

INVARIANCE RELATIONS OF GI/G/1 QUEUEING SYSTEMS WITH PREEMPTIVE-RESUME LAST-COME-FIRST-SERVED QUEUE DISCIPLINE

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Abstract This paper is concerned with the GI/G/1 queueing system under the assumption of last-come-first-served queue discipline where each customer enters the service immediately upon his arrival. If a customer service is interrupted because of the arrival of another customer, his remaining service requirement remains unchanged until the server can attend to him again. Let (p_n) and (r_n) , $n = 0, 1, 2, \dots$, be the equilibrium distributions of the number of customers in the system, when the system is considered 'at any time' and when it is considered at epochs immediately before successive arrivals respectively. It is shown that a relation between the sequences (p_n) and (r_n) is the same as that of the GI/M/1 queueing system with first-come-first-served queue discipline. Also it is shown that the remaining service requirement of the customer being served at any time is independent of the state of the system and it has the distribution function of the 'residual' service requirement. Furthermore a simple expression of the distribution function of the nearest time after any time is given.

GI/G/1 QUEUE; LAST-COME-FIRST-SERVED; INVARIANCE RELATION

1. Introduction

Customers arrive at a station at epochs $\dots, a_{-2}, a_{-1}, a_0, a_1, a_2, \dots$, where the interarrival times $T_n = a_n - a_{n-1}$, $n=0, \pm 1, \pm 2, \dots$, are independent and identically distributed (i.i.d.) random variables (r.v.'s) having a distribution function (d.f.) $A(x)$ with a finite mean λ^{-1} . There is a single server and also waiting facility for infinitely many customers. The service requirements $\dots, S_{-1}, S_0, S_1, \dots$ of successive arriving customers are i.i.d. r.v.'s having a d.f. $B(x)$ with a finite mean μ^{-1} . The r.v.'s $T_{\pm 1}, T_{\pm 2}, \dots$ and $S_0, S_{\pm 1}, S_{\pm 2}, \dots$ are stochastically independent. Let $\rho = \lambda/\mu$; we assume that ρ is strictly less than one.

We partition the waiting facility into positions labeled by the integers $1, 2, \dots$, and we call the server 'position 1.' The system, then, operates as follows: Whenever there are m customers present (i) each arriving customer occupies position 1, while customers previously in positions $1, 2, \dots, m$ move to positions $2, 3, \dots, m+1$ respectively, (ii) when a customer departs from the system, customers in positions $2, 3, \dots, m$ move to positions $1, 2, \dots, m-1$ respectively. If a customer service is interrupted because of the arrival of another customer, his remaining service requirement remains unchanged until the server can attend to him again.

This system has been studied by Fakinos [1] and Yamazaki [7]. Fakinos, by considering the system at epochs immediately before successive arrivals, proved that the equilibrium distribution of the number of customers in the system is geometric while the remaining service requirements of customers present are i.i.d. r.v.'s. Yamazaki, by considering the system at epochs immediately after successive departures as well as at epochs immediately before successive arrivals, proved the same result as Fakinos' in both cases. Moreover, Yamazaki proved that the distribution of the time between a departure and the next arrival epochs is independent of the state of the system.

It is the purpose of this paper to extend the results obtained by Fakinos and Yamazaki for the case of "any time", that is, to derive three invariance relations valid at any time for the GI/G/1 queueing system described above. Let (p_m) and (r_m) , $m=0, 1, 2, \dots$, be the equilibrium distributions of the number of customers in the system, when the system is considered at any time and when it is considered at epochs immediately before successive arrivals respectively. The first relation, then, is a simple relation between the sequences (p_m) and (r_m) , $m=0, 1, 2, \dots$, which is identical with the well-known relation for the GI/M/1 queueing system with first-come-first-served queue discipline. The second is concerned with the remaining service requirement of the customer being served. It is shown that this remaining service requirement is a r.v. independent of the state of the system, with the d.f. of the 'residual' service requirement. The last is a simple expression of the d.f. of the nearest arrival time after any time.

2. Preliminaries

We consider here the queueing system described in the preceding section, where every customer behaves always alone, that is, we exclude a case which has batch arrivals or customers having 0 service requirement. Let $Q(t)$ be the number of customers in the system at any time t , let $X_i(t)$ be the remaining

service requirement of the customer occupying position i at t , $i=1,2,\dots,Q(t)$, and let $Y(t)$ be the nearest arrival time after t . Then, the process

$$Z(t) = (Q(t); X_1(t), X_2(t), \dots, X_{Q(t)}(t); Y(t))$$

is a Markov process representing the state of the system. $Z(t)$ ($-\infty < t < \infty$) is defined on a probability space $(\Omega, \mathfrak{E}, P)$. Throughout this paper it is assumed that $Z(t)$ is time continuous strictly stationary and ergodic for P .

Now, let us consider two Palm measures, P_a and P_d , defined with respect to the sequences $\{a_n\}_{n=-\infty}^{\infty}$ and $\{d_n\}_{n=-\infty}^{\infty}$ respectively, where $\{d_n\}$ is the sequence of departure times. $P_a(P_d)$ is a probability measure of the space (Ω, \mathfrak{E}) , which may be thought as a conditional distribution of P under the condition $\{a_1=0\}(\{d_1=0\})$. By the stationarity of $Z(t)$, the intensity of departures from the system is equal to λ , which is the intensity of arrivals to the system. Then, P_a and P_d are uniquely determined from the stationary of P and they are strictly stationary for the sequences $\{a_n\}$ and $\{d_n\}$ respectively, that is, for any $E \in \mathfrak{E}$

$$P_a(E) = P_a(E_{a_n}), \quad P_d(E) = P_d(E_{d_n}) \quad \text{for all integer } n,$$

where E_{a_n} (E_{d_n}) is the event E with respect to an epoch a_n (d_n) (c.f. for example Franken et al. [2]).

Consider a random walk defined by $U_{n+1} = U_n + V_n$, $n=0,1,2,\dots$, where $V_n = S_n - T_{n+1}$, $n=0,1,2,\dots$. The points at which the random walk first jumps above its latest maximum value are called 'ascending ladder indices'. Since $\rho < 1$, it is clear that $U_n \rightarrow -\infty$ as $n \rightarrow \infty$. Therefore, there will exist a (finite) integer K such that K is the largest ascending ladder index for U_n . Then, the distribution for K is geometric (see Kleinrock [4]):

$$(1) \quad \Pr(k=n) = (1-\sigma)\sigma^n \quad (n=0,1,2,\dots),$$

where $1-\sigma \equiv \Pr(U_n \leq U_0; n=1,2,\dots)$. Given that $K=k$ and the successive ladder indices are the points n_1, n_2, \dots, n_k , we put

$$J_i = U_{n_i} - U_{n_{i-1}} \quad (i=1,2,\dots,k; n_0=0).$$

It is well-known that J_1, J_2, \dots, J_k are i.i.d. r.v.'s having d.f. $F(x) \equiv \Pr(J_i \leq x)$. On the basis of the random walk, we define $I(x)$ for $x \geq 0$ as:

$$1 - I(x) = \Pr(U_n \leq -x; n=1,2,\dots \mid U_0=0, K=0).$$

Let us introduce the following notation for the probability distributions:

$$p_m = \Pr(Q(t)=m), \quad r_m = P_a(Q(a_n-0)=m), \quad q_m = P_d(Q(d_n+0)=m).$$

Then, we have the following lemma:

Lemma 1. (Yamazaki [7]).

$$(2) \quad P_a(Q(a_n-0)=m; X_1(a_n-0)\leq x_1, X_2(a_n-0)\leq x_2, \dots, X_m(a_n-0)\leq x_m; Y(a_n-0)\leq y) \\ = r_m^A(y) \prod_{j=1}^m F(x_j),$$

$$(3) \quad P_d(Q(d_n+0)=m; X_1(d_n+0)\leq x_1, X_2(d_n+0)\leq x_2, \dots, X_m(d_n+0)\leq x_m; Y(d_n+0)\leq y) \\ = q_m^I(y) \prod_{j=1}^m F(x_j),$$

where

$$(4) \quad r_m = q_m = (1 - \sigma)\sigma^m \quad \text{for } m=0,1,2,\dots$$

Based on (2)-(4) the behaviour at any time of the system is discussed in the next section. We will also need the following lemma to do this, which is one of the basic results concerning 'point processes' (for proof, see [2]):

Lemma 2. For any t and $h \rightarrow 0 + 0$,

$$(5) \quad \Pr\{\text{the number of arrivals to the system in } (t, t+h] = 1\} \\ = \Pr\{\text{the number of departures from the system in } (t, t+h] = 1\} \\ = \lambda h + o(h),$$

$$(6) \quad \Pr\{\text{the number of arrivals and/or departures in } (t, t+h] > 1\} \\ = o(h).$$

3. Invariance Relations

Following the same approach as Kopocińska and Kopociński [5], we derive three invariance relations at any time for the GI/G/1 queueing system with queue discipline described in Section 1. One is a simple relation between the sequences (p_m) and (r_m) , $m=0,1,2,\dots$, and the others are simple expressions of distributions of two marginal processes for the process $Z(t)$, that is, of processes $(Q(t); X_1(t))$ and $(Q(t); Y(t))$. In this section it is assumed that the d.f.'s $A(x)$ and $B(x)$ are absolutely continuous.

Observing the state of the process $Z(t)$ at the epochs t and $t+h$ ($h>0$), we find from the stationarity of the process and Lemma 2 that

$$(7) \quad \Pr(Q(t+h)=0; Y(t+h)\leq y) \\ = \Pr(Q(t)=0; Y(t)\leq y+h) - \Pr(Q(t)=0; Y(t)\leq h) + \\ \Pr(Q(t)=1; X_1(t)\leq h; h<Y(t)\leq y+h) + o(h),$$

$$\begin{aligned}
 (8) \quad & \Pr(Q(t+h)=m; X_1(t+h)\leq x; Y(t+h)\leq y) \\
 &= \Pr(Q(t)=m; X_1(t)\leq x+h; Y(t)\leq y+h) \\
 &\quad - \Pr(Q(t)=m; X_1(t)\leq h; h<Y(t)\leq y+h) \\
 &\quad - \Pr(Q(t)=m; h<X_1(t)\leq x+h; Y(t)\leq h) \\
 &\quad + \Pr(Q(t)=m+1; X_1(t)\leq h, X_2(t)\leq x; h<Y(t)\leq y+h) \\
 &\quad + \Pr(Q(t)=m-1; X_1(t)>h; Y(t)\leq h; S\leq x, T\leq y) + o(h) \\
 &\quad \text{for } m=1,2,\dots,
 \end{aligned}$$

where S and T are a new service requirement and an interarrival time, respectively, distributed independently of all other r.v.'s.

In an immediately consequence of (5), we have

$$(9) \quad \lim_{h \rightarrow 0} \frac{1}{h} \Pr(Y(t)\leq h) = \lim_{h \rightarrow 0} \frac{1}{h} \Pr(Q(t)>0; X_1(t)\leq h) = \lambda.$$

Since the two Palm measures, P_a and P_d can be interpreted as conditional distributions of P (cf. Section 2), for example $\Pr(Q(t)=m; h<X_1(t)\leq x+h \mid Y(t)\leq h)$ is the probability that an arriving customer in $(t, t+h]$ finds the system in the state $(Q=m; X_1\leq x)$, we find that

$$(10) \quad \lim_{h \rightarrow 0} \Pr(Q(t)=m; h<X_1(t)\leq x+h \mid Y(t)\leq h) = P_a(Q(a_n-0)=m; X_1(a_n-0)\leq x).$$

Similarly we can obtain the following:

$$\begin{aligned}
 (11) \quad & \lim_{h \rightarrow 0} \Pr(Q(t)=m; X_2(t)\leq x; h<Y(t)\leq y+h \mid Q(t)>0; X_1(t)\leq h) \\
 &= P_d(Q(d_n+0)=m-1; X_1(d_n+0)\leq x; Y(d_n+0)\leq y).
 \end{aligned}$$

From (9), (10) and Lemma 1 we have the following:

$$\begin{aligned}
 (12) \quad & \lim_{h \rightarrow 0} \frac{1}{h} \Pr(Q(t)=m; X_1(t)\leq h; h<Y(t)\leq y+h) \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \Pr(Q(t)>0; X_1(t)\leq h) \cdot \Pr(Q(t)=m; h<Y(t)\leq y+h \mid Q(t)>0; X_1(t)\leq h) \\
 &= \lambda \cdot P_d(Q(d_n+0)=m-1; Y(d_n+0)\leq y) \\
 &= \lambda q_{m-1} I(y) \qquad \qquad \qquad (\text{cf. (3)}).
 \end{aligned}$$

$$\begin{aligned}
 (13) \quad & \lim_{h \rightarrow 0} \frac{1}{h} \Pr(Q(t)=m; X_1(t)\leq h, X_2(t)\leq x; h<Y(t)\leq y+h) \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \Pr(Q(t)>0; X_1(t)\leq h) \cdot \Pr(Q(t)=m; X_2(t)\leq x; h<Y(t)\leq y+h \mid \\
 &\quad Q(t)>0; X_1(t)\leq h)
 \end{aligned}$$

$$\begin{aligned}
 &= \lambda \cdot P_d (Q(d_n + 0) = m - 1; X_1(d_n + 0) \leq x; Y(d_n + 0) \leq y) \\
 &= \lambda q_{m-1} I(y) F(x) \qquad \qquad \qquad \text{(cf. (3))},
 \end{aligned}$$

$$\begin{aligned}
 (14) \quad &\lim_{h \rightarrow 0} \frac{1}{h} \Pr (Q(t) = m; h < X_1(t) \leq x+h; \leq Y(t) \leq h) \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \Pr (Y(t) \leq h) \cdot \Pr (Q(t) = m; h < X_1(t) \leq x+h \mid Y(t) \leq h) \\
 &= \lambda \cdot P_a (Q(a_n - 0) = m; X_1(a_n - 0) \leq x) \\
 &= \lambda r_m F(x) \qquad \qquad \qquad \text{(cf. (2))},
 \end{aligned}$$

$$\begin{aligned}
 (15) \quad &\lim_{h \rightarrow 0} \frac{1}{h} \Pr (Q(t) = m; X_1(t) > h; Y(t) \leq h; S \leq x, T \leq y) \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \Pr (Y(t) \leq h) \cdot \Pr (Q(t) = m; X_1(t) > h; S \leq x, T \leq y \mid Y(t) \leq h) \\
 &= \lambda \cdot \lim_{h \rightarrow 0} \Pr (Q(t) = m; X_1(t) > h \mid Y(t) \leq h) \cdot \Pr (S \leq x) \cdot \Pr (T \leq y) \\
 &= \lambda \cdot P_a (Q(a_n - 0) = m) \cdot B(x) \cdot A(y) \\
 &= \lambda r_m B(x) A(x) \qquad \qquad \qquad \text{(cf. (2))},
 \end{aligned}$$

Since $B(x)$ and $A(y)$ are absolutely continuous, the joint distribution $Q(t)$, $X_1(t)$ and $Y(t)$ possesses densities with respect to the second and third variables (see, for example Tumura [6]). Forming difference quotients in (7) and (8), taking their limits as $h \rightarrow 0$, and using (12)–(15) we get the following:

$$(16) \quad 0 = \frac{\partial}{\partial y} \Pr (Q(t) = 0; Y(t) \leq y) - \lambda (r_0 - q_0 I(y)),$$

$$\begin{aligned}
 (17) \quad 0 &= \frac{\partial}{\partial y} \Pr (Q(t) = m; X_1(t) \leq x; Y(t) \leq y) + \\
 &\quad \frac{\partial}{\partial x} \Pr (Q(t) = m; X_1(t) \leq x; Y(t) \leq y) - \\
 &\quad \lambda (q_{m-1} I(y) + r_m F(x) - q_m F(x) I(y) - r_{m-1} B(x) A(y)) \quad m \geq 1.
 \end{aligned}$$

Substituting $q_m = r_m$ and $r_m = \sigma \cdot r_{m-1}$ (cf. (4)) into (17), we can obtain

$$\begin{aligned}
 (18) \quad &\frac{\partial}{\partial y} \Pr (Q(t) = m; X_1(t) \leq x; Y(t) \leq y) + \frac{\partial}{\partial x} \Pr (Q(t) = m; X_1(t) \leq x; Y(t) \leq y) \\
 &= \lambda r_{m-1} (I(y) + \sigma F(x) \bar{I}(y) - B(x) A(y)) \qquad \qquad \qquad m \geq 1,
 \end{aligned}$$

where $\bar{I}(y) = 1 - I(y)$.

On the basis of (16) and (18) we have the following theorem:

Theorem.

$$(19) \quad p_0 = 1 - \rho, \quad p_m = \rho r_{m-1} \quad (m \geq 1),$$

$$(20) \quad \Pr(Q(t)=m; X_1(t) \leq x) = p_m B^*(x),$$

$$(21) \quad \Pr(Q(t)=0; Y(t) \leq y) = p_0 I^*(y),$$

$$\Pr(Q(t)=m; Y(t) \leq y) = p_m \rho^{-1} (A^*(y) - p_0 I^*(y)) \quad (m \geq 1),$$

where for any d.f. $G(x)$ with $G(0) = 0$ and $\int_0^\infty x dG(x) < \infty$,

$$G^*(x) \equiv \frac{\int_0^x (1 - G(x)) dx}{\int_0^\infty x dG(x)}.$$

Proof: Taking the limit of (18) as $y \rightarrow \infty$, we can get

$$(22) \quad \frac{\partial}{\partial x} \Pr(Q(t)=m; X_1(t) \leq x) = \lambda r_{m-1} (1 - B(x)) \quad (m \geq 1).$$

Integrating (22) over 0 to ∞ gives (19). (20) can be obtained by integrating (22) over 0 to x and using (19). Integrating (16) over 0 to ∞ yields

$$(23) \quad \int_0^\infty \bar{I}(y) dy = \frac{1 - \rho}{\lambda \cdot r_0}.$$

The former expression of (21) can be easily obtained by integrating (16) over 0 to y and using (23). Taking the limit of (18) as $x \rightarrow \infty$ and integrating it over 0 to y , the latter expression of (21) can be derived.

Remarks

(i) Much work has been performed with queue discipline in relation to product-form solutions of the equilibrium equations in queueing network settings, under assumption that the arrival process to the system is a Poisson Process. One of the common properties resulting from the queue disciplines is that the remaining service requirement of the customer being served at any time at a node in the queueing network is independent of the state of the node and it has the d.f. of the 'residual' service requirement (see, for example, Kelly [3]).

From this viewpoint (20) is a generalization of the case of an arbitrary arrival process.

(ii) The relation between the sequences (p_m) and (r_m) given in (19) is identical with the well-known relation for the GI/M/1 queueing system with first-come-first-served (FCFS) queue discipline. From this viewpoint, (19) is a generalization of the case of a general service distribution.

(iii) We can generalize to a certain extent the queue discipline (ii) of Section 1 as follows (see, [1]): When departing customer leaves m customers in the system, the customer to be next is the customer occupying position i at the epoch, $i=2,3,\dots,m+1$, with probability $p_{i,m}$, where $p_{2,m} + p_{3,m} + \dots + p_{m+1,m} = 1$, and customers previously in positions $i+1, i+2, \dots, m+1$ move to positions $i, i+1, \dots, m$ respectively.

Following the same approach as for the proof of the theorem of [7] it can be easily checked that Lemma 1 remains valid when the system operates under the above and (i) of Section 1. Hence the theorem of Section 3 remains valid under this discipline.

(iv) Moments of $Q(t)$ (involving unknown σ) can be obtained from (4) and (19); for example, the expectation of $Q(t)$ as

$$(24) \quad E[Q(t)] = \frac{\rho}{1-\sigma} .$$

Since it is clear that

Pr (an arrival finds the system idle in the GI/G/1 queue with the queue discipline described in Section 1)
 = Pr (an arrival finds the system idle in the GI/G/1 queue with FCFS queue discipline), we can interpret σ in (24) as

$$\sigma = \text{Pr (an arrival finds the system busy in the GI/G/1 queue with FCFS queue discipline)} .$$

Therefore, if we can find σ by some methods, for example, by a numerical calculation of the GI/G/1 queue with FCFS, we can easily compare the moments of $Q(t)$ with those of $Q^*(t)$, where $Q^*(t)$ is the number of customers in the system at any time t in the steady state under FCFS queue discipline. In other words, whenever σ can be found, it is possible to judge whether introducing the pre-emptive-resume queue discipline is efficient or not.

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