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THE LEXICO-BOUNDED FLOW ALGORITHM FOR SOLVING THE MINIMUM COST PROJECT SCHEDULING PROBLEM WITH AN ADDITIONAL LINEAR CONSTRAINT

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Abstract This paper proposes an algorithm for solving the minimum cost project scheduling problem with an additional linear constraint whose right hand side is a parameter. Instead of putting the additional constraint, we add it to the objective function, where it is multiplied by a parameter. First, we get an optimal solution when a parameter is zero, and next, increase it to infinity. To do so, we iterate to find two dimensional flow on the given arrow diagram. The bounds for each arrow flow are determined by the current solution. The first elements are related to the costs, and the second to the coefficients in the additional constraint. A lexico-bounded flow is defined as the flow such that each arrow flow is within the bounds in the lexicographical ordering. When a lexico-bounded flow exists, the parameter is increased. Otherwise, the function of the additional constraint is increased. Our algorithm is almost dual to the lexico-shortest route algorithm for the minimum cost flow problem with an additional linear constraint. For example, a loop is replaced by a cutset. Our algorithm is also applicable to a project scheduling problem with two objectives by combining them with a parameter.

1. Introduction

The project scheduling problems with additional linear constraints occur when there exist divisible activities or when some activities use a common resource, and some algorithms are presented for getting the length of critical path [6, 8, 9, 11]. This paper presents an algorithm for solving the minimum cost project scheduling problem with an additional linear constraint, whose right hand side is a parameter. The author has presented an algorithm for the minimum cost flow problem with an additional linear constraint, called the lexico-shortest route algorithm [10]. By transforming it in the dual way, like replacing a loop by a cutset, we get our algorithm.

Let

$\eta(t)\geqq\theta$

be the additional constraint, where θ is a parameter. Instead of putting it, we add $\phi\cdot\eta(t)$ to the objective function, where ϕ is a parameter. First, we get an optimal solution for $\phi=0$, and next, increase ϕ to infinity. To do so, we iterate to find two dimensional flow on the given arrow diagram. The bounds for each arrow flow are determined by the current solution. The first elements are related to the costs, and the second to the coefficients in $\eta(t)$. A lexico-bounded flow is defined as the flow such that each arrow flow is within the bounds in the lexicographical ordering. When a lexico-bounded flow exists, ϕ is increased. Otherwise, $\eta(t)$ is increased. We prove the validity of our algorithm by the complementary slackness conditions.

2. Problem Formulation

Let us consider an arrow diagram for a project, with n nodes, numbered 1, 2,..., n, where node 1 represents the start and node n the termination. Let A be the set of jobs (pairs of nodes), and for each job (i,j), the standard time h_{ij}, and the shortest time g_{ij} are given, where $h_{ij}\geqq g_{ij}$. Furthermore, for (i,j) such that $h_{ij}>g_{ij}$, the cost for shortening the time by a unit c_{ij} is given. Conveniently, let $c_{ij}=0$ for (i,j) such that $h_{ij}=g_{ij}$. Then, the minimum cost project scheduling problem with an additional linear constraint is formulated as follows:

$P_0(\theta)$: Minimize

$$z_0(t) = \sum_{(i,j)} c_{ij}(h_{ij}-t_{ij})$$

subject to

$$(2.1)\qquad g_{ij} \leqq t_{ij} \leqq h_{ij} \qquad ((i,j)\varepsilon A),$$

$$(2.2)\qquad v_i + t_{ij} \leqq v_j \qquad ((i,j)\varepsilon A),$$

$$(2.3)\qquad v_n - v_1 = p_0,$$

$$(2.4)\qquad \sum_{(i,j)} b_{ij}\, t_{ij} \geqq \theta,$$

where p_0 (total duration) is a given positive number, and θ is a parameter. We assume that $b_{ij}\geqq 0$ for any $(i,j)\varepsilon A$, and that $b_{ij}=0$ for (i,j) such that $h_{ij}=g_{ij}$.

Now, let

$$\eta(t)=\sum_{(i,j)} b_{ij}t_{ij},$$

and we shall consider the following parametric programming problem with a parameter ϕ instead of θ.

P(ϕ): Minimize

$$z(t,\phi) = z_0(t)-\phi\eta(t)$$

subject to (2.1), (2.2) and (2.3).

Let

$$c^*_{ij}(\phi) = c_{ij}+\phi b_{ij} \qquad \text{for } (i,j)\varepsilon \text{A}.$$

Then,

$$z(t,\phi) = \sum_{(i,j)} c^*_{ij}(\phi)\ (h_{ij}-t_{ij})+c_0,$$

where c_0 is a constant.

Hence, P(ϕ) is a normal minimum cost project scheduling problem when ϕ is fixed. The relations between the solutions of two problems are stated by the following theorems.

Theorem 1. Let $(\bar{v},\bar{t})$ be an optimal solution of P(0). Then it is also optimal to $P_0(\theta)$ for any θ such that $\theta\leqq\eta(\bar{t})$.

Theorem 2. Let $(\bar{v},\bar{t})$ be an optimal solution of P(ϕ) for some positive ϕ. Then it is also optimal to $P_0(\bar{\eta})$, when $\bar{\eta}=\eta(\bar{t})$.

It is proved in the same way as theorem 2 of [10].

3. Dual problem and Complementary Slackness Conditions

We consider the dual problem to P(ϕ).

D(ϕ): Maximize

$$w = -p_0\cdot q-\Sigma h_{ij}x^+_{ij} + \Sigma g_{ij}x^-_{ij}+c'_0$$

subject to

$$\text{(3.1)} \qquad \sum_j x_{ij}-\sum_j x_{ji} = \begin{cases} q & (i=1), \\ 0 & (i=2,3,\dots,n-1), \\ -q & (i=n), \end{cases}$$

$$x_{ij}+x^+_{ij}-x^-_{ij} = c^*_{ij}(\phi) \qquad ((i,j)\varepsilon\text{A}),$$

$$x_{ij},\ x^+_{ij},\ x^-_{ij} \geqq 0 \qquad ((i,j)\varepsilon\text{A}),$$

where c'_0 is a constant.

Note that q is a variable. Now, we add an arrow $(n,1)$ conveniently, and let $\text{A*}=\text{A}\cup\{(n,1)\}$. By replacing q with x_{n1}, (3.1) is rewritten as follows.

(3.2) $\sum_j x_{ij} - \sum_j x_{ji} = 0 \qquad (i=1,2,\ldots,n).$

It means that (x_{ij}) is a circulation flow. In any optimal solution,

$$
(3.3)\qquad \begin{aligned} x^+_{ij} &= \max(c^*_{ij}(\phi)-x_{ij},0), \\ x^-_{ij} &= \max(x_{ij}-c^*_{ij}(\phi),0), \end{aligned}
$$

are satisfied for any $(i,j)\varepsilon A$.

The complementary slackness conditions for $P(\phi)$ and $D(\phi)$ are as follows: For any $(i,j)\varepsilon A$,

$$
(3.4)\qquad \begin{cases} x_{ij}(v_j-v_i-t_{ij})=0, \\ x^+_{ij}(t_{ij}-h_{ij})=0, \\ x^-_{ij}(t_{ij}-g_{ij})=0. \end{cases}
$$

In any optimal solution,

$$
(3.5)\qquad t_{ij} = \min(v_j-v_i,h_{ij})
$$

is satisfied.

Let

$$
(3.6)\qquad \begin{cases} A_{p1} = \{(i,j)\,|\,(i,j)\varepsilon A,\ v_j-v_i>h_{ij}\}, \\ A_{p2} = \{(i,j)\,|\,(i,j)\varepsilon A,\ v_j-v_i=h_{ij}>g_{ij}\}, \\ A_{p3} = \{(i,j)\,|\,(i,j)\varepsilon A,\ h_{ij}>v_j-v_i>g_{ij}\}, \\ A_{p4} = \{(i,j)\,|\,(i,j)\varepsilon A,\ v_j-v_i=g_{ij}\}. \end{cases}
$$

Then, (3.4) is replaced by the following (3.7). (See [5].)

$$
(3.7)\qquad \begin{cases} x_{ij}=0 & \text{for } (i,j)\varepsilon A_{p1}, \\ 0\leqq x_{ij}\leqq c^*_{ij}(\phi) & \text{for } (i,j)\varepsilon A_{p2}, \\ x_{ij}=c^*_{ij}(\phi) & \text{for } (i,j)\varepsilon A_{p3}, \\ x_{ij}\geqq c^*_{ij}(\phi) & \text{for } (i,j)\varepsilon A_{p4}. \end{cases}
$$

Next, we let

$$
(3.8)\qquad \begin{cases} A_{d1} = \{(i,j)\,|\,(i,j)\varepsilon A,\ x_{ij}=0\}, \\ A_{d2} = \{(i,j)\,|\,(i,j)\varepsilon A,\ 0<x_{ij}<c^*_{ij}(\phi)\}, \\ A_{d3} = \{(i,j)\,|\,(i,j)\varepsilon A,\ x_{ij}=c^*_{ij}(\phi)\}, \\ A_{d4} = \{(i,j)\,|\,(i,j)\varepsilon A,\ x_{ij}>c^*_{ij}(\phi)\}, \end{cases}
$$

Then, (3.4) can be replaced by the following (3.9), too.

$$
(3.9) \quad \begin{cases} v_j - v_i \geqq h_{ij} & \text{for } (i,j) \varepsilon A_{d1}, \\ v_j - v_i = h_{ij} & \text{for } (i,j) \varepsilon A_{d2}, \\ h_{ij} \geqq v_j - v_i \geqq g_{ij} & \text{for } (i,j) \varepsilon A_{d3}, \\ v_j - v_i = g_{ij} & \text{for } (i,j) \varepsilon A_{d4}. \end{cases}
$$

4. Lexico-bounded Flow

Suppose that we have an optimal solution of $P(\bar{\phi})$, $(\bar{v},\bar{t})$. Let us show that $(\bar{v},\bar{t})$ is optimal to $P(\phi)$ for some $\phi>\bar{\phi}$, or that an optimal solution of $P(\bar{\phi})$ such that $\eta(t)>\eta(\bar{t})$ exists.

We assign two dimentional real ξ_{ij} for each arrow $(i,j)\varepsilon A^*$. Its upper bound α_{ij} and lower bound β_{ij} are determined by Table 1.

Table 1. α_{ij} and β_{ij}

(i,j)	α_{ij}	β_{ij}
in $A_{p1}(\bar{v}_j-\bar{v}_i>h_{ij})$	$(0,0)$	$(0,0)$
in $A_{p}2(\bar{v}_j-\bar{v}_i=h_{ij}>g_{ij})$	$(c^*_{ij}(\bar{\phi}),b_{ij})$	$(0,0)$
in $A_{p3}(h_{ij}>\bar{v}_j-\bar{v}_i>g_{ij})$	$(c^*_{ij}(\bar{\phi}),b_{ij})$	$(c^*_{ij}(\bar{\phi}),b_{ij})$
in $A_{p4}(\bar{v}_j-\bar{v}_i=g_{ij})$	(∞,∞)	$(c^*_{ij}(\bar{\phi}),b_{ij})$
$(n,1)$	(∞,∞)	$(0,0)$

Definition 1. (ξ_{ij}) is called a *lexico-bounded flow (LB flow)* if it satisfies the following conditions:

(a) $\sum_j \xi_{ij} - \sum_j \xi_{ji} = 0 \qquad (i=1,2,\ldots,n).$

(b) For each $(i,j)\varepsilon A$, ξ_{ij} is not greater than α_{ij} and is not less than β_{ij} in the lexicographical ordering.

Hereafter, we use the lexicographical ordering. Furthermore, let $\xi^{(k)}_{ij}$, $\alpha^{(k)}_{ij}$ and $\beta^{(k)}_{ij}$ (k=1,2) represent the k-th elements of ξ_{ij}, α_{ij} and β_{ij} respectively.

First, we shall consider the condition for existence of an LB flow.

Let

$$N=\{1, 2, \ldots, n\}.$$

For any subset of N, say N_0, let

$$C^+(N_0,\bar{N}_0) = \{(i,j)\,|\,(i,j)\varepsilon A^*,\ i\varepsilon N_0,\ j\varepsilon\bar{N}_0\},$$

$$C^-(N_0,\bar{N}_0) = \{(i,j)\,|\,(i,j)\varepsilon A^*,\ i\varepsilon\bar{N}_0,\ j\varepsilon N_0\},$$

and

$$C(N_0,\bar{N}_0) = (C^+(N_0,\bar{N}_0),\ C^-(N_0,\bar{N}_0)),$$

where $\bar{N}_0=N-N_0$. $C(N_0,\bar{N}_0)$, is called a *cutset separating N_0 and $\bar{N}_0$* (with the direction from N_0 to $\bar{N}_0$).

For a cutset $C(N_0,\bar{N}_0)$, $\gamma(N_0,\bar{N}_0)$ is defined by

$$\gamma(N_0,\bar{N}_0) = \sum_{(i,j)\varepsilon C^+}\alpha_{ij} - \sum_{(i,j)\varepsilon C^-}\beta_{ij},$$

where C^+ and C^- are abbreviations of $C^+(N_0,\bar{N}_0)$ and $C^-(N_0,\bar{N}_0)$ respectively. When a given cutset is obvious, we represent it by γ briefly, and let $\gamma=(\gamma_1,\gamma_2)$.

Theorem 3. For any cutset $C(N_0,\bar{N}_0)$, $\gamma_1\geqq 0$.

Proof: $\gamma_1 = \sum_{(i,j)\varepsilon C^+}\alpha^{(1)}_{ij} - \sum_{(i,j)\varepsilon C^-}\beta^{(1)}_{ij}$,

and there exists (x_{ij}) that satisfies (3.2) and

$$\alpha^{(1)}_{ij}\geqq x_{ij}\geqq\beta^{(1)}_{ij} \qquad ((i,j)\varepsilon A),$$

which is equivalent to (3.7). From the circulation flow existence theorem [2], $\gamma_1\geqq 0$.

Q.E.D.

Definition 2. A cutset $C(N_0,\bar{N}_0)$ is called a *lexico-negative cutset (LN cutset)* if $\gamma<(0,0)$.

Theorem 4. An LB flow exist if and only if there exists no LN cutset.

Next, we suppose that there exists an LB flow.

Definition 3. The LB flow which maximizes ξ_{n1} is called the *L-max flow*.

Definition 4. The cutset which minimizes γ among cutsets such that $1\varepsilon N_0$ and that $n\varepsilon\bar{N}_0$ is called the *L-min cutset*.

Theorem 5. ξ_{n1} of the L-max flow is equal to γ of the L-min cutset. This is the max-flow min-cutset theorem in the lexicographical ordering.

We shall show that we can increase ϕ when there exists an LB flow, (ξ_{ij}). Let

(4.1) $$\lambda_{ij} = (c^*_{ij}(\bar{\phi}),\ b_{ij}) - \xi_{ij}, \qquad ((i,j)\varepsilon A_{p2}\cup A_{p4}),$$

$$A_1 = \{(i,j)\,|\,(i,j)\varepsilon A_{p2},\ \xi^{(2)}_{ij}<0\},$$

$$A_2 = \{(i,j)\,|\,(i,j)\varepsilon A_{p2}\cup A_{p4},\ \lambda_{ij}^{(1)}\cdot\lambda_{ij}^{(2)}<0\},$$

and determine $\Delta\phi_1$, $\Delta\phi_2$, $\Delta\phi$ as follows:

$$(4.2)\quad \begin{cases} \Delta\phi_1 = \min\limits_{(i,j)\varepsilon A_1} (-\xi_{ij}^{(1)}/\xi_{ij}^{(2)}), \\ \Delta\phi_2 = \min\limits_{(i,j)\varepsilon A_2} (-\lambda_{ij}^{(1)}/\lambda_{ij}^{(2)}), \\ \Delta\phi\ = \min\ (\Delta\phi_1,\Delta\phi_2). \end{cases}$$

If A_k(k=1 or 2) is empty, let $\Delta\phi_k=\infty$.

Theorem 6. If there exists an LB flow (ξ_{ij}), for any δ such that $0\leqq\delta\leqq\Delta\phi$, $(\bar{v},\bar{t})$ is optimal to $P(\bar{\phi}+\delta)$.

Proof: Let

$$(4.3)\qquad x_{ij} = \xi_{ij}^{(1)} + \delta\xi_{ij}^{(2)} \qquad ((i,j)\varepsilon A).$$

We shall show that x satisfies (3.7) for $\phi=\bar{\phi}$, $(v,t)=(\bar{v},\bar{t})$.

Every arrow is in one of four subsets A_{pk}(k=1,2,3,4).

Case 1: $(i,j)\varepsilon A_{p1}$.

As $\xi_{ij}=(0,0)$, $x_{ij}=0$.

Case 2: $(i,j)\varepsilon A_{p2}$.

$$(0,0) \leqq \xi_{ij} \leqq (c^*_{ij}(\bar{\phi}),\ b_{ij}) = \alpha_{ij}.$$

If $\xi_{ij}^{(2)}\geqq 0$, $x_{ij}\geqq 0$ for any δ. If $\xi_{ij}^{(2)}<0$, $\xi_{ij}^{(1)}>0$ and $x_{ij}\geqq 0$ for δ such that $\delta\leqq-\xi_{ij}^{(1)}/\xi_{ij}^{(2)}$. Define u_{ij} by

$$(4.4)\qquad u_{ij} = (c^*_{ij}(\bar{\phi})+\delta b_{ij}) - x_{ij},$$

Then, $\quad u_{ij} = \lambda_{ij}^{(1)} + \delta\lambda_{ij}^{(2)},$

and

$$\lambda_{ij} = \alpha_{ij}-\xi_{ij} \geqq (0,0).$$

If $\lambda_{ij}^{(2)}\geqq 0$, $u_{ij}\geqq 0$ for any δ. If $\lambda_{ij}^{(2)}<0$, $\lambda_{ij}^{(1)}>0$ and $u_{ij}\geqq 0$ for δ such that $\delta\leqq-\lambda_{ij}^{(1)}/\lambda_{ij}^{(2)}$.

Case 3: $(i,j)\varepsilon A_{p3}$.

$$\alpha_{ij} = \beta_{ij} = (c^*_{ij}(\bar{\phi}),\ b_{ij}).$$

Hence, $\quad x_{ij} = c^*_{ij}(\bar{\phi})+\delta b_{ij} = c^*_{ij}(\bar{\phi}+\delta).$

Case 4: $(i,j)\varepsilon A_{p4}$.

$$\xi_{ij} \geqq (c^*_{ij}(\bar{\phi}),\ b_{ij}) = \beta_{ij}.$$

So, $\lambda_{ij} \leqq (0,0)$. Define u_{ij} by (4.4).

If $\lambda^{(2)}_{ij} \leqq 0$, $u_{ij} \leqq 0$ for any δ. If $\lambda^{(2)}_{ij} > 0$, $\lambda^{(1)}_{ij} < 0$, and $u_{ij} \leqq 0$ for δ such that $\delta \leqq -\lambda^{(1)}_{ij}/\lambda^{(2)}_{ij}$.

We consider four cases together. Then, x defined by (4.3) satisfies (3.7) for δ such that $0 \leqq \delta \leqq \Delta\phi$.

Q.E.D.

Next, assume that there exists an LN cutset, $C(N_0,\bar{N}_0)$. Since $\alpha_{n1} = (\infty,\infty)$, $1 \varepsilon N_0$ or $n \varepsilon \bar{N}_0$. We shall prove that if $1 \varepsilon N_0$ and $n \varepsilon \bar{N}_0$, $C(N'_0,\bar{N}'_0)$ is also an LN cutset where $N'_0 = N_0 + \{n\}$. Let

$$A_1 = \{(i,n) \mid (i,n) \varepsilon A,\ i \varepsilon N_0\}$$

and

$$A_2 = \{(i,n) \mid (i,n) \varepsilon A,\ i \varepsilon \bar{N}_0\}.$$

Then, $C^+(N'_0,\bar{N}'_0) = C^+(N_0,\bar{N}_0) - A_1$

and

$$C^-(N'_0,\bar{N}'_0) = C^-(N_0,\bar{N}_0) + A_2 - \{(n,1)\}.$$

Hence, $\gamma(N'_0,\bar{N}'_0) = \gamma(N_0,\bar{N}_0) - \sum_{A_1} \alpha_{ij} - \sum_{A_2} \beta_{ij} + \beta_{n1} < (0,0).$

That is, $C(N'_0,\bar{N}'_0)$ is an LN cutset. From now, we do not consider LN cutsets such that $1 \varepsilon N_0$ and $n \varepsilon \bar{N}_0$.

From theorem 3, $\gamma_1 = 0$ and $\gamma_2 < 0$. Let

$$C_1 = \{(i,j) \mid (i,j) \varepsilon C^+(N_0,\bar{N}_0) \cap A_{p1}\},$$

$$C_2 = \{(i,j) \mid (i,j) \varepsilon C^+(N_0,\bar{N}_0) \cap (A_{p2} \cup A_{p3})\},$$

and

$$C_3 = \{(i,j) \mid (i,j) \varepsilon C^-(N_0,\bar{N}_0) \cap (A_{p3} \cup A_{p4})\}.$$

Define Δv by

$$\Delta v = \min\{ \min_{(i,j)\varepsilon C_1} (\bar{v}_j - \bar{v}_i - h_{ij}),\quad \min_{(i,j)\varepsilon C_2} (\bar{v}_j - \bar{v}_i - g_{ij}),\quad \min_{(i,j)\varepsilon C_3} (h_{ij} - \bar{v}_j + \bar{v}_i)\}. \tag{4.5}$$

Then, we get the following theorem.

Theorem 7. If there exists an LN cutset $C(N_0,\bar{N}_0)$, for any ε such that $0 \leqq \varepsilon \leqq \Delta v$, there exists an optimal solution of $P(\bar{\phi})$ with

$$\eta(t) = \eta(\bar{t})+\varepsilon|\gamma_2|.$$

Proof: Let $\bar{x}$ be an optimal solution of $D(\bar{\phi})$. Then, $(v,t)=(\bar{v},\bar{t})$ satisfies (3.9) for $\phi=\bar{\phi}$, $x=\bar{x}$. Let

$$(4.6)\qquad v_j = \begin{cases} v_j & \text{if } j\varepsilon N_0, \\ \bar{v}_j-\varepsilon & \text{if } j\varepsilon \bar{N}_0, \end{cases}$$

and

$$(4.7)\qquad t_{ij} = \begin{cases} \bar{t}_{ij}-\varepsilon & \text{if } (i,j)\varepsilon C_2, \\ \bar{t}_{ij}+\varepsilon & \text{if } (i,j)\varepsilon C_3, \\ \bar{t}_{ij} & \text{otherwise}. \end{cases}$$

We show that (v,t) satisfies (3.9) for $x=\bar{x}$, too. Since $\gamma_1=0$,

$$x_{ij}=\alpha_{ij}^{(1)} \qquad \text{if } (i,j)\varepsilon C^+(N_0,\bar{N}_0),$$

and

$$x_{ij}=\beta_{ij}^{(1)} \qquad \text{if } (i,j)\varepsilon C^-(N_0,\bar{N}_0).$$

Therefore, $A_{d2}\cap C^+$ and $A_{d4}\cap C^-$ are empty. That is, there exists neither $(i,j)\varepsilon A_{d2}$ with $v_j-v_i<h_{ij}$, nor $(i,j)\varepsilon A_{d4}$ with $v_j-v_i>g_{ij}$. Hence, (v,t) defined by (4.6) and (4.7) satisfies (3.9) for $x=\bar{x}$, and it is optimal to $P(\bar{\phi})$. Then,

$$\eta(t) = \eta(\bar{t}) - \varepsilon\cdot\gamma_2 = \eta(\bar{t}) + \varepsilon|\gamma_2|.$$

Q.E.D.

If $1\varepsilon\bar{N}_0$, $v_1-\bar{v}_1=-\varepsilon$. To hold that $v_1=0$, let

$$(4.6')\qquad v_j = \begin{cases} \bar{v}_j+\varepsilon & \text{if } j\varepsilon N_0, \\ \bar{v}_j & \text{if } j\varepsilon \bar{N}_0. \end{cases}$$

5. Lexico-bounded Flow Algorithm

Now, we show our algorithm, called the *lexico-bounded flow algorithm (LB flow algorithm)*. It has two phases. The first phase is for getting what maximizes η among optimal solutions of P(0), and the second is for increasing ϕ or η.

LB flow algorithm

Phase 1:

Step 1. Let $t_{ij}=h_{ij}$ for each $(i,j)\varepsilon A$, and obtain the earliest starting node time v_j for each node j. If $v_n-v_1\leqq p_0$, stop.

Step 2. For each $(i,j)\varepsilon A^*$, determine α_{ij} and β_{ij} by table 1, where $\bar{\phi}=0$ and

$\bar{v}_j$ is the current value of v_j. Find the L-max flow. If it does not exist (ξ_{n1} is infinite), stop.

Step 3. Let $C(N_0,\bar{N}_0)$ be the L-min cutset. Determine Δv by (4.5) and let $\varepsilon=\min(\Delta v,\bar{v}_n-p_0)$, and get the new values of (v,t) by (4.6) and (4.7). If $v_n=p_0(\varepsilon=v_n-p_0)$, go to phase 2. Otherwise, go back to step 2.

Phase 2:

Step 1. For each $(i,j)\varepsilon A^*$, determine α_{ij} and β_{ij} by table 1. Find an LB flow. If it exists, go to step 2. Otherwise (if an LN cutset exists), go to step 3.

Step 2. Let ξ be the LB flow. Determine λ_{ij} by (4.1), and $\Delta\phi$ by (4.2). It $\Delta\phi=\infty$, stop. Otherwise, let $\phi=\bar{\phi}+\Delta\phi$, and go back to step 1.

Step 3. Let $C(N_0,\bar{N}_0)$ be the LN cutset. Determine Δv by (4.5). Get an improved solution by (4.6) (or (4.6')) and (4.7) for $\varepsilon=\Delta v$. Go back to step 1.

The procedure in phase 1 corresponds to the critical path method for the usual minimum cost project scheduling problem [7]. If we stop at step 1 of phase 1 (the length of the critical path for the standard times is not greater than p_0), (v,t) is optimal to P(0). (Let $v_n=p_0$ if $v_n<p_0$.) For θ such that $\theta>\eta(t)=\Sigma b_{ij}h_{ij}$, $P_0(\theta)$ is infeasible (from the assumption that $b_{ij}\geqq 0$). If we stop at step 2 of phase 1, P(0) is infeasible. (Therefore, $P_0(\theta)$ is infeasible, too.)

The L-max flow at the end of phase 1 is an LB flow at the start of phase 2. So, in the first iteration of phase 2, we always go to step 2.

After the first iteration, at step 1, we use a labeling procedure to find an LB flow or an LN cutset. Suppose that we always keep (ξ_{ij}) which satisfies that

$$\sum_j \xi_{ij}-\sum_j \xi_{ji}=0 \qquad (i=1,2,\ldots,n).$$

When ϕ is increased at step 3 of the former iteration, for (i,j) with positive b_{ij}, $c^*_{ij}(\phi)$ is increased and it is possible that $\xi_{ij}<\beta_{ij}$. Assume that $\xi_{ts}<\beta_{ts}$. Then, we must increase ξ_{ts}. We call a path from node s to node t a *flow augmenting path (FA path)* with respect to (ξ_{ij}) if $\xi_{ij}<\alpha_{ij}$ on any forward arrow and $\xi_{ij}>\beta_{ij}$ on any reverse arrow of the path. If there exists an FA path from node s to node t, we can increase the flow on it and ξ_{ts}. For finding an FA path we use a labeling procedure like in usual maximum flow problems [1, 2, 3]. If node t is labeled, there exists an FA path. Otherwise, let N_0 be the set of nodes labeled at termination. Then,

$$s\varepsilon N_0 \text{ and } t\varepsilon\bar{N}_0.$$

For the cutset $C(N_0,\bar{N}_0)$,

$$\xi_{ij}\geqq\alpha_{ij} \quad \text{if } (i,j)\varepsilon C^+,$$

and

$$\xi_{ij} \leq \beta_{ij} \quad \text{if } (i,j) \varepsilon C^{-}.$$

As $\xi_{ts} < \beta_{ts}$,

$$C(N_0,\bar{N}_0) = \Sigma_{C^+}\alpha_{ij} - \Sigma_{C^-}\beta_{ij} < \Sigma_{C^+}\xi_{ij} - \Sigma_{C^-}\xi_{ij} = (0,0).$$

That is, $C(N_0,\bar{N}_0)$ is an LN cutset. Here, note that $1 \varepsilon N_0$ if and only if $n \varepsilon N_0$ because $\alpha_{n1} > \xi_{n1} > \beta_{n1}$.

6. Illustrative Example

To illustrate our algorithm, consider an arrow diagram shown in Fig. 1. We use a labeling procedure with "first labeled first scanned rule".

Phase 1.

We set $t_{ij}=h_{ij}$ for each $(i,j)\varepsilon A$, and get the earliest starting node times, which are shown in Fig. 2. As $v_5-v_1=26>20=p_0$, we shorten v_5-v_1 to p_0, and get an optimal solution of P(0) in Fig. 3.

Phase 2.

Iteration 1. We determine α_{ij} and β_{ij} for (v,t) in Fig. 3. In Fig. 4, for each branch (i,j), (c^*_{ij}, b_{ij}) and its condition (the range of v_j-v_i) are shown. (Refer to Fig. 3 for meanings of branch symbols.) We can know α_{ij} and β_{ij} by them. For example, for (1,2), $\alpha_{12}=\beta_{12}=(9,2)$ as $h_{12}>v_2-v_1>g_{12}$. For (1,3), $\alpha_{13}=(1,4)$ and $\beta_{13}=(0,0)$ as $v_3-v_1=h_{13}$.
Since there exists an LB flow, which is the L-max flow at termination of phase 1, we go to step 2. A_1 is empty, $A_2=\{(2,3),(2,4)\}$, and

$$\Delta\phi = \min(-(3-5)/(1-0),\ -(5-4)/(0-2)) = 0.5.$$

Therefore, ϕ is increased to 0.5.
Iteration 2. See Fig. 5. There exists an LN cutset. $N_0=\{1,2,3,5\}$, and $\gamma=(0,-2)$. C_1 is empty, $C_2=\{(2,4)\}$ and $C_3=\{(4,5)\}$. Hence,

$$\Delta v = \min(13-5-6,\ 10-20+13) = 2,$$

$$v_4 = 13-2 = 11,$$

$$t_{24} = 8-2 = 6,$$

and

$$\eta = 79+2\times 2 = 83.$$

The new schedule is shown in Fig. 5(b).
Iteration 3. See Fig. 6. There exist an LB flow. A_1 is empty, $A_2=\{(2,3)\}$,

$$\Delta\phi = -(3.5-5)/(1-0) = 1.5,$$

and

ϕ = 0.5+1.5 = 2.

Iteration 4. See Fig. 7. There exists an LN cutset. N_0={1,3,5}, and γ=(0,-1). C_1 is empty, C_2={(1,2)} and C_3={(2,3),(4,5)}.

Δv = min(5-0-4, 9-12+5, 10-20+11) = 1.

The new schedule is shown in Fig. 7(b).

Iteration 5. See Fig. 8. There exists an LB flow. A_1={(1,3)}, and A_2 is empty.

$\Delta\phi$ = 1, and ϕ = 2+1 = 3.

Iteration 6. See Fig. 9. There exists an LN cutset. N_0={3}, and γ=(0,-1). C_1 is empty, C_2={(3,5)}, and C_3={(2,3)}. (Note that (1,3) does not belong to C_3.)

Δv = min(20-12-6, 9-12+4) = 1.

The new schedule is shown in Fig. 9(b). (As 1 belongs to $\bar{N}_0$, we use (4.6').)

Iteration 7. See Fig. 10. There exists an LB flow, but both A_1 and A_2 and are empty. Therefore, $\Delta\phi=\infty$ and we terminate our algorithm.

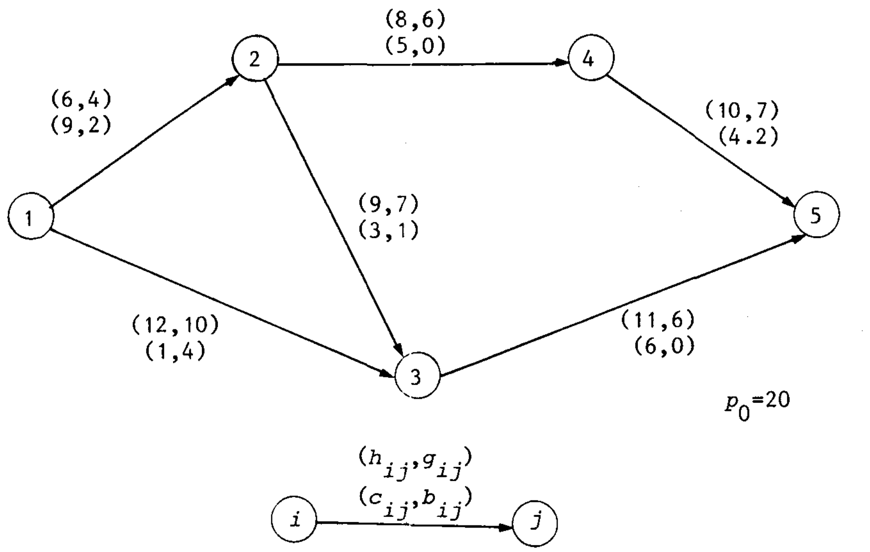


Fig. 1. Arrow diagram.

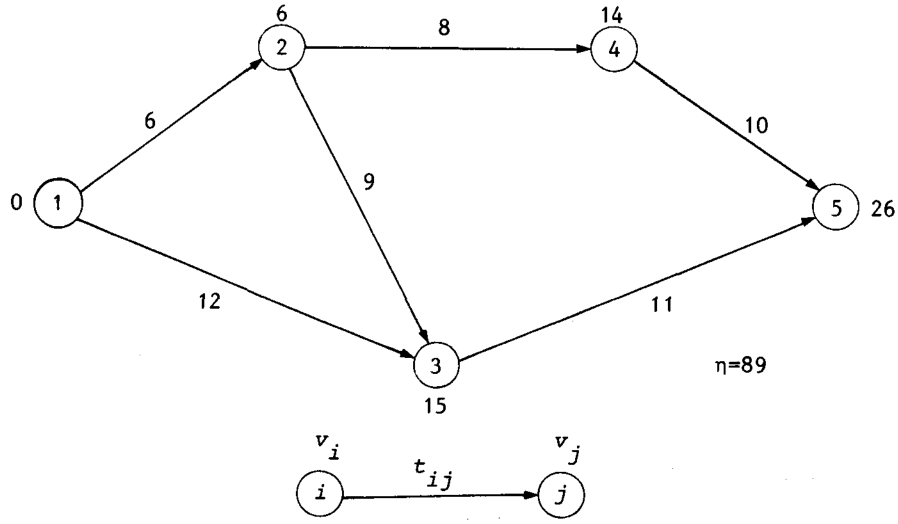


Fig. 2. Standard time scheduling.

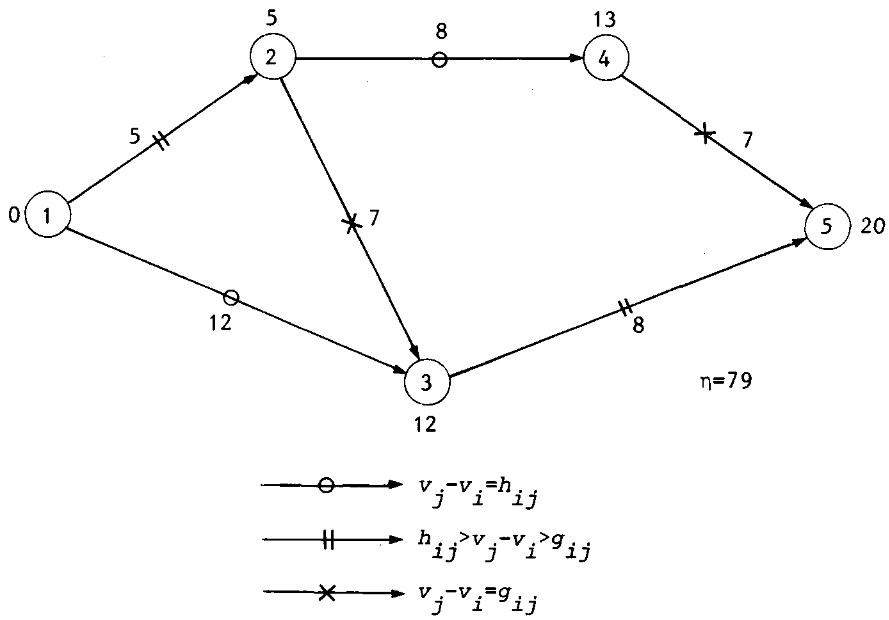


Fig. 3. An optimal solution of P(0).

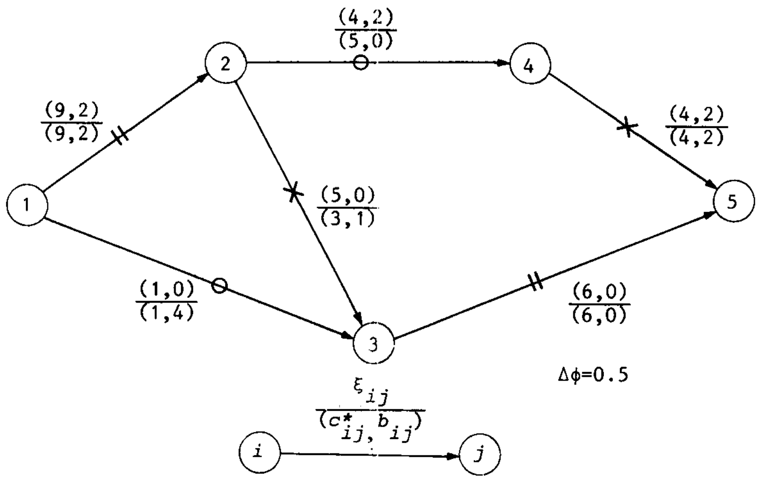
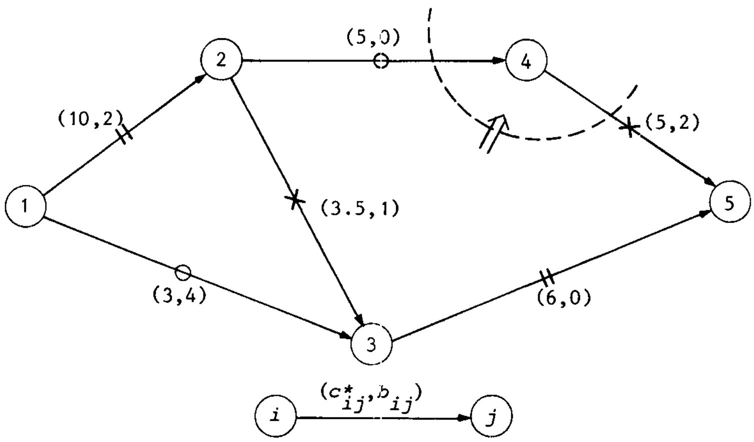
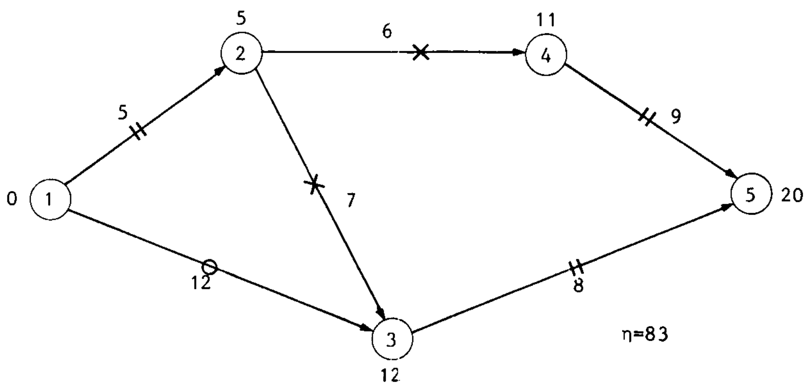


Fig. 4. LB flow of iteration 1(φ=0).



(a) LN cutset.



(b) Schedule.

Fig. 5. Iteration 2(φ=0.5).

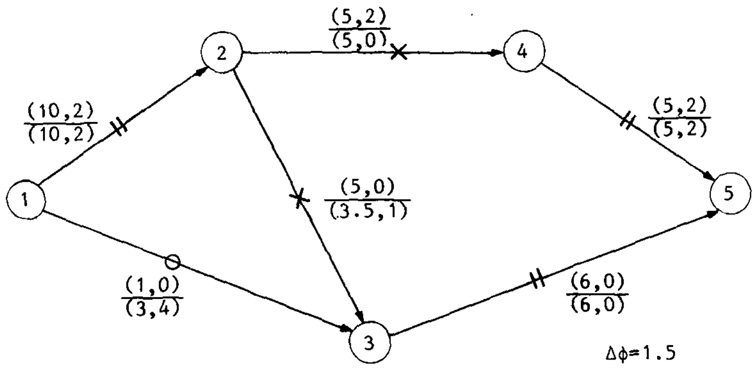
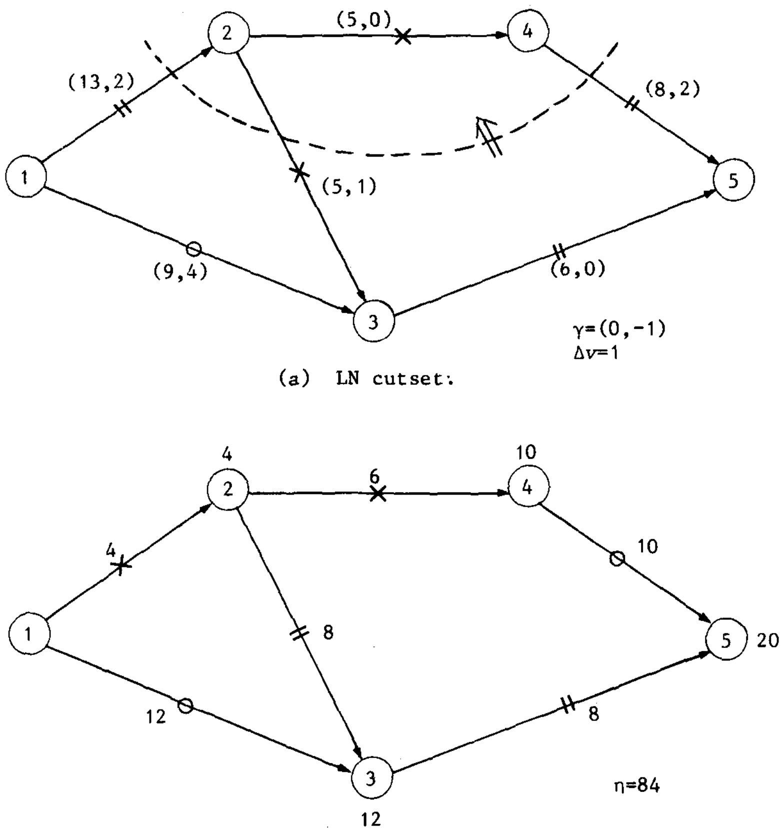


Fig. 6. LB flow of iteration 3(φ=0.5).



(a) LN cutset.

(b) Schedule.

Fig. 7. Iteration 4(φ=2).

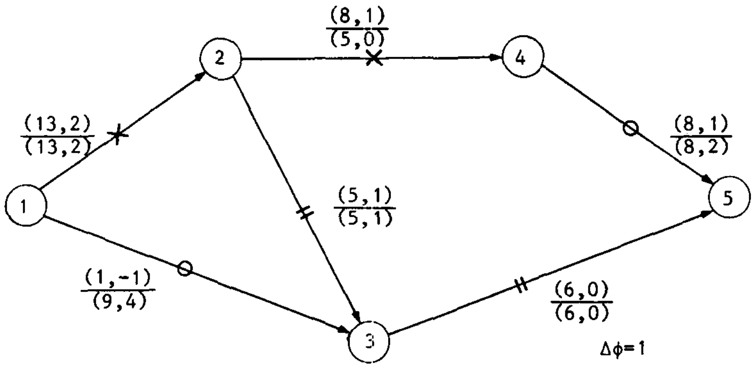
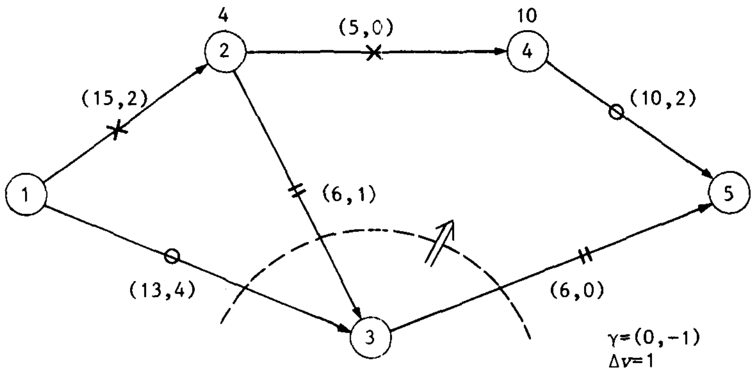
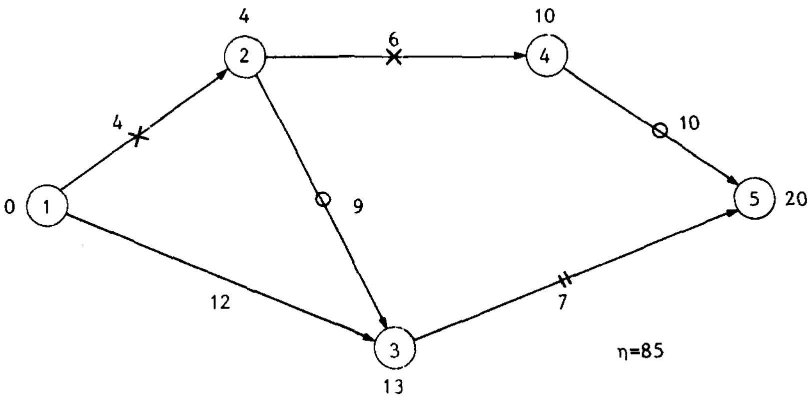


Fig. 8. LB flow of interation 5(ϕ=2).



(a) LN cutset.



(b) Schedule.

Fig. 9. Iteration 6(ϕ=3).

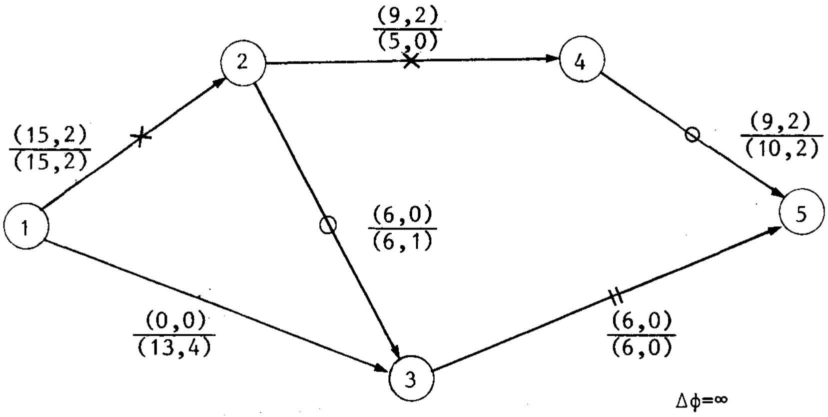


Fig. 10. LB flow of iteration 7(φ=3).

Fig. 11 shows the locus of (ϕ,η). For (ϕ,η) on the locus it, there exists an optimal solution of $P(\phi)$ with $\eta=\Sigma b_{ij}t_{ij}$, which is optimal to $P_0(\eta)$. On a horizontal segment, the solution of $P(\phi)$ does not vary. For a point on a vertical segment, we can obtain a solution of $P_0(\eta)$ by linear interpolation of two solutions of $P(\phi)$ corresponding to terminal points of the segment.

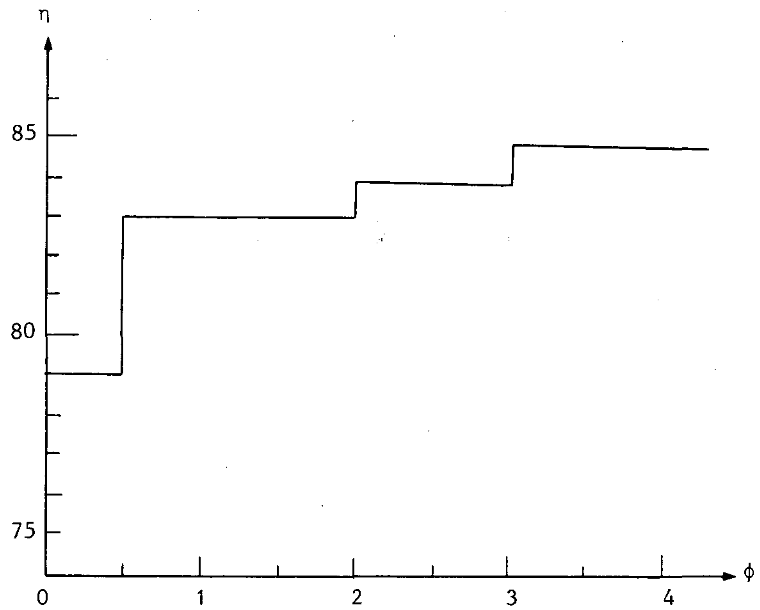


Fig. 11. Locus of (ϕ,η).

7. The Case with Negative b's

When some negative b's exist, we must modify our algorithm. Suppose that $c_{kl}>0$ and $b_{kl}<0$. Let

$$\phi_{kl} = -c_{kl}/b_{kl}.$$

If $\phi>\phi_{kl}$, $c^*_{kl}(\phi)<0$, and hence t_{kl} is always equal to g_{kl}. ((3.5) is not necessarily satisfied.) Therefore, we replace h_{kl} with g_{kl}, and $(c^*_{kl}(\phi),b_{kl})$ with (0,0) at $\phi=\phi_{kl}$. ((k,l) stays in $A_{p1}\cup A_{p4}$ after that.) To stop at $\phi=\phi_{kl}$, in step 3 of phase 2, replace $\Delta\phi$ in (4.2) by

$$\Delta\phi = \min(\Delta\phi_1, \Delta\phi_2, \Delta\phi_3),$$

where

$$\Delta\phi_3 = -c^*_{kl}(\bar{\phi})/b_{kl}.$$

When two or more negative b's exist, $\Delta\phi_3$ is defined by

$$\Delta\phi_3 = \min_{(i,j)\varepsilon A_3} (-c^*_{ij}(\bar{\phi})/b_{ij}),$$

wjere

$$A_3 = \{(i,j)\,|\,(i,j)\varepsilon A,\ b_{ij}<0\}.$$

8. Concluding Remark

Our algorithm is also applied to a project scheduling problem with two objectives. Suppose that we wish to minimize

$$z_c(t) = \Sigma c_{ij}(h_{ij}-t_{ij}),$$

and

$$z_b(t) = \Sigma b_{ij}(h_{ij}-t_{ij})$$

subject to (2.1), (2.2) and (2.3). Let us combine them as

$$z(t) = z_c(t)+\phi\cdot z_b(t).$$

Then, the problem to be solved is $P(\phi)$.

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