Journal of the Operations Research
Society of Japan
Vol. 27, No. 3, September 1984

STABLE SETS FOR SIMPLE GAMES WITH ORDINAL PREFERENCES*

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(Received December 19, 1984)

Abstract In this paper, stable sets for simple games with ordinal preferences are studied in the case where the number of alternatives is finite. It is shown that the condition for the existence of a nonempty core for any possible combination of players' preferences is also a necessary and sufficient condition for the existence of a unique stable set. Moreover, a necessary and sufficient condition that proper simple games have at least one stable set is presented.

1. Introduction

In Nakamura [3], a necessary and sufficient condition was provided for simple games to have a nonempty core for any possible combination of players' preferences, in case the number of alternatives is finite. Recently, Ferejohn and McKelvey [2] studied a necessary condition for a social choice function satisfying stable set property to exist. The purpose of this paper is to show the condition given by Nakamura in [3] is also a necessary and sufficient condition that simple games have a unique stable set for any combination of preferences in case of finite number of alternatives, and moreover to present a necessary and sufficient condition that proper simple games with finite number of alternatives have at least one stable set.

After reviewing basic definitions relating to simple games to be used in this paper in Section 2, we will provide some preliminary theorems and lemmas in Section 3. In Section 4, we will show the condition given by Nakamura in [3] is also a necessary and sufficient condition that a unique stable set exist. In Section 5, we will describe a necessary and sufficient condition for the existence of stable sets for proper simple games. Several discussions will be

* This research was supported in part by the Japanese Education Ministry through Kagakukenkyuhi-Hojokin.

provided in Section 6 as concluding remarks.

2. Basic Definitions

A simple game is an ordered pair $G = (N,\mathcal{W})$ where $N = \{1,\ldots,n\}$ is a set of players, and $\mathcal{W}$ is a set of winning coalitions satisfying (i) $\emptyset \notin \mathcal{W}$, (ii) $N \in \mathcal{W}$, and (iii) $S \in \mathcal{W}$ and $S \subset T \rightarrow T \in \mathcal{W}$ (monotonicity). We say G is proper if $S \in \mathcal{W} \rightarrow N - S \notin \mathcal{W}$, and G is weak if $V = \cap \{S|S\in\mathcal{W}\} \neq \emptyset$. Here the members of V are called veto players.

Let Ω be a nonempty set of alternatives. Throughout this paper, Ω is assumed to be a finite set, i.e., $|\Omega| < \infty$ where $|\Omega|$ denotes a cardinality of Ω. Let D denote a set of all weak order preference relations on Ω, i.e., complete, reflexive, and transitive binary preference relations on Ω. Let $R^N = \{R^i\}_{i\in N}$ where $R^i \in D$ for all $i \in N$, and D^N be a set of all such R^N. For any $x, y \in \Omega$, we define P^i and I^i, for any $i \in N$, by $x\ P^i\ y \leftrightarrow (x\ R_i\ y$ and $\sim y\ R_i\ x)$, and $x\ I^i\ y \leftrightarrow (x\ R^i\ y$ and $y\ R^i\ x)$, where $\sim$ denotes a negation. We easily notice that P^i and I^i are both transitive, and P^i is irreflexive and asymmetric. Here we remark that we could develop our theory based on acyclic strict preference relations on Ω as was done in Nakamura [3].

Take any $x, y \in \Omega$ $(x \neq y)$ and any $R^N \in D^N$, we say x dominates y w.r.t. R^N, written by $x \operatorname{dom}(R^N)\ y$, if there is a set $S \in \mathcal{W}$ such that $x\ P^i\ y$ for all $i \in S$. This set S is called an effective set for this domination. It is easily seen that, for any $R^N \in D^N$, $\operatorname{dom}(R^N)$ is irreflexive, and moreover, in case G is proper, $\operatorname{dom}(R^N)$ is asymmetric. Take any $R^N \in D^N$. If a subset $\{x_1,\ldots,x_k\}$ of Ω satisfies

$$x_2 \operatorname{dom}(R^N)\ x_1,\ x_3 \operatorname{dom}(R^N)\ x_2,\ \ldots\ ,\ x_k \operatorname{dom}(R^N)\ x_{k-1}, \text{ and}$$
$$x_1 \operatorname{dom}(R^N)\ x_k,$$

then we say that $\{x_1,\ldots,x_k\}$ forms a cycle w.r.t. $\operatorname{dom}(R^N)$. We say $\operatorname{dom}(R^N)$ is acyclic if there is no subset of Ω forming a cycle w.r.t. $\operatorname{dom}(R^N)$.

In case G is not weak, let $\Sigma = \{\mathcal{U} \subseteq \mathcal{W} \mid \cap \{S|S\in\mathcal{U}\} = \emptyset\}$, and define a number $\nu(G)$ by $\nu(G) = \min\{\ |\mathcal{U}|\ \mid \mathcal{U}\in\Sigma\ \}$. Here we note that if G is proper, then we have $\nu(G) \geqq 3$, from the monotonicity. Moreover, from Lemma 2.1 and Corollary 2.2 of Nakamura [3], we have (i) $\nu(G) \leqq n$, and (ii) $\nu(G) = n$ if and only if $\mathcal{W} = \{N, N-\{1\},\ldots,N-\{n\}\}$.

The core for G w.r.t. R^N, denoted by $C(G,R^N)$, is a subset of Ω such that

$$C(G,R^N) = \{x \in \Omega \mid \sim y \operatorname{dom}(R^N)\ x \text{ for any } y \in \Omega\}.$$

The stable set for G w.r.t. R^N, denoted by $K(G,R^N)$, is a subset of Ω satisfying

(i) for any $x, y \in K(G,R^N)$, $\sim x \operatorname{dom}(R^N)\ y$ and $\sim y \operatorname{dom}(R^N)\ x$, and (ii) for any $y \notin K(G,R^N)$, there exists an $x \in K(G,R^N)$ such that $x \text{ dom}(R^N)\ y$.

(i), (ii) are called internal, and external stability, respectively.

3. Preliminary Theorems and Lemmas

We first state three theorems due to von Neumann and Morgenstern [6], Richardson [5], and Nakamura [3].

Theorem 3.1 (von Neumann and Morgenstern [6]). Let $G = (N,\mathcal{W})$ be a simple game and Ω be a finite set. Take any $R^N \in D^N$. Then we have the following:

(i) If $\text{dom}(R^N)$ is acyclic, then the core $C(G,R^N) \neq \emptyset$.

(ii) If $\text{dom}(R^N)$ is acyclic, then there exists a unique stable set $K(G,R^N)$.

Theorem 3.2 (Richardson [5]). Let $G = (N,\mathcal{W})$ be a simple game and Ω be a finite set. Take any $R^N \in D^N$. Then if there is no set $\{x_1,\ldots,x_k\} \subseteqq \Omega$ with k being odd which forms a cycle w.r.t. $\text{dom}(R^N)$, then there exist a stable set $K(G,R^N)$.

Theorem 3.3 (Nakamura [3]). Let $G = (N,\mathcal{W})$ be a simle game and Ω be a finite set. Then $C(G,R^N) \neq \emptyset$ for any $R^N \in D^N$ if and only if (G is weak) or ($\nu(G) > |\Omega|$), and if $|\Omega| \geqq n$ and $C(G,R^N) \neq \emptyset$ for any $R^N \in D^N$, then G is weak.

Now we will prepare some lemmas which will be used in the following sections. All the proofs of the following lemmas will be given in the appendix.

Lemma 3.1. If there is an $R^N \in D^N$ and a set $\Omega' = \{x_1,\ldots,x_k\} \subseteqq \Omega$ forming a cycle w.r.t. $\text{dom}(R^N)$, then G is not weak and $\nu(G) \leqq k$.

From this lemma, we easily notice that in case G is not weak, for any $R^N \in D^N$, there is no set $\{x_1,\ldots,x_\ell\} \subseteqq \Omega$ with $\ell < \nu(G)$ which forms a cycle w.r.t. $\text{dom}(R^N)$.

Lemma 3.2. Assume $\nu(G) = k \leqq |\Omega|$. Then there is an $R^N \in D^N$ and a set $\Omega' = \{x_1,\ldots,x_k\} \subseteqq \Omega$ such that

(i) $x_p \text{ dom}(R^N)\ x_q$ if and only if $p = q + 1 \pmod k$ for any $x_p, x_q \in \Omega'$,

(ii) $x_p \text{ dom}(R^N)\ x$ for any $x_p \in \Omega'$ and any $x \in \Omega - \Omega'$, and

(iii) $\sim x \text{ dom}(R^N)\ y$ for any $x \in \Omega - \Omega'$ and any $y \in \Omega$.

The next lemma is based on Lemma 5 of Ferejohn and McKelvey [2].

Lemma 3.3. Assume $4 \leqq \nu(G) = k \leqq |\Omega| - 1$. Then there is an $R^N \in D^N$ and a set $\Omega' = \{x_1,\ldots,x_k,x_{k+1}\} \subseteqq \Omega$ such that

(i) $x_p \text{ dom}(R^N) x_q$ if and only if $p = q + 1 \pmod{k + 1}$ for any $x_p, x_q \in \Omega'$,

(ii) $x_p \text{ dom}(R^N)\ x$ for any $x_p \in \Omega'$ and any $x \in \Omega - \Omega'$, and

(iii) $\sim x \operatorname{dom}(R^N)\, y$ for any $x \in \Omega - \Omega'$ and any $y \in \Omega$.

4. Uniqueness of Stable Sets

In this section, we will show that the condition given in Theorem 3.3 is also a necessary and sufficient condition that a unique stable set exist for any $R^N \in D^N$.

Theorem 4.1. Let $G = (N,\mathcal{W})$ be a simple game and Ω be a finite set. Then there exists a unique stable set $K(G,R^N)$ for any $R^N \in D^N$ if and only if (G is weak) or ($\nu(G) > |\Omega|$), and if $|\Omega| \geqq n$ and there exists a unique stable set $K(G,R^N)$ for any $R^N \in D^N$, then G is weak.

Proof: (Necessity) Suppose G is not weak and $\nu(G) = k \leqq |\Omega|$. Then from Lemma 3.2, there exists an R^N and a set $\Omega' = \{x_1,\ldots,x_k\} \subseteqq \Omega$ satisfying the properties (i), (ii), and (iii) in Lemma 3.2.

From (ii) and (iii), if there exists a stable set $K(G,R^N)$ for this R^N, then $K(G,R^N)$ must be included in Ω'. Therefore from (i), we easily see that if k is even, there are two stable sets, $K(G,R^N)$, i.e., $\{x_1,x_3,\ldots,x_{k-1}\}$ and $\{x_2,x_4,\ldots,x_k\}$, and if k is odd, there is no stable set $K(G, R^N)$. For details, see von Neumann and Morgenstern [6]. Therefore there is no unique stable set for this R^N.

(Sufficiency) Suppose there does not exist a unique stable set $K(G,R^N)$ for some $R^N \in D^N$. Then from Theorem 3.1 (ii), $\operatorname{dom}(R^N)$ must be not acyclic, i.e., there exists some set $\{x_1,\ldots,x_k\} \subseteqq \Omega$ forming a cycle w.r.t. $\operatorname{dom}(R^N)$. Therefore from Lemma 3.1, we obtain that G is not weak and $\nu(G) \leqq k$. Since $\{x_1,\ldots,x_k\} \subseteqq \Omega$, $\nu(G) \leqq k \leqq |\Omega|$.

The second part of the theorem is clear from the fact that $\nu(G) \leqq n$.

Q.E.D.

Combining the theorem above and Theorem 3.3, we have the following theorem.

Theorem 4.2. Let $G = (N,\mathcal{W})$ be a simple game and Ω be a finite set. Then there exists a unique stable set $K(G,R^N)$ for any $R^N \in D^N$ if and only if $C(G, R^N) \neq \emptyset$ for any $R^N \in D^N$.

Proof: This easily follows from Theorems 3.3 and 4.1. Q.E.D.

5. Existence of Stable Sets for Proper Simple Games

In this section, we will describe a necessary and sufficient condition that proper simple games have at least one stable set for any $R^N \in D^N$ in case Ω is a finite set

Theorem 5.1. Let $G = (N,W)$ be a proper simple game and Ω be a finite set. Then there exists at least one stable set $K(G,R^N)$ for any $R^N \in D^N$ if and only if (G is weak) or ($\nu(G) > |\Omega|$) or ($\nu(G) = |\Omega|$ and $|\Omega|$ is even). Moreover, in case there exists a stable set $K(G,R^N)$ for any $R^N \in D^N$, if $|\Omega| > n$ or $|\Omega| = n$ with odd n, then G is weak, and if $|\Omega| = n$ with even n, then G is weak or $W = \{N,N-\{1\},\ldots,N-\{n\}\}$.

Proof: We first note that $\nu(G) \geqq 3$ since the game is proper.

(Necessity) Suppose G is not weak and $\nu(G) = k \leqq |\Omega|$. Since $\nu(G) = k \leqq |\Omega|$, from Lemma 3.2, there exists an R^N and a set $\Omega' = \{x_1,\ldots,x_k\} \subseteqq \Omega$ satisfying the properties (i), (ii), and (iii) in Lemma 3.2. Thus if $\nu(G) = |\Omega|$ and $|\Omega|$ is odd, or $\nu(G) \leqq |\Omega| - 1$ and k is odd, then we easily see, from the argument in the proof of Theorem 4.1, that there is no stable set $K(G,R^N)$ for this R^N. When $\nu(G) \leqq |\Omega| - 1$ and k is even, since $k \leqq |\Omega| - 1$ and $k \geqq 4$, we obtain, from Lemma 3.3, an R'^N and a set $\Omega'' = \{x_1,\ldots,x_k,x_{k+1}\} \subseteqq \Omega$ satisfying the properties (i), (ii), and (iii) in Lemma 3.3. Since k + 1 is odd, again using the same argument as in the proof of Theorem 4.1, we obtain that there is no stable set $K(G,R^N)$ for this R'^N.

(Sufficiency) Suppose there is an R^N such that there is no stable set $K(G,R^N)$ for this R^N. Then from Theorem 3.2, we must have a set $\{x_1,\ldots,x_k\} \subseteqq \Omega$ with k being odd which forms a cycle w.r.t. $dom(R^N)$. Hence from Lemma 3.1, we have that G is not weak and $\nu(G) \leqq k \leqq |\Omega|$. Now suppose $\nu(G) = |\Omega|$. Then again from Lemma 3.1, for any $R^N \in D^N$, $dom(R^N)$ is acyclic, or any set forming a cycle w.r.t. $dom(R^N)$ must coincide with Ω. Therefore we must have that $|\Omega|$ is odd.

The second part of the theorem easily follows from the properties of $\nu(G)$ mentioned just below its definition. Q.E.D.

6. Concluding Remarks

In this paper, we have investigated stable sets for simple games with finite set of alternatives. We have shown that the condition given by Nakamura in [3] for the existence of a nonempty core is also a necessary and sufficient condition that there exist a unique stable set for any combination of players' preferences, and moreover described a necessary and sufficient condition that there exist at least one stable set in case the game is proper. From this condition, we notice that, in case the number of alternative is finite but greater than the number of players, simple games must be weak, i.e., have at least one veto player, in order to have stable sets for any combination of players' preferences, similarly as in the case of the core.

If the game is not proper, we might have the case of $\nu(G) = 2$ in Theorem 5.1, and thus Lemma 3.3 cannot be applicable for such an occasion. Therefore we need some more conditions in addition to those given in Theorem 5.1, in order to obtain a necessary and sufficient condition for the existence of stable sets for simple games including nonproper ones. Though for simple games, it seems reasonable to assume games to be proper, in order to generalize our results to general characteristic function form games with ordinal preferences such as studied in Ishikawa and Nakamura [1] and Polishchuk [4], we must investigate such cases.

This problem, together with the properties of stable sets for simple games having infinite number of alternatives, will be studied in future papers.

Appendix

Proof of Lemma 3.1: Assume $\{x_1,\ldots,x_k\} \subseteqq \Omega$ forms a cycle w.r.t. $\mathrm{dom}(R^N)$. Then we have $x_{p+1}\ \mathrm{dom}(R^N) x_p$ for all $p = 1,\ldots,k$ (mod k). Let $S_2,\ldots,S_k$, and S_1 be effective sets for these dominations. Then we must have $\cap\{S_p | p=1,\ldots, k\} = \emptyset$. In fact, if $\cap\{S_p | p=1,\ldots,k\} \neq \emptyset$, then we can take some $i \in \cap\{S_p | p= 1,\ldots,k\}$. For this i, we have $x_{p+1}\ P^i\ x_p$ for all $p = 1,\ldots,k$ (mod k), which contradicts the fact that P^i is transitive. Therefore $\cap\{S_p | p=1,\ldots,k\} = \emptyset$. Since $S_p \in \mathcal{W}$ for all $p = 1,\ldots,k$ and $\cap\{S_p | p=1,\ldots,k\} = \emptyset$, we have G is not weak and $\{S_p | p=1,\ldots,k\} \in \Sigma$. Hence from the definition of $\nu(G)$, we obtain $\nu(G) \leqq k$. Q.E.D.

Proof of Lemma 3.2: Let $\mathcal{U} \in \Sigma$ be such that $\nu(G) = |\mathcal{U}|$. Since $\nu(G) = k \leqq |\Omega|$, we have $k = |\mathcal{U}| \leqq |\Omega|$. Hence there is a one to one mapping ψ from $\mathcal{U}$ into Ω. Let $\Omega' = \psi(\mathcal{U})$. Without loss of generality, let $\mathcal{U} = \{S_1,\ldots,S_k\}$, $\Omega' = \{x_1, \ldots,x_k\}$, and $x_p = \psi(S_p)$ for all $p = 1,\ldots,k$. Note that $S_p \in \mathcal{W}$ (for all $p = 1,\ldots,k$) and $\cap\{S_p | p=1,\ldots,k\} = \emptyset$.

Using the idea given in Nakamura [3], define $R^i \in D$ for each $i \in N$ in the following manner. Take any $i \in N$. Since $\cap\{S_p | p=1,\ldots,k\} = \emptyset$, there is some p such that $i \notin S_p$. Let S_{p*} be one of such S_p, and define R^i by,

$$x_{p+1}\ P^i\ x_p \text{ for all } p \neq p* - 1 \text{ (mod k), i.e.,}$$

$$x_{p*-1}\ P^i\ x_{p*-2}\ P^i \ldots P^i\ x_1\ P^i\ x_k\ P^i \ldots P^i\ x_{p*+1}\ P^i\ x_{p*},$$

$$x_{p*}\ P^i\ x \text{ for any } x \in \Omega - \Omega',$$

$$x\ I^i\ y \text{ for any } x,\ y \in \Omega - \Omega',$$

and for any other $x, y \in \Omega$, define R^i so that R^i may be a weak order.

Let $R^N = \{R^i\}_{i \in N}$. From the definition of R^i above, we have for any $i \in N$,

$$x_p\ P^i\ x \text{ for any } x_p \in \Omega' \text{ and any } x \in \Omega - \Omega',$$

and $$x\ I^i\ y \text{ for any } x,\ y \in \Omega - \Omega'.$$

Hence we easily obtain (ii) and (iii).

Now we will show (i). First we will prove the sufficiency. Take any $x_p \in \Omega'$, and any $i \in S_p$ where $x_p = \psi(S_p)$. Since $i \in S_p$, we must have $i \notin S_q$ for some $q \neq p$. Therefore from the construction of R^i above, we have $x_p P^i x_{p-1}$ (mod k). For details, see Nakamura [3]. This shows that x_p dom(R^N) x_{p-1} since $S_p \in \mathcal{W}$. Therefore the sufficiency holds.

To show the necessity, we assume that there are some x_p and x_q such that x_p dom(R^N) x_q and $p \neq q + 1$ (mod k). Then it is easily seen that we have some set $\Omega'' \subsetneqq \Omega'$ forming a cycle w.r.t. dom(R^N), which contradicts Lemma 3.1 since $\nu(G) = k$. Thus we have shown (i). Q.E.D.

Proof of Lemma 3.3: Let $\Omega' = \{x_1,\dots,x_k\} \subsetneqq \Omega$, $\mathcal{U} = \{S_1,\dots,S_k\}$, and $R^N = \{R^i\}_{i \in N}$ be as defined in Lemma 3.2, where $4 \leqq k \leqq |\Omega| - 1$.

Take any $x \in \Omega - \Omega'$ and let it be x_{k+1}. Let $\Omega'' = \{x_1,\dots,x_k,x_{k+1}\}$. Take any $i \in N$, and, following the idea given in Ferejohn and McKelvey [2], define $R'^i \in D$ in the following manner. Let $T_p = \{i \in N \mid x_q P^i x_p \text{ for all } x_q \in \Omega' - \{x_p\}\}$ for all $p = 1,\dots,k$. From the definition of R^i in the proof of Lemma 3.2, if $i \in T_p$, then we must have

$$x_{p-1}\ P^i\ x_{p-2}\ P^i\ \dots\ P^i\ x_1\ P^i\ x_k\ P^i\ \dots\ P^i\ x_p,$$

and moreover, any $i \in N$ must be in exactly one of these T_p.

If $i \in T_p$ where $p \neq 1,\ k-1$, then we define P'^i by

$$x_{p-1}\ P'^i\ x_{p-2}\ P'^i\ \dots\ P'^i\ x_1\ P'^i\ x_{k+1}\ P'^i\ x_k\ P'^i\ \dots\ P'^i\ x_p.$$

If $i \in T_1$, then let

$$x_{k+1}\ P'^i\ x_k\ P'^i\ x_{k-1}\ P'^i\ \dots\ P'^i\ x_1.$$

If $i \in T_{k-1}$, then let

$$x_{k-2}\ P'^i\ x_{k-3}\ P'^i\ \dots\ P'^i\ x_2\ P'^i\ x_k\ P'^i\ x_1\ P'^i\ x_{k-1}\ P'^i\ x_{k+1}.$$

For any $i \in N$, define

$$x_p\ P'^i\ x \text{ for any } x_p \in \Omega'' \text{ and any } x \in \Omega - \Omega'',$$

and $$x\ I'^i\ y \text{ for any } x,\ y \in \Omega - \Omega''.$$

For any $i \in N$ and any other $x, y \in \Omega$, define R'^{i} so that R'^{i} may be a weak order. Here note that for any $i \notin T_{k-1}$,

$$x\ P^{i}\ y \leftrightarrow x\ P'^{i}\ y \text{ for any } x, y \in \Omega - \{x_{k+1}\}.$$

Let $R'^{N} = \{R'^{i}\}_{i\in N}$. Then (ii) and (iii) easily follow. We will first show the sufficiency of (i). If $q = 1,\ldots, k-1$, then this easily holds since $x_{q+1}\ P^{i}\ x_q \leftrightarrow x_{q+1}\ P'^{i}\ x_q$ for all $i \in N$ and $x_{q+1}\ \mathrm{dom}(R^{N})\ x_q$. Assume $q = k$. Noting that $k \geqq 4$, we have $x_{k-1}\ P^{i}\ x_{k-2} \leftrightarrow x_{k+1}\ P'^{i}\ x_k$ for all $i \in N$. Therefore $x_{k+1}\ \mathrm{dom}(R'^{N})\ x_k$ holds since $x_{k-1}\ \mathrm{dom}(R^{N})\ x_{k-2}$. Finally, when $q = k + 1$, $x_1\ \mathrm{dom}(R'^{N})\ x_{k+1}$ easily follows since $x_1\ P^{i}\ x_k \leftrightarrow x_1\ P'^{i}\ x_{k+1}$ for all $i \in N$ and $x_1\ \mathrm{dom}(R^{N})\ x_k$. Thus the sufficiency is proved.

Now we will show the necessity of (i). Suppose $x_p\ \mathrm{dom}(R'^{N})\ x_q$ and $p \neq q + 1 \pmod{k + 1}$. If $p \neq q + 2 \pmod{k + 1}$, then it is easily seen that we have some set $\Omega''' \subsetneqq \Omega''$ with $|\Omega'''| \leqq k - 1$ forming a cycle w.r.t. $\mathrm{dom}(R'^{N})$, which contradicts Lemma 3.1 since $\nu(G) = k$. Thus we assume $p = q + 2 \pmod{k + 1}$. If $q = 1,\ldots,k-2$, then it is clear, from $k \geqq 4$, that $\sim x_{q+2}\ \mathrm{dom}(R'^{N})\ x_q$ since $x_{q+2}\ P^{i}\ x_q \leftrightarrow x_{q+2}\ P'^{i}\ x_q$ for all $i \in N$ and $\sim x_{q+2}\ \mathrm{dom}(R^{N})\ x_q$. Assume $q = k - 1$. Noting that $k \geqq 4$, we have $x_k\ P^{i}\ x_{k-2} \leftrightarrow x_{k+1}\ P'^{i}\ x_{k-1}$ for all $i \in N$. Hence $\sim x_{k+1}\ \mathrm{dom}(R'^{N})\ x_{k-1}$ since $\sim x_k\ \mathrm{dom}(R^{N})\ x_{k-2}$. In case $q = k + 1$, we have $x_2\ P^{i}\ x_k \leftrightarrow x_2\ P'^{i}\ x_{k+1}$ for all $i \in N$ since $k \geqq 4$. Thus we have $\sim x_2\ \mathrm{dom}(R'^{N})\ x_{k+1}$ since $\sim x_2\ \mathrm{dom}(R^{N})\ x_k$. Finally consider the case of $q = k$. Suppose $x_1\ \mathrm{dom}(R'^{N})\ x_k$, and let S be an effective set for this domination. From the construction of R'^{i} above, we must have $i \notin T_{k-1}$ for all $i \in S$. For each $i \in N$, define R''^{i} in the following manner. For any $i \in T_p$ $(p \neq k - 1)$, let $R''^{i} = R^{i}$. For any $i \in T_{k-1}$, let

$$x_k\ P''^{i}\ x_{k-1}\ P''^{i}\ \ldots\ P''^{i}\ x_1,$$

$$x_1\ P''^{i}\ x \text{ for any } x \in \Omega - \Omega',$$

$$x\ I''^{i}\ y \text{ for any } x, y \in \Omega - \Omega',$$

and for any other $x, y \in \Omega$, define R''^{i} so that R''^{i} may be a weak order. Let $R''^{N} = \{R''^{i}\}_{i\in N}$. Then we have

$$x_{q+1}\ P'^{i}\ x_q \rightarrow x_{q+1}\ P''^{i}\ x_q \text{ for all } q = 1,\ldots,k \pmod{k},$$

and thus

$$x_{q+1}\ \mathrm{dom}(R''^{N})\ x_q \text{ for all } q = 1,\ldots,k \pmod{k}.$$

Here we notice that from the construction of R''^N, we have x_k P'^i $x_{k-1} \leftrightarrow$ x_k P''^i x_{k-2} for all $i \in N$. Hence we must have x_k dom(R''^N) x_{k-2} since x_k dom(R'^N) x_{k-1}. Therefore we have a set $\{x_1,\ldots,x_{k-2},x_1\}$ which forms a cycle w.r.t. dom(R''^N), which contradicts Lemma 3.1 since $\nu(G) = k$. Hence $\sim x_1$ dom(R'^N) x_k, and thus the necessity of (i) is proved. Q.E.D.

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