process

We have characterized this process as a semi-Markov process and have derived a semi-Markov matrix.

(v) Departure Process

We have characterized this process as a semi-Markov process and have derived a semi-Markov matrix.

(vi) Transient Behavior of the Number of Customers in the System

We have derived the transition probability for the bivariate stochastic process $\{Y(t), W(t)\}$ where $Y(t)$ and $W(t)$ signify the number of customers in the system and the arrival phase state at time t, respectively.

Appendix. Accepted Arrival Process and Departure Process for the PH/M/S/K Queue

We have considered the process of arrivals which are accepted by the PH/M/S/K queue, and have termed this process the Accepted Arrival Process. In the loss system PH/M/S/S queue, this process is called the Carried Arrival Process.[8]

Let $\{\tau_i; i = 1, 2, \ldots\}$ denote the sequence of times at which arrivals are accepted by the PH/M/S/K queue, and let $Y(t)$ denote the number of customers in this system at time t. Since $\{\tau_i; i = 1, 2, \ldots\}$ is the subset of times at which customers arrive, $\{Y(\tau_i); i = 1, 2, \ldots\}$ forms an embedded Markov chain. We characterize the accepted arrivals as a semi-Markov process which depends on the number of customers in the system, and derive a semi-Markov matrix $(q_{ln}(t))$ $1 \leqq l,\ n \leqq K$,

where

(A.1) $$q_{ln}(t)dt \triangleq P\{Y(\tau_i)=n, t<\tau_i-\tau_{i-1}<t+dt \mid Y(\tau_{i-1})=l\}.$$

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For $l<K$, since the next arrival is always accepted and there is a pure death process between accepted arrivals, we obtain

(A.2)
$$q_{ln}(t)=\begin{cases} H^{+}F_{l}^{+}(t), & n=l+1,\\ H^{+}F_{l}^{-}(t)*\cdots\cdots*F_{n}^{-}(t)*F_{n-1}^{+}(t), & 1\leq n\leq l,\\ 0, \quad \text{otherwise}, & \end{cases}$$
$1\leq l<K$.

For $l=K$, since the next arrival need not be accepted and overflows might occur between accepted arrivals, we obtain

(A.3)
$$q_{Kn}(t)=\begin{cases} H^{+}F_{K}^{\pm}(t)*F_{K-1}^{+}(t), & n=K,\\ H^{+}F_{K}^{\pm}(t)*F_{K-1}^{-}(t)*\cdots\cdots*F_{n}^{-}(t)*F_{n-1}^{+}(t), & 1\leq n<K,\\ 0, \quad \text{otherwise}. & \end{cases}$$

Next, we consider the departure process. Let $\{\delta_i; i=1, 2, \ldots\}$ denote the sequence of times at which customers in the queue depart. Since $\{Y(\delta_i); i = 1, 2, \ldots\}$ does not constitute a Markov chain, we then consider the bivariate Markov chain $\{Y(\delta_i), W(\delta_i); i=1, 2, \ldots\}$.

We characterize the departures as a semi-Markov process which depends on the number of customers in the system and the arrival phase state and derive a semi-Markov matrix $(q_{ln}(t;j,k))$ $0\leq l,\ n<K,\ 1\leq j,\ k\leq m$,

where

(A.4)
$$q_{ln}(t;j,k)dt \triangleq P\{Y(\delta_i)=n,\ W(\delta_i)=k,\ t<\delta_i-\delta_{i-1}<t+dt \mid Y(\delta_{i-1})=l,\ W(\delta_{i-1})=j\}\ .$$

For $n<K-1$, since there is a pure birth process between departures, we obtain

$(q_{ln}(t;j,k))$ $1\leq j,\ K\leq m$

(A.5)
$$=\begin{cases} F_{l}^{-}(t), & 1\leq l\leq K-1,\ n=l-1\leq K-2,\\ F_{l}^{+}(t)*H^{+}F_{l+1}^{+}(t)*\cdots*H^{+}F_{n}^{+}(t)*H^{+}F_{n+1}^{-}(t), & 0\leq l\leq K-1,\ 1\leq n\leq K-2,\\ 0, & n<l-1\leq K-2\ . \end{cases}$$

For $n=K-1$, since overflows might occur between departures, we obtain

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$$(q_{l,K-1}(t))_{1\leqq j,\ K\leqq m} = F_l^+(t) *_H {}^+F_{l+1}^+(t) * \cdots\cdots *_H {}^+F_{K-1}^+(t) *_H {}^+F_K^{\pm}(t) , \qquad 0\leqq l\leqq K-1 . \tag{A.6}$$

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References

[1] U. N. Bhat: Some Problems in Finite Queue, *Lecture Notes in Eco. and Math. Syst.*, No.98, PP.139-156, Springer, New York, 1974.

[2] K. L. Chung: *Markov Chains with Stationary Transition Probabilities*, Second Ed., Springer, Berlin, 1967.

[3] E. Çinlar: *Introduction to Stochastic Processes*, Pretice-Hall, New Jersey, 1975.

[4] E. Çinlar and R. L. Disney: Streams of Overflows from a Finite Queue, *Oper. Res.*, 15, PP.131-138, 1967.

[5] V. P. Franken: Erlangsche Formeln für Semi-Markovschen Eingang, *Elec. Infor. Kyb.*, 4, SS. 197-204, 1968.

[6] M. Fujiki and E. Gambe: *Teletraffic Theory*, Maruzen, Tokyo, 1980. (In Japanese)

[7] H. Heffes: On the Output of a GI/M/N Queueing System with Interrupted Input, *Oper. Res.*, 24, PP.530-542, 1976.

[8] H. Heffes and J. M. Holtzman: Peakedness of Traffic Carried by a Finite Trunk Group with Renewal Input, *B. S. T. J.*, 52, PP.1617-1642, 1973.

[9] A. Kuczura: Piecewise Markov Processes, *SIAM J. Appl. Math.*, 24, PP.169-181, 1973.

[10] G. M. Laslett: Characterising the Finite Capacity GI/M/1 Queue with Renewal Output, *Manage. Sci.*, 22, PP.106-110, 1975.

[11] F. Machihara: Traffic Characteristics of Loss System with Two Independent Renewal Inputs, *Trans. IECE Japan*, 63-B, PP.1071-1078, 1980. (In Japanese)

[12] J. Matsumoto and Y. Watanabe: Theoretical Method for the Analysis of Queueing System with Overflow Traffic, *Trans. IECE Japan*, 64-B, PP.536-543, 1981. (In Japanese)

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[13] V. Nollau: *Semi-Markovsche Prozesse*, Akademie-Verlag, Berlin, 1980.

[14] S. N. Raju and U. N. Bhat: Recursive Relations in the Computation of the Equilibrium Results of Finite Queues, *TIMS Studies in Manage. Sci.*, Vol.7 (Algorithmic Methods in Probability), PP.247-270, North-Holland, New York, 1971.

[15] L. Takács: *Introduction to the Theory of Queues*, Oxford Univ. Press, New York, 1962.

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