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TWO-AND THREE-MACHINE FLOW-SHOP MAKESPAN SCHEDULING PROBLEMS WITH ADDITIONAL TIMES SEPARATED FROM PROCESSING TIMES

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Abstract This paper considers two- and three-machine flow-shop makespan scheduling problems with additional times separated from processing times.

For two-machine case we investigate the optimality of the passing of jobs and give an optimal algorithm in permutation case with five independent times, that is, setup and removal times, start and stop lags and transportation times. It generalizes known algorithms. And for a three-machine case with independent setup times alone, we present several sufficient optimality conditions for a specified sequence ω_1 and clarify not only their mutual relation but also the superiority of ω_1 as an approximate sequence. Those results generalize the results in usual three-machine flow-shop case.

1. Introduction

Historically, for the two-machine flow-shop makespan scheduling problem, Maggu et al [10] extended well known Johnson criterion [7] in the usual two-machine flow-shop makespan scheduling problem for constructing an optimal sequence to two-machine *permutation* flow-shop case with arbitrary time lags (start lag and stop lag) and transportation times of jobs between machines, and on the other hand, Sule [18] and Maruyama and Nabeshima [11] separately extended Johnson [7] and Yoshida and Hitomi [20] results to the similar two-machine *permutation* scheduling problem with separated sequence-independent setup times and removal (clean up) times.

For a three-machine flow-shop makespan scheduling problem with sequence-independent setup times, Yoshida and Hitomi [20] has shown that the passing of jobs may produce a sequence having minimum makespan, and Maruyama and Nabeshima [11] proposed two solvable cases under some restraints on removal times for similar problem with sequence-independent setup times and removal times.

Practical significances of those additional times are obvious and are described in detail in the papers [12], [20], [10] and [18].

Then in the first sections (§§ 2.1-2.3) in this paper we investigate a general two-machine flow-shop makespan scheduling problem with such five additional times under the same assumptions as in the usual two-machine flow-shop case [4] [2] and show by simple examples that the passing of jobs may produce an optimal schedule. Next, we give, in permutation scheduling case, an optimal algorithm, that is, a criterion for constructing an optimal sequence in our general problem. Our algorithm includes known algorithms as special cases.

In sections §§ 3.1-3.7, this paper deals with a three-machine permutation makespan scheduling problem with sequence-independent setup times and, in order to cralify not only the relation among polynomially solvable cases [6] [5] but also the superiority of a specified sequence ω_1 as an approximate sequence, we identify several new sufficient optimality conditions for ω_1.

In this paper, the following notations are used. Those are deterministically given where $i=1,2,\ldots,n$ and $t=1,2$ in § 2, $t=1,2,3$ in § 3 ($P_{i,t}$, S_i^t).

$P_{i,t}$: Sequence-independent processing time of job i on the t-th machine M_t.

S_i^t : Sequence-independent setup times of M_t for job i. Setup can start prior to the processing of the job i if machine is idle.

R_i^t : Sequence-independent removal times of M_t for job i. Removal follows directly after the processing of the job i.

D_i : Start lag of job i which requires that the processing of job i on M_2 must not start sooner than D_i time-units after the starting of its processing on M_1.

E_i : Stop lag of job i which requires that the processing of job i on M_2 must not complete sooner than E_i time-units after the completion of its processing on M_1.

t_i : Transportation time of job i which is required to transport job i directly after the completion of its processing on M_1 to M_2 for its processing on M_2.

ω_1 : A specified sequence ω_1 for three-machine case with sequence-independent setup time alone is constructed by a single criterion: if

(1) $$\min\,[S_i^1+P_{i,1}+P_{i,2}-S_i^3,\ P_{j,2}+P_{j,3}] \le \min\,[S_j^1+P_{j,1}+P_{j,2}-S_j^3,\ P_{i,2}+P_{i,3}]$$

holds, then job i precedes job j where either ordering is optimal provided the equality holds.

2. Passing of Jobs and An Optimal Permutation Algorithm in General Two-Machine Case with Five Separated Independent Times

2.1 On the Passing of Jobs

When additional times are sequence-independent setup time alone, it is sufficient to consider only permutation schedules as demonstrated in Yoshida and Hitomi's paper [20].

Also, if additional times are sequence-independent removal times alone, the same fact hold. cf. [4] Chap. 5, [2] Chap. 7.

On the other hand, we can illustrate by examples that the passing of a job may produce an optimal schedule in the cases where at least one additional times among five such times exist, except the above S_i^t case and R_i^t case. For example, next example shows the above fact for the (S_i^t, R_i^t) case:

Example. (S_i^t, R_i^t) case. We consider two jobs 1 and 2 with related data given by Table 1.

Table 1

job	S_i^1	$P_{i,1}$	R_i^1	S_i^2	$P_{i,2}$	R_i^2
1	2	2	2	2	3	2
2	2	2	12	11	2	2

Permutation schedules 1 2 and 2 1 have makespan M(1 2)=24 and M(2 1)=25 respectively, but a schedule (1 2 on M_1, 2 1 on M_2) gives smaller makespan $M\binom{1\ 2}{2\ 1}=22$. In fact it is optimal since $M\binom{2\ 1}{1\ 2}=40$.

Similarly we can easily give the examples showing that $M\binom{1\ 2}{2\ 1}$ is the smallest, for the other cases.

Then, in order to identify a simple optimality criterion, in the next sections we seek for an optimal schedule among the set of permutation schedules, that is, sequences in our general problem.

2.2 The Optimal Algorithm for General Permutation Case

The optimal algorithm for general problem restricted to permutation schedules consists of the following steps:

Step 1. Let T_i denote the revised start lag, defined by

$$T_i = \max\,[D_i,\ \delta(t_i),\ P_{i,1}+E_i-P_{i,2}],$$

where

$$\delta(t_i) = \begin{cases} P_{i,1}+t_i & \text{for } t_i > 0 \\ 0 & \text{for } t_i = 0. \end{cases}$$

Step 2. Let G and H be two fictitious machines having respective processing times for job i as G_i and H_i, where G_i and H_i are defined by

$$G_i = S_i^1 - S_i^2 + T_i,\quad H_i = T_i - P_{i,1} + P_{i,2} - R_i^1 + R_i^2.$$

Step 3. Construct an optimal sequence by Johnson's working procedure [7] [2] in usual two-machine n-job problem for the reduced problem in step 2.

Step 4. An optimal sequence constructed in step 3 also gives an optimal sequence for the original general permutation scheduling problem.

By this algorithm we have an optimal sequence in polynomial time O(n log n).

Particular Studies

As special cases of our algorithm, we have the following:

For every job i.

(1) If $D_i=P_{i,1}$, $E_i=P_{i,2}$ and $t_i=0$, then the algorithm reduces to the algorithm presented separately by Sule [18], and Maruyama and Nabeshima [11].

(2) If $S_i^t=0$ and $R_i^t=0$, then the algorithm reduces to the algorithm by Maggu et al [10] where $t_i'=T_i-P_{i,1}$.

(3) If $S_i^t=0$, $R_i^t=0$, $D_i=P_{i,1}$ and $E_i=P_{i,2}$, then the algorithm reduces to the algorithm by Maggu and Das [9].

(4) If $S_i^t=0$, $R_i^t=0$ and (a) $t_i=0$, or (b) $D_i \geq P_{i,1}+t_i$ and $E_i \geq t_i+P_{i,2}$, then the algorithm reduces to the algorithm by Mitten [12] and Johnson [8].

(5) If $S_i^t=0$, $R_i^t=0$, $D_i=P_{i,1}$, $E_i=P_{i,2}$ and $t_i=0$, then the algorithm reduces to the algorithm by Johnson [7] and Bellman [1].

2.3 Proof of Optimal Algorithm

We compare the makespans of two sequences $S=\pi\ell ij\pi'$ and $S'=\pi\ell ji\pi'$ where S' is a sequence obtained only by interchanging two adjacent jobs i and j in S, and π and π' are subsequences or may be empty.

Let $C_t(i)$ be the finishing time (here finish means end of removal) of job i on M_t.

Then, we have for S

$$C_1(i) = C_1(\ell)+S_i^1+P_{i,1}+R_i^1, \tag{2}$$

$$\begin{aligned} C_2(i) &= \max\ [\max\{C_2(\ell)+S_i^2,\ C_1(\ell)+S_i^1+D_i,\ C_1(\ell)+S_i^1+\delta(t_i)\} \\ &\qquad +P_{i,2},\ C_1(\ell)+S_i^1+P_{i,1}+E_i] + R_i^2 \\ &= S_i^2+P_{i,2}+R_i^2+ \max\ [C_2(\ell),\ C_1(\ell)+S_i^1-S_i^2+T_i] \end{aligned} \tag{3}$$

where $C_t(\ell)=0$ $(t=1,2)$ when i, j are the first two jobs of S.

Using relations (2) and (3), finally we obtain

$$C_1(ij) \equiv C_1(j) = C_1(\ell)+S_i^1+P_{i,1}+R_i^1+S_j^1+P_{j,1}+R_j^1, \tag{4}$$

and

$$\begin{aligned} C_2(ij) \equiv C_2(j) &= S_j^2+P_{j,2}+R_j^2+S_i^2+P_{i,2}+R_i^2 \\ &\quad + \max\ [C_2(\ell),\ C_1(\ell)+S_i^1-S_i^2+T_i,\ C_1(\ell)+S_i^1+P_{i,1}+R_i^1 \\ &\qquad +S_j^1+T_j-S_j^2-S_i^2-P_{i,2}-R_i^2]. \end{aligned} \tag{5}$$

Similarly we obtain for S'

$$C_1(ji) \equiv C_1(i) = C_1(\ell)+S_j^1+P_{j,1}+R_j^1+S_i^1+P_{i,1}+R_i^1, \tag{6}$$

and

$$\begin{aligned} C_2(ji) \equiv C_2(i) &= S_i^2+P_{i,2}+R_i^2+S_j^2+P_{j,2}+R_j^2 \\ &\quad + \max\ [C_2(\ell),\ C_1(\ell)+S_j^1-S_j^2+T_j,\ C_1(\ell)+S_j^1+P_{j,1}+R_j^2 \\ &\qquad +S_i^1+T_i-S_i^2-S_j^2-P_{j,2}-R_j^2]. \end{aligned} \tag{7}$$

Since $C_1(ij) = C_1(ji)$ holds from (4) and (6), if $C_2(ij) \le C_2(ji)$ holds, then makespan of S, M(S), is not larger than M(S') because generally $C_2(i)$ is a nondecreasing function of $C_2(\ell)$ and $C_1(\ell)$ where job i is a next job of job ℓ, as is clear from (3).

Substituting (5) and (7) into $C_2(ij) \le C_2(ji)$, if an inequality:

$$\begin{aligned} &\max\ [S_i^1-S_i^2+T_i,\ S_i^1+P_{i,1}+R_i^1+S_j^1+T_j-S_j^2-S_i^2-P_{i,2}-R_i^2] \\ &\quad \le \max\ [S_j^1-S_j^2+T_j,\ S_j^1+P_{j,1}+R_j^1+S_i^1+T_i-S_i^2-S_j^2-P_{j,2}-R_j^2] \end{aligned} \tag{8}$$

holds, then $C_2(ij) \le C_2(ji)$ follows.

Subtracting $S_i^1-S_i^2+T_i+S_j^1-S_j^2+T_j$ from both sides of (8), it follows

$$\max\ [-S_j^1+S_j^2-T_j,\ -T_i+P_{i,1}-P_{i,2}+R_i^1-R_i^2]$$
$$\leq \max\ [-S_i^1+S_i^2-T_i,\ -T_j+P_{j,1}-P_{j,2}+R_j^1-R_j^2],$$

that is, our criterion:

$$\min\ (G_i,\ H_j) \leq \min\ (G_j,\ H_i). \qquad \text{Q.E.D.}$$

3. Solvable Cases and An Approximate Sequence ω_1 in Three-Machine Case with Sequence-Independent Setup Times Alone

3.1 An Expression of Makespan

Let every job be available at time zero and be processed on three machines M_1, M_2, M_3 in that order and no passing of jobs be allowed.

For any sequence $\omega=(1,2,\ldots,n-1,n)$ in which number i means i-th job in ω, completion time $C_t(i)$ of the processing of job i on M_t is determined by reccurrence relation:

$$C_t(i) = \max\ [C_{t-1}(i),\ C_t(i-1)+S_i^t] + P_{i,t},\ (i=1,\ldots,n,\ t=1,2,3)$$

where $C_t(i-1)+S_i^t$ denotes completion time of the setup for job i on M_t, and $C_t(0)=0$, $C_0(i)=0$.

Then, makespan of ω, $M(\omega)$, can be expressed by critical path approach.

$$(9)\qquad M(\omega) = \max_{0\leq r\leq s\leq n}\ [\sum_{i=1}^{r}(S_i^1+P_{i,1})+P_{r,2}+\sum_{i=r+1}^{s}(S_i^2+P_{i,2})+P_{s,3}+\sum_{i=s+1}^{n}(S_i^3+P_{i,3})].$$

Notice that this is a machine-base expression of the makespan.

3.2 A Relation Between $M(\omega)$ and $m(\omega)-\sum_{i=1}^{n}(P_{i,2}-S_i^3)$

As preliminary for deriving our results, we need this relation.

From (9) it follows

$$(9)'\qquad M(\omega) = \sum_{i=1}^{r^\circ}(S_i^1+P_{i,1})+P_{r^\circ,2}+\sum_{i=r^\circ+1}^{s^\circ}(S_i^2+P_{i,2})+P_{s^\circ,3}+\sum_{i=s^\circ+1}^{n}(S_i^3+P_{i,3}),$$

where, in a case $r^\circ\geq 1$, r° and s° denote the corner jobs on the first machine M_1 and the second machine M_2 respectively along any critical path for $M(\omega)$.

Therefore, we can express $M(\omega)$ as

$$(10)\qquad M(\omega) = \max_{r^\circ\leq s\leq n}\ [\sum_{i=1}^{r^\circ}(S_i^1+P_{i,1})+P_{r^\circ,2}+\sum_{i=r^\circ+1}^{s}(S_i^2+P_{i,2})+P_{s,3}+\sum_{i=s+1}^{n}(S_i^3+P_{i,3})]$$
$$\equiv A(\omega;\ r^\circ,s),$$

and

$$(11)\qquad M(\omega) = \max_{0\le r\le s^\circ} \Big[\sum_{i=1}^{r} (S_i^1+P_{i,1})+P_{r,2}+ \sum_{i=r+1}^{s^\circ} (S_i^2+P_{i,2})+P_{s^\circ,3}+ \sum_{i=s^\circ+1}^{n} (S_i^3+P_{i,3})\Big]$$

$$\equiv A(\omega;\ r,s^\circ)$$

respectively.

Next, we consider *a related two-machine problem without setup times* in which every job i has its processing times $\sum_{t=1}^{2}(S_i^t+P_{i,t}-S_i^{t+1})=S_i^1+P_{i,1}+P_{i,2}-S_i^3$ and $\sum_{t=2}^{3} P_{i,t}$ on the first and second machines respectively. This problem corresponds to the two-machine problem with setup times in which every job i has its setup times S_i^1 separated from processing times $P_{i,1}+P_{i,2}$ on the first machine and its setup times S_i^3 separated from processing times $P_{i,2}+P_{i,3}$ on the second machine. cf. [20]

Then, for any sequence $\omega=(1,2,\ldots,n-1,n)$, the makespan of ω, $m(\omega)$, in the related two-machine problem is expressed by

$$(12)\qquad m(\omega) = \max_{0\le u\le n} \Big[\sum_{i=1}^{u} (S_i^1+P_{i,1}+P_{i,2}-S_i^3) + \sum_{i=u}^{n} (P_{i,2}+P_{i,3})\Big]$$

$$= \sum_{i=1}^{u^\circ} (S_i^1+P_{i,1}+P_{i,2}-S_i^3) + \sum_{i=u^\circ}^{u} (P_{i,2}+P_{i,3}),$$

where, in a case $u^\circ\ge 1$, u° is a corner job on the first machine along any critical path for $m(\omega)$ and in a case $u^\circ=0$ which can happen when $S_1^1+P_{1,1}+P_{1,2}-S_1^3 \le 0$, critical path for $m(\omega)$ exists on the second machine alone.

From (12), we have

$$(13)\qquad m(\omega)-\sum_{i=1}^{n} (P_{i,2}-S_i^3) = \sum_{i=1}^{u^\circ} (S_i^1+P_{i,1})+P_{u^\circ,2}+P_{u^\circ,3}+ \sum_{i=u^\circ+1}^{n} (S_i^3+P_{i,3})$$

$$\equiv B(\omega;\ u^\circ).$$

Then, from (10) and (13), it follows

$\forall\{r^\circ,\ s^\circ\}$, $\forall u^\circ\ge 0$;

$$M(\omega) - \{m(\omega) - \sum_{i=1}^{n} (P_{i,2}-S_i^3)\}$$

$$(14)\qquad = A(\omega;\ r^\circ,\ s) - B(\omega;\ u^\circ)$$

$$\ge A(\omega;\ u^\circ,\ s) - B(\omega;\ u^\circ)$$

$$= \max_{u^\circ\le s\le n} \Big[\sum_{i=u^\circ+1}^{s} (S_i^2+P_{i,2}) - \sum_{i=u^\circ}^{s-1} (S_{i+1}^3+P_{i,3})\Big]$$

$$(15)\qquad \equiv D(\omega;\ u^\circ,\ s).$$

Similarly we have from (11) and (13)

$$(16)\qquad M(\omega) - \{m(\omega) - \sum_{i=1}^{n} (P_{i,2} - S_i^3)\}$$

$$\geq A(\omega;\, r,\, u^\circ) - B(\omega;\, u^\circ)$$

$$= \max_{0 \leq r \leq u^\circ} \left[\sum_{i=r}^{u^\circ - 1} (S_{i+1}^2 + P_{i,2}) - \sum_{i=r+1}^{u^\circ} (S_i^1 + P_{i,1}) \right]$$

$$\equiv E(\omega;\, r,\, u^\circ).$$

Here, if we put $s=u^\circ$ and $r=u^\circ$ to the expressions in the brackets of (15) and (16) respectively, those expressions become zero. Therefore, $D(\omega;\, u^\circ,\, s) \geq 0$ and $E(\omega;\, r,\, u^\circ) \geq 0$ follow.

From (15) and (16) we obtain

$$(17)\qquad M(\omega) \geq m(\omega) - \sum_{i=1}^{n} (P_{i,2} - S_i^3) + \max\, [D(\omega;\, u^\circ,\, s),\, E(\omega;\, r,\, u^\circ)]$$

$$\geq m(\omega) - \sum_{i=1}^{n} (P_{i,2} - S_i^3).$$

On the other hand, it follows from (14)

$$M(\omega) - \{m(\omega) - \sum_{i=1}^{n} (P_{i,2} - S_i^3)\}$$

$$\leq A(\omega;\, r^\circ,\, s) - B(\omega;\, r^\circ) = D(\omega;\, r^\circ,\, s).$$

Similarly we obtain from (16)

$$M(\omega) - \{m(\omega) - \sum_{i=1}^{n} (P_{i,2} - S_i^3)\}$$

$$\leq A(\omega;\, r,\, s^\circ) - B(\omega;\, s^\circ) = E(\omega;\, r,\, s^\circ).$$

From the above it follows

$$(18)\qquad M(\omega) \leq m(\omega) - \sum_{i=1}^{n} (P_{i,2} - S_i^3) + \min\, [D(\omega;\, r^\circ,\, s),\, E(\omega;\, r,\, s^\circ)].$$

Finally, let us define $M(\omega;\, r=s=u)$ $(0 \leq u \leq n)$ as

$$M(\omega;\, r=s=u) = \sum_{i=1}^{u} (S_i^1 + P_{i,1}) + P_{u,2} + P_{u,3} + \sum_{i=u+1}^{n} (S_i^3 + P_{i,3}),$$

that is, the makespan of ω in original three-machine problem when in case $u \geq 1$ both corner jobs r and s on the respective machines M_1 and M_2 are assumed to be identical and equal to u. If u=0, it corresponds to the critical path on the last machine M_3.

Then, we have from (13)

$$(19)\qquad \forall u(0\le u\le n),\ \forall u^\circ;\ M(\omega;\ r=s=u) \le m(\omega) - \sum_{i=1}^{n}(P_{i,2}-S_i^3)$$

$$= \max_{0\le u\le n} M(\omega;\ r=s=u) = M(\omega;\ r=s=u^\circ) \le M(\omega).$$

So that from (17), (18) and (19) we have the following inequalities (20) in Theorem 1:

Theorem 1. For any sequence $\omega=(1,2,\ldots,n-1,n)$ in our three-machine problem, we have

$\forall\{r^\circ,\ s^\circ\}$ $(0\le r^\circ\le s^\circ\le n)$ for $M(\omega)$, $\forall u^\circ(0\le u^\circ\le n)$ for $m(\omega)$, $\forall u(0\le u\le n)$;

$$(20)\qquad M(\omega;\ r=s=u) \le m(\omega) - \sum_{i=1}^{n}(P_{i,2}-S_i^3) = \max_{0\le u\le n} M(\omega;\ r=s=u)$$

$$= M(\omega;\ r=s=u^\circ) \le m(\omega) - \sum_{i=1}^{n}(P_{i,2}-S_i^3) + \max\Big[\max_{u^\circ\le s\le n}\Big\{\sum_{i=u^\circ+1}^{s}(S_i^2+P_{i,2}) - \sum_{i=u^\circ}^{s-1}(S_{i+1}^3+P_{i,3})\Big\},\ \max_{0\le r\le u^\circ}\Big\{\sum_{i=r}^{u^\circ-1}(S_{i+1}^2+P_{i,2}) - \sum_{i=r+1}^{u^\circ}(S_i^1+P_{i,1})\Big\}\Big]$$

$$\le M(\omega) \le m(\omega) - \sum_{i=1}^{n}(P_{i,2}-S_i^3) + \min\Big[\max_{r^\circ\le s\le n}\Big\{\sum_{i=r^\circ+1}^{s}(S_i^2+P_{i,2}) - \sum_{i=r^\circ}^{s-1}(S_{i+1}^3+P_{i,3})\Big\},\ \max_{0\le r\le s^\circ}\Big\{\sum_{i=r}^{s^\circ-1}(S_{i+1}^2+P_{i,2}) - \sum_{i=r+1}^{s^\circ}(S_i^1+P_{i,1})\Big\}\Big].$$

3.3 Fundamental Result derived from Theorem 1

From Theorem 1 we obtain the following lemma which shows equivalent conditions for any sequence.

In the next section those equivalent conditions are shown to become sufficient optimality conditions for a specified sequence ω_1 which is constructed by using a single criterion (1).

Lemma. For any sequence $\omega=(1,2,\ldots,n-1,n)$ in our three-machine problem, the following conditions (21), 1), 2), 3) and 4) where r° and s° are for $M(\omega)$ and u° is for $m(\omega)$ are equivalent to each other.

$$(21)\qquad m(\omega) - \sum_{i=1}^{n}(P_{i,2}-S_i^3) = M(\omega).$$

1). $\exists s°;\ \max_{0 \le r \le s°} [\sum_{i=r}^{s°-1} (S_{i+1}^2+P_{i,2}) - \sum_{i=r+1}^{s} (S_i^1+P_{i,1})] = 0,$

or $\exists r°;\ \max_{r° \le s \le n} [\sum_{i=r°+1}^{s} (S_i^2+P_{i,2}) - \sum_{i=r°}^{s-1} (S_{i+1}^3+P_{i,3})] = 0.$

2). $\exists\{r°,\ s°\};\ \sum_{i=r°}^{s°-1} (S_{i+1}^2+P_{i,2}) \not> \sum_{i=r°+1}^{s°} (S_i^1+P_{i,1}),$

or $\sum_{i=r°+1}^{s°} (S_i^2+P_{i,2}) \not> \sum_{i=r°}^{s°-1} (S_{i+1}^3+P_{i,3}),$

3). $\exists\{r°,\ s°\},\ \exists u°;\ r°=s°=u°.$

4). $\exists u(0 \le u \le n);\ M(\omega;\ r=s=u) = M(\omega).$

In this case, it follows $u=u°$ for some $u°$.

Proof: 1). Condition 1) yields (21) from (20).

2). Max-terms in 1) take their maxima when $r=r°$ and $s=s°$ respectively. Since max-terms in 1) are not negative in itself and 2) yields their non-positivities, 1) follows from 2).

3). If 3) holds, 2) follows. Next if (21) holds, we have a critical path expression of $M(\omega)$ from (13)

$$M(\omega) = \sum_{i=1}^{u°} (S_i^1+P_{i,1}) + P_{u°,2} + P_{u°,3} + \sum_{i=u°+1}^{n} (S_i^3+P_{i,3}).$$

This shows that there exists some critical path for $M(\omega)$ which has corner job set $\{r°,\ s°\}$ satisfying $r°=s°=u°$, that is, 3) holds.

From the above, we have 3) $\Rightarrow$ 2) $\Rightarrow$ 1) $\Rightarrow$ (21) $\Rightarrow$ 3).

4). 4) is equivalent to (21) from (20).

So that the conclusion of the lemma follows. Q.E.D.

3.4 Sufficient Optimality Conditoins for A Specified Sequence ω_1

Using Lemma, we obtain equivalent fundamental (weakest) sufficient optimality conditions for a specified sequence ω_1 as shown in the next theorem:

Theorem 2. If any one of the following equivalent fundamental conditions (F); 1), i.e. (F'); 2), i.e. (F''); 3), i.e. (F'''), holds for a specified sequence $\omega_1=(1,2,\ldots,n-1,n)$ where $r°$ and $s°$ are for $M(\omega_1)$ and $u°$ is for $m(\omega_1)$, then ω_1 becomes an optimal sequence in the problem:

$$m(\omega_1) - \sum_{i=1}^{n} (P_{i,2}-S_i^3) = M(\omega_1). \tag{F}$$

1). $\exists\{r^\circ, s^\circ\};\ \sum_{i=r^\circ}^{s^\circ-1} (S_{i+1}^2+P_{i,2}) \not> \sum_{i=r^\circ+1}^{s^\circ} (S_i^1+P_{i,1}),$

or $\sum_{i=r^\circ+1}^{s^\circ} (S_i^2+P_{i,2}) \not> \sum_{i=r^\circ}^{s^\circ-1} (S_{i+1}^3+P_{i,3}).$ (F')

2). $\exists\{r^\circ, s^\circ\}, \exists u^\circ;\ r^\circ=s^\circ=u^\circ.$ (F'')

3). $\exists u(0\le u\le n);\ M(\omega_1;\ r=s=u) = M(\omega_1).$ (F''')

In this case $u=u^\circ$ holds for some u°.

Proof: Since the equivalence of (F), (F'), (F'') and (F''') follows from that of conditions (21), 2), 3) and 4) in Lemma, it is sufficient only to prove that (F) yields the optimality of ω_1.

Then, from $m(\omega_1) \le m(\omega_0)$ and $M(\omega_0) \le M(\omega_1)$ for an optimal sequence ω_0 and also from (20) for $\omega=\omega_0$, we have

$$m(\omega_1) - \sum_{i=1}^{n} (P_{i,2}-S_i^3) \le m(\omega_0) - \sum_{i=1}^{n} (P_{i,2}-S_i^3) \le M(\omega_0) \le M(\omega_1).$$

Therefore, (F) yields $M(\omega_0) = M(\omega_1)$, that is, ω_1 is an optimal sequence.

Q.E.D.

Remark. *Bound of* $M(\omega_1)-M(\omega_0)$. Directly from the above proof, a bound of the deviation of an approximate value $M(\omega_1)$ by an approximate sequence ω_1 from an unknown optimal value $M(\omega_0)$ is given by

$$M(\omega_1)-M(\omega_0) \le M(\omega_1)-\{m(\omega_1)-\sum_{i=1}^{n} (P_{i,2}-S_i^3)\}.$$

Next, applying a fundamental condition (F'), we obtain the following concrete sufficient optimality conditions for ω_1, that are stronger than the fundamental conditions proposed in Theorem 2:

Theorem 3. If any one of the following conditions $(\tilde{A}_2)$, $(\tilde{A})$, $(\tilde{A}_3)$ and $(\tilde{A}_4)$ including corner jobs r°, s° for $M(\omega_1)$ holds for a specified sequence $\omega_1=(1,2,\dots,n-1,n)$ in our three-machine problem, ω_1 becomes an optimal sequence in the problem:

1). $\exists s^\circ, \forall i(0\le i\le s^\circ-1);\ S_{i+1}^2+P_{i,2} \le S_{i+1}^1+P_{i+1,1},$ $(\tilde{A}_2)$

or $\exists r^\circ, \forall i(r^\circ \le i \le n-1);\ S^2_{i+1}+P_{i+1,2} \le S^3_{i+1}+P_{i,3}$.

2). $\exists s^\circ;\ \max_{0\le i\le s^\circ-1}(S^2_{i+1}+P_{i,2}) \le \min_{1\le i\le s^\circ}(S^1_i+P_{i,1})$,

or $\exists r^\circ;\ \max_{r^\circ+1\le i\le n}(S^2_i+P_{i,2}) \le \min_{r^\circ\le i\le n-1}(S^3_{i+1}+P_{i,3})$. $(\tilde{A})$

3). $\exists\{r^\circ, s^\circ\};\ S^2_{r^\circ+1}+P_{r^\circ,2} \le S^3_{r^\circ+1}+P_{r^\circ,3}$,

$\forall i(r^\circ+1\le i\le s^\circ-1),\ \max(S^2_i, S^2_{i+1})+P_{i,2} \le \min(S^1_i+P_{i,1}, S^3_{i+1}+P_{i,3})$, $(\tilde{A}_3)$

$S^2_{s^\circ}+P_{s^\circ,2} \le S^1_{s^\circ}+P_{s^\circ,1}$.

4). $\exists\{r^\circ, s^\circ\};\ \forall i(r^\circ\le i\le s^\circ)$;

$\max(S^2_i, S^2_{i+1})+P_{i,2} \le \min(S^1_i+P_{i,1}, S^3_{i+1}+P_{i,3})$. $(\tilde{A}_4)$

Proof: 1). $(\tilde{A}_2)$ yields (F'). 2). $(\tilde{A})$ yields $(\tilde{A}_2)$.
3). Assume that (F') in Theorem 2 does not follow from $(\tilde{A}_3)$, then we have for $\omega_1=(1,2,\ldots,n-1,n)$

$\forall\{r^\circ, s^\circ\}$;

$$(22)\qquad \sum_{i=r^\circ}^{s^\circ-1}(S^2_{i+1}+P_{i,2}) > \sum_{i=r^\circ+1}^{s^\circ}(S^1_i+P_{i,1}),$$

and

$$(23)\qquad \sum_{i=r^\circ+1}^{s^\circ}(S^2_i+P_{i,2}) > \sum_{i=r^\circ}^{s^\circ-1}(S^3_{i+1}+P_{i,3}).$$

In (22) and (23) we have from $(\tilde{A}_3)$

$$\sum_{i=r^\circ+1}^{s^\circ}(S^2_i+P_{i,2}) \le \sum_{i=r^\circ+1}^{s^\circ}(S^1_i+P_{i,1}) \text{ and } \sum_{i=r^\circ}^{s^\circ-1}(S^2_{i+1}+P_{i,2}) \le \sum_{i=r^\circ}^{s^\circ-1}(S^3_{i+1}+P_{i,3})$$

respectively. Therefore, from (22) and (23) they follow

$$P_{r^\circ,2}-P_{s^\circ,2} > 0 \quad\text{and}\quad P_{s^\circ,2}-P_{r^\circ,2} > 0$$

respectively. This is a contradiction. So that $(\tilde{A}_3)$ must yield (F').
4). $(\tilde{A}_4)$ yields $(\tilde{A}_3)$. Q.E.D.

Remark. $(\tilde{A}_2)$ and $(\tilde{A}_3)$ are weaker than $(\tilde{A})$ and $(\tilde{A}_4)$ respectively and are stronger than (F') $\Leftrightarrow$ (F) from the above proof.

Next, we give concrete conditions (A_2), (A), (A_3) and (A_4) for ω_1 which do not contain corner jobs $r°$ and $s°$ for $M(\omega_1)$ and are stronger than the respective conditions $(\tilde{A}_2)$, $(\tilde{A})$, $(\tilde{A}_3)$ and $(\tilde{A}_4)$ presented in Theorem 3. The results are obvious from Theorem 3:

Theorem 4. The followings are sufficient optimality conditions for $\omega_1=(1,2,\ldots,n-1,n)$:

1). $\forall i\,(0\le i\le n-1)$: $S^2_{i+1}+P_{i,2} \le S^1_{i+1}+P_{i+1,1}$,

or $S^2_{i+1}+P_{i+1,2} \le S^3_{i+1}+P_{i,3}$. (A_2)

2). $\max_{1\le i\le n}(S^2_i+P_{i-1,2}) \le \min_{1\le i\le n}(S^1_i+P_{i,1})$,

or $\max_{1\le i\le n}(S^2_i+P_{i,2}) \le \min_{1\le i\le n}(S^3_i+P_{i-1,3})$. (A)

3). $S^2_1 \le S^3_1$.

$\forall i\,(1\le i\le n-1)$; $\max(S^2_i,\ S^2_{i+1})+P_{i,2} \le \min(S^1_i+P_{i,1},\ S^3_{i+1}+P_{i,3})$,

$S^2_n+P_{n,2} \le S^1_n+P_{n,1}$. (A_3)

4). $\forall i\,(0\le i\le n)$; $\max(S^2_i,\ S^2_{i+1})+P_{i,2} \le \min(S^1_i+P_{i,1},\ S^3_{i+1}+P_{i,3})$. (A_4)

Remark. (A_2) and (A_3) are weaker than (A) and (A_4) respectively.

3.5 A Condition (G)

We show here an additional condition (G) which has similar inequalities to a fundamental condition (F'), but is slightly stronger than (F').

A condition (G) is

$$\forall r,\ \forall s\,(0\le r\le s\le n);\quad \sum_{i=r}^{s-1}(S^2_{i+1}+P_{i,2}) \le \sum_{i=r+1}^{s}(S^1_i+P_{i,1}),$$

$$\text{or}\quad \sum_{i=r+1}^{s}(S^2_i+P_{i,2}) \le \sum_{i=r}^{s-1}(S^3_{i+1}+P_{i,3}). \qquad (G)$$

Then next theorem holds:

Theorem 5. (G) is stronger than (F') and is equivalent to a condition (A_2) and is weaker than a condition (A_3).

Proof: Obviously (G) yields (F'). Next, (A_2) yields (G) and inversely (G) for every $r=s-1$ $(s=1,2,\ldots,n)$ yields (A_2). Therefore, (G) is equivalent to (A_2). Finally, by following similar reasoning to the proof of 3) in Theorem 3, we have the result that (A_3) yields (G). Q.E.D.

3.6 Numerical Examples Illustrating Solvable Cases

We present three simple examples for illustrating some solvable cases, in which the first two are for our problem with setup times and the last one is for the usual flow-shop case.

Example 1. *(Condition A_2)*

Consider a problem (n=6, m=3) given in Table 2. Although condition (A) does not hold, observing ω_1=(1,2,3,4,5,6) and (1,2,4,3,5,6), we find that ω_1=(1,2,3,4,5,6) satisfies the second inequalities of weaker condition (A_2). cf. Table 2. So that this ω_1 is an optimal sequence. Here another ω_1 is not optimal $(M(\omega_1)=49)$.

Table 2. Example 1.

i	1	2	3	4	5	6
S_i^1	4	3	5	3	1	4
$P_{i,1}$	5	3	4	3	4	1
S_i^2	3	2	3	4	1	2
$P_{i,2}$	2	5	3	5	4	1
S_i^3	6	4	4	3	1	1
$P_{i,3}$	5	3	6	4	3	1
$S_i^1+P_{i,1}+P_{i,2}-S_i^3$	5	7	8	8	8	5
$P_{i,2}+P_{i,3}$	7	8	9	9	7	2

For an optimal ω_1, obviously the weakest conditions (F): $M(\omega_1)=46=47-(20-19)=m(\omega_1)-\sum_{i=1}^{6}(P_{i,2}-S_i^3)$ and (F''): $\exists\{r°, s°\}$, $\exists u°$; $r°=s°=u°=1,2$ or 3, are satisfied. cf. Table 3 where C_t (t=1,2,3) denotes completion times of setup in parentheses and processing of every job and C_{12} and C_{23} mean completion times of every job on the first and second machines in related two-machine problem respectively, and also each sequence of arrows shows related critical path for $M(\omega_1)$ and $m(\omega_1)$.

Table 3. Example 1.

ω_1	$r°,s°$ 1	$r°,s°$ 2	$r°,s°$ 3	$s°$ 4	$s°$ 5	6
C_1	(4) 9(12) →	15(20) →	24(27)	30(31)	35(39)	40
	↓	↓	↓			
C_2	(3)11(13)	20(23)	27(31) →	36(37) →	41(43)	44
	↓	↓	↓	↓	↓	
C_3	(6)16(20) →	23(27) →	33(36) →	40(41) →	44(45) →	46 = $M(\omega_1)$
C_{12}	5 →	12 →	20	28	36	41
	↓	↓	↓			
C_{23}	12 →	20 →	29 →	38 →	45 →	47 = $m(\omega_1)$
	$u°$	$u°$	$u°$			

Example 2. (*Case where $r°=s°=u°=0$ holds*)

If we put $S_1^2=14$ in Example 1 (Table 2), then under $S_1^3\geq 16$ second inequalities of (A_2) is also satisfied by $\omega_1=(1,2,3,4,5,6)$ and so this ω_1 remains optimal.

In this case we have $S_1^2 > S_1^1+P_{1,1}$ and $S_1^3 \geq \max(S_1^1+P_{1,1}, S_1^2)+P_{1,2}$ for $M(\omega_1)$ and $S_1^1+P_{1,1}+P_{1,2}-S_1^3 \leq -5$ for $m(\omega_1)$, which means that critical paths for $M(\omega_1)$ and $m(\omega_1)$ exist on M_2 and M_3 $(S_1^3=16)$ where $r°=0$, or on $M_3(S_1^3\geq 16)$ where $r°=s°=0$, and on the second machine where $u°=0$ respectively.

Specifically in a case $S_1^2=14$ and $S_1^3=16$ in Table 2, we have (F):

$$M(\omega_1)=51=42-(20-29)=m(\omega_1)-\sum_{i=1}^{6}(P_{i,2}-S_i^3) \text{ and (F''): } \exists\{r°, s°\};\ r°=s°=u°=0.$$

Example 3. (*Usual flow-shop case* [14])

For a problem (n=6, m=3) given in Table 4, Johnson condition (A) does not hold, but for $\omega_1=(1,2,3,4,5,6)$ first inequalities of weaker condition (A_2): $\forall i(1\leq i\leq n-1)$; $P_{i,2}\leq P_{i+1,1}$, is satisfied. cf. Table 4. So ω_1 is optimal. Obviously, they hold (F): $M(\omega_1)=18=31-13=m(\omega_1)-\sum_{i=1}^{6}P_{i,2}$, and (F''): $\exists\{r°, s°\}$; $r°=s°=u°=3$ cf. Table 5.

Table 4. Example 3.

i	1	2	3	4	5	6
$P_{i,1}$	1	2	3	4	3	2
$P_{i,2}$	2	3	4	2	1	1
$P_{i,3}$	2	3	3	2	2	1
$P_{i,1}+P_{i,2}$	3	5	7	6	4	3
$P_{i,2}+P_{i,3}$	4	6	7	4	3	2

Table 5. Example 3.

ω_1	$r°$ 1	$r°$ 2	$r°, s°$ 3	4	5	6	
C_1	1	3	6	10	13	15	
C_2	3	6	10	12	14	16	
C_3	5	9	13	15	17	18	$= M(\omega_1)$
C_{12}	3	8	15	21	25	28	
C_{23}	7	14	22	26	29	31	$= m(\omega_1)$
			$u°$				

3.7 Inclusion Relation among Proposed Conditions

Figure 1 depicts an inclusion relation among the proposed conditions for a specified sequence ω_1 in our NP-complete (hard) three-machine flow-shop problem with independent setup times.

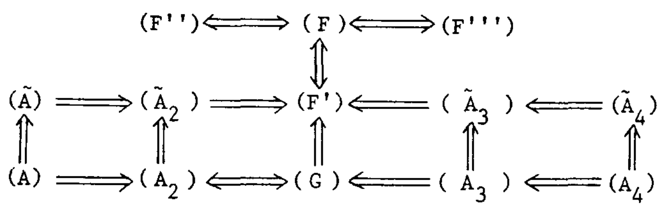

Fig. 1. An Inclusion Relation (m=3).

The weakest condition which may be most useful is (F'') and that which soes not contain corner jobs r° and s° is (G) or equivalently (A_2).

It setup times are assumed to be included in the processing times, Figure 1 reduces to an inclusion relation among conditions for a Johnson approximation sequence ω_1 in the usual flow-shop case (m=3) where (A), (A_4), (F) and (F'') are the conditions separately found by S.M.Johnson [7], Burns and Rooker [3], Szwarc [19] and Raghavachari [16] respectively, whereas our results show their mutual relation including other conditions.

4. Conclusion

First we investigated the optimality of the passing of jobs and proposed optimal permutation algorithm in general two-machine flow-shop makespan scheduling problem with five independent times, that is, setup and removal times, start and stop lags and transportation times, separated from processing times.

Proposed algorithm includes known algorithm as special cases and constructs an optimal sequence in polynomial-time O(n log n).

Next we considered NP-complete three-machine flow-shop makespan scheduling problem with separated sequence-independent setup times alone in order to identify solvable cases and an approximate sequence.

Proposed sufficient optimality conditions for a specified sequence ω_1 identify the optimality of ω_1 in polynomial time O(n log n).

Therefore, several types of polynomially solvable cases and their inclusion relation and also the superiority of that sequence ω_1 as an approximate sequence were clarified in our NP-complete problem.

Note that our conditions always yield an optimal sequence ω_1 having a critical path with corner jobs r° and s° such that (F''): $r^\circ=s^\circ=u^\circ$ $(0\le u^\circ\le n)$ holds, that is, generally $(2\le u^\circ\le n-1)$ having a critical path with two horizontal and one vertical segments as exemplified in Table 3 ($r^\circ=s^\circ=2$, or 3) and Table 5 ($r^\circ=s^\circ=3$).

As already noted, our results are reduced to the results in the usual NP-complete three-machine permutation flow-shop makespan problem and extends known solvable cases when we put every setup times zero. cf. [13] [14] [17]

As further extensions of the results for the present three-machine case, we remark that 1. our results can be further generalized for general NP-complete m-machine (m≥3) permutation case, having, for m≥4, additional conditions derived by applying the transformation method [2] first proposed in [15], and 2. including further the transportation times, time lags and/or suitably restricted removal times in our problem, we can obtain more general complicated results.

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