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RELATIONSHIPS BETWEEN TIME/CUSTOMER STATIONARY CHARACTERISTICS OF TANDEM QUEUES ATTENDED BY A SINGLE SERVER

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Abstract This paper employs the methods recently developed for queues with non-independent interarrival and service times in order to investigate tandem queues attended by a single server, where the primary focus is upon relationships between time-stationary and customer-stationary queue length characteristics. Examples are given how these relationships can be applied.

1. Introduction

A rather comprehensive queueing models is considered, in which the arriving customers are served by a single server, but in an ordered sequence of c service stage ($1 \leq c \leq \infty$) they can obtain a random number of service transactions, which in general is larger than one and bounded by c. In difference from the usually made *i.i.d.* assumption it is only assumed that the govering sequence of interarrival and service times is stationary and ergodic. The main interest is put on the limiting behaviour of the queue length process $(Q^o(t))$, where $(Q^o(t)) = (Q_1^o,\ldots,Q_c^o(t))$ and $Q_j^o(t)$ denotes the number of customers in stage j at time t. In particular, so-called time-stationary limit distributions of $(Q^o(t))$ are considered, when the time t continuously goes to infinity, as well as customer-stationary limit distributions, when the limits are taken over various embedded epochs such as arrival or departure epochs. It is mentioned that the classical concept of embedding can be concerned with a general point process approach if the govering sequence satisfies some mixing condition stated in Section 3. In Section 4 relationships between various time-stationary and customer-stationary queue length distributions are given, which are applied in Section 5 to several concrete systems considered in the literature. Namely, using these relationships it is possible to determine a wide spectrum of stationary characteristics in a simple way starting from some given ones. For example, in the case of the system with alternating properties considered in 5.1, it is not necessary to investigate every of the interesting stationary queue length distributions separately. But, from the customer-stationary queue length distributions at departure epochs several further stationary queue length distributions can be immediately obtained. In this way it is possible to determine first those stationary queue length distributions separately, the derivation of which is relatively simple, and after that to obtain further interesting stationary queue length distributions by using the relationships stated in Section 4.

2. The tandem model

In the queueing system considered in this paper, each arriving customer requires a certain number of sequential types of service, all of which are performed by a single server. Such a situation appears, for example, in the case when a single server serves the arriving customers at several service stages (see Fig.1).

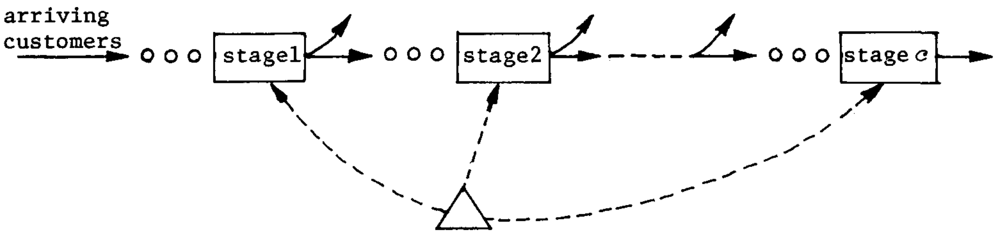

Fig. 1

Thus, we assume that the system consists of an ordered sequence of c service stages ($1 \leq c \leq \infty$). An arriving customer waits for service in stage 1 unless the server begins to serve him. After termination of his service in the first stage the customer can leave the system or go to stage 2 for receiving the second service type, and so on. After completing the service in stage c (if $c < \infty$) the customer leaves the system in any case. Thereby, under a service of type r we understand a service in the rth stage and, furthermore, we say that a customer is of type m if he leaves the system immediately after termination of his service in the mth stage ($1 \leq r,m \leq c$). When a service of type r is completed the server chooses the next service type. to continue his work, according to some selection rule $u(r;\ k_1,k_2,\ldots,k_c)$ depending on the type r of service which is just completed and on the vector $(k_1,k_2,\ldots,k_c)$, where k_j denotes the number of customers waiting in the system for recieving their service of type j; $j = 1,2,\ldots,c$. Obviously, $u(r;\ k_1,k_2,\ldots,k_c) = j$ implies that $k_j > 0$.

The temporal behaviour of system is essentialy influenced by the so-called govering sequence $\{[A_n,B_n]\}$ with

$B_n = (m_n;\ B_{n,1},\ldots,B_{n,m_n},0,\ldots,0)$ where

A_n - the time interval between the arrival epochs of the nth and the $(n+1)$th customer,

m_n - the type of the nth customer,

$B_{n,j}$ - the service time of the nth customer in stage j.

We assume that

(I) $\{[A_n,B_n]\}$ is strictly stationary and ergodic,

(II) $\Pr(0<A_n<\infty) = \Pr(0<B_{n,j}<\infty) = \Pr(1\leq m_n\leq\infty) = 1$,

$EA_n < \infty$, $E(\sum_{j=1}^{m_n} B_{n\ j}) < \infty$.

3. Stationary queue length distributions

Under the condition that the first customer arrives at time zero finding the system empty we consider the process $(Q^o(t))$ with $Q^o(t) = (Q_1^o(t),\ldots,Q_c^o(t))$, where $Q_j^o(t)$ denotes the number of customers in stage j at time t. In particular, we are interested in the limiting behaviour of $(Q^o(t))$ and, therefore, we consider the following limits:

(1) $p_{1,\ldots,c}(k_1,\ldots,k_c) = \lim_{t\to\infty} \Pr(Q^o(t) = (k_1,\ldots,k_c))$,

(2) $p^l_{1,\ldots,c}(k_1,\ldots,k_c) = \lim_{n\to\infty} \Pr(Q^o(T_n^l-0) = (k_1,\ldots,k_c))$

for $l = (i_{j_1},\ldots,i_{j_h})$,

(3) $p^l_{1,\ldots,c}(k_1,\ldots,k_c) = \lim_{n\to\infty} \Pr(Q^o(T_n^l+0) = (k_1,\ldots,k_c))$

for $l = (o_{j_1},\ldots,o_{j_h})$, $(d_{j_1},\ldots,d_{j_h})$ or $(s_{j_1},\ldots,s_{j_h})$,

where the T_n^l, $n = 1,2,\ldots$, denote the arrival epochs at any stage of $j_1,\ldots,j_h$ if $l = (i_{j_1},\ldots,i_{j_h})$, the termination epochs of service times at any stage of $j_1,\ldots,j_h$ if $l = (o_{j_1},\ldots,o_{j_h})$, the departure epochs after service completions in any stage of $j_1,\ldots,j_h$ if $l = (d_{j_1},\ldots,d_{j_h})$, the service beginning at any stage of $j_1,\ldots,j_h$ if $l = (s_{j_1},\ldots,s_{j_h})$; $1 \leq h \leq c$.

Moreover, sequence $\{T_n^l\}$ with mixed indeces, for example $l = (d_{j_1},d_{j_2},$

$\ldots, d_{j_{h-1}}, o_{j_h})$, have an analogous meaning and can be handled in the same manner. Note that $\{T_n^{(o_j)}\} = \{T_n^{(i_j+1)}\} \cup \{T_n^{(d_j)}\}$ for $j=1,\ldots,c-1$ and $T_n^{(o_c)} = T_n^{(d_c)}$.

From the arguments used in [13] (see also [2], chapt. 2) we get that the condition

$$(4)\qquad E(\sum_{j=1}^{m} B_{1,j}) < EA_1$$

is necessary and sufficient to ensure that with probability one infinitely many arriving customers find the system empty and that, consequently, for $l = (i_1)$ the limit in (2) exists and leads to a proper probability distribution. To have this ergodicity property also for the other limits appearing in (1) - (3) we need, beyond (4), some further condition on the govering sequence $\{[A_n,B_n]\}$.

In [11] (see also [9]) the following mixing condition has been shown to be sufficient for the existence and properness of the remaining limits in (1) - (3). Let $\Phi_0 = \{[T_n^{(i_1)},B_n]\}$ denote the marked point process of the arrival epochs $T_n^{(i_1)}$ at stage 1 marked by the corresponding service time vector B_n. Because of the stationarity of the govering sequence $\{[A_n,B_n]\}$ we can extend Φ_0, by pointwise shifting, to a marked point process on the whole real line R, which will be also denoted by Φ_0. Let M_K denote the set of all counting measure on $R\times K$ which are locally finite in the first component, where $K = \{1,2,\ldots,c\}\times R_+^\infty$ is the mark space of Φ_0. On the σ-algebra $\mathfrak{M}_K$ $(=\mathfrak{M}_{-\infty}^{\infty})$ of subsets of M_K, where the σ-algebra $\mathfrak{M}_a^b$ is generated by sets of the form $\{\psi : \psi = \{[t_n,b_n]\}\in M_K,\ \mathrm{card}\,\{n : [t_n,b_n]\in C\} = k\}$ with $C\subset(a,b)\times K$, we define the probability distribution P by

$$(5)\qquad P(.) = (EA_1)^{-1}E\{\int_{T_1^{i_1}(=0)}^{T_2^{i_1}} I_{(.)}(\mathfrak{T}^u\Phi_0)du\},$$

where I_X denotes the indicator of the set X, and $\mathfrak{T}^u$ the shift operator $\mathfrak{T}^u\psi = \{[t_n-u,b_n]\}$, $I_X(\psi) = 1$ if $\psi\in X$ and $I_X(\psi) = 0$ if $\psi\notin X$.

Because of the invariance of Φ_0 with respect to pointwise shifting it can be shown that P defined by (5) is invariant with respect to the shifting $\mathfrak{T}^u$ for every fixed $u\in R$, see [2], [5], [13]. The above mentioned mixing condition ensuring the existence and properness of every limit in (1) - (3) has the form

$$(6)\qquad \sup_{Y\in\mathfrak{M}_0^\infty} |P(X\cap\mathfrak{T}^{-t}Y) - P(X)P(Y)| \xrightarrow[t\to\infty]{} 0$$

for every $X\in\mathfrak{M}_{-\infty}^u$ and for every $u \geqq 0$. It can be shown that (6) is satisfied, for example, for every P induced by a govering sequence consisting of completely independent random variables (r.v.'s) if the interarrival time distribution is not arithmetic, i.e., if for every $a > 0$ we have $\sum_{j=1}^{\infty} \Pr(A_1 = aj) < 1$. Moreover, it suffices to assume only that the sequence $\{A_n\}$ and $\{B_n\}$ are independent and that each of them satisfies some mixing condition (see [9], [11]).

4. General relationships

In this section we investigate relationships between limit queue length probabilities as given in (1) - (3) provided that they exists. This is the case if, for example, the conditions (4) and (6) are satisfied. Furthermore, we assume that with probability one it is not possible that an interarrival time and a service time can be finished simultaneously. Let

$(p_{j_1,\dots,j_e}(k_1,\dots,k_e))$ and $(p^l_{j_1,\dots,j_e}(k_1,\dots,k_e))$ with

$$p_{j_1,\dots,j_e}(k_1,\dots,k_e) = \lim_{t\to\infty} \Pr(Q^o_{j_r}(t) = k_r \ ;\ r = 1,\dots,e)$$

and

$$p^l_{j_1,\dots,j_e}(k_1,\dots,k_e) = \lim_{n\to\infty} \Pr(Q^o_{j_r}(T^l_n \pm 0) = k_r \ ;\ r = 1,\dots,e)$$

denote the marginal distributions ($1 \leq e \leq c$) of the corresponding limit distributions defined in (1) - (3). Furthermore, let

$$p_{(j_1,\dots,j_e)}(k) = \lim_{t\to\infty} \Pr(\sum_{r=1}^{e} Q^o_{j_r}(t) = k)$$

and

$$p^l_{(j_1,\dots,j_e)}(k) = \lim_{n\to\infty} \Pr(\sum_{r=1}^{e} Q^o_{j_r}(T^l_n \pm 0) = k).$$

Following the arguments used for standard queueing systems with general stationary ergodic govering sequence (see, for example, [4], [8], [15] as well as chap. 4 in [2]), we get relationships between these stationary queue length characteristics. Concerning generalization to further non-standard queueing systems, which partially are contained in the model considered in this paper (see Sect. 5), we refer to [1], [7], [10], [11] as well as to chapt. 5 of [2]. For shortness, we will omit the proofs of most results stated in this section because they are rather similar to those provided in the cited literature. To illustrate the proving technique used there, in the appendix we sketch the proof of theorem 1 as well as of the relationships (10) and (13).

Theorem 1. We have

$$(7)\quad p_{(j,j+1,\ldots,j+h)}^{(i_j,i_{j+1},\ldots,i_{j+h})}(k) = p_{(j,j+1,\ldots,j+h)}^{(o_j,o_{j+1},\ldots,o_{j+h})}(k)$$

$$(8)\quad p_{(j,j+1,\ldots,j+h)}^{(i_j)}(k) = p_{(j,j+1,\ldots,j+h)}^{(d_j,d_{j+1},\ldots,d_{j+h-1},o_{j+h})}(k)$$

and, if $\Pr(j \leqq m_1 \leqq j+h) = 0$,

$$(9)\quad p_{(j,j+1,\ldots,j+h)}^{(i_j)}(k) = p_{(j,j+1,\ldots,j+h)}^{(o_{j+h})}(k) .$$

Theorem 2. We have

$$(10)\quad p_{1,\ldots,c}^{(o_1,\ldots,o_{j_r})}(k_1,\ldots,k_c) = \sum_{e=1}^{r} \frac{\Pr(m_1 \geqq j_e)}{\sum_{f=1}^{r} \Pr(m_1 \geqq j_f)} p_{1,\ldots,c}^{(o_{j_e})}(k_1,\ldots,k_c)$$

and, in particular,

$$(10')\quad p_{1,\ldots,c}^{(o_1,\ldots,o_c)}(k_1,\ldots,k_c) = \sum_{e=1}^{c} \frac{\Pr(m_1 \geqq e)}{Em_1} p_{1,\ldots,c}^{(o_e)}(k_1,\ldots,k_c)$$

Analogously,

$$(11)\quad p_{1,\ldots,c}^{(i_{j_1},\ldots,i_{j_r})}(k_1,\ldots,k_c) = \sum_{e=1}^{c} \frac{\Pr(m_1 \geqq j_e)}{\sum_{f=1}^{r} \Pr(m_1 \geqq j_f)} p_{1,\ldots,c}^{(i_{j_e})}(k_1,\ldots,k_c)$$

and, in particular,

$$(11')\quad p_{1,\ldots,c}^{(i_1,\ldots,i_c)}(k_1,\ldots,k_c) = \sum_{e=1}^{c} \frac{\Pr(m_1 \geqq e)}{Em_1} p_{1,\ldots,c}^{(i_e)}(k_1,\ldots,k_c)$$

Furthermore,

$$(12)\quad p_{1,\ldots,c}^{(d_{j_1},\ldots,d_{j_r})}(k_1,\ldots,k_c) = \sum_{e=1}^{r} \frac{\Pr(m_1 = j_e)}{\sum_{f=1}^{r} \Pr(m_1 = j_f)} p_{1,\ldots,c}^{(d_{j_e})}(k_1,\ldots,k_c)$$

and, in particular,

$$(12')\quad p_{1,\ldots,c}^{(d_1,\ldots,d_c)}(k_1,\ldots,k_c) = \sum_{e=1}^{c} \Pr(m_1 = e) p_{1,\ldots,c}^{(d_e)}(k_1,\ldots,k_c)$$

as well as

$$(13)\quad p_{1,\ldots,c}^{(o_r)}(k_1,\ldots,k_c) = \frac{\Pr(m_1 = r)}{\Pr(m_1 \geqq r)} p_{1,\ldots,c}^{(d_r)}(k_1,\ldots,k_c) + \frac{\Pr(m_1 \geqq r)}{\Pr(m_1 \geqq r)} p_{1,\ldots,c}^{(i_{r+1})}(k'_1,\ldots,k'_c),$$

where $k'_j = k_j + 1$ for $j = r$, $k'_j = k_j - 1$ for $j = r+1$ and $k'_j = k_j$ otherwise.

Theorem 3. Let $\Pr(m_n = r) > 0$ and let the decision that a customer leaves the system immediately after service in stage r or that he goes to stage $r+1$ be independent of the remaining components of the govering sequence, i.e., let

$$m_n = \max\{k:\ 1 \leq k \leq c,\ C_{n,j} = 1 \text{ for } j \in \{2,\ldots,k\}\},$$

where the {0,1}-valued random variables $C_{n,j}$ have the property that the $C_{n,r+1}$ are independent of each other as well as independent of the other $C_{n,j}$ ($j \neq r+1$) and of the interarrival and service times. Then we have for $r < c$

$$(14)\qquad p_{1,\ldots,c}^{(d_r)}(k_1,\ldots,k_c) = p_{1,\ldots,c}^{(o_r)^-}(k_1,\ldots,k_c),$$

where

$$p_{1,\ldots,c}^{(o_r)^-}(k_1,\ldots,k_c) = \lim_{n\to\infty} \Pr(\tilde{Q}^o(T_n^{o_r}+0) = k_j;\ j=1,\ldots,c),$$

$$\tilde{Q}_j^o(t) = Q_j^o(t) - I_{\{r+1\}}(t)Y_{n,r} \quad \text{and}$$

$$Y_{n,r} = \begin{cases} 0 & \text{if } T_n^{o_r} \text{ is a departure epoch,} \\ 1 & \text{otherwise.} \end{cases}$$

Furthermore,

$$(15)\qquad p_{1,\ldots,c}^{(d_r)}(k_1,\ldots,k_c) = p_{1,\ldots,c}^{(i_{r+1})}(\tilde{k}_1,\ldots,\tilde{k}_c),$$

where $\tilde{k}_j = k_j + I_{\{r\}}(j)$.

Theorem 4. If the sequence $\{A_n\}$ consists of independent exponenetially distributed r.v.'s, which are independent of the sequence $\{B_n\}$, we have

$$(16)\qquad p_{1,\ldots,c}(k_1,\ldots,k_c) = p_{1,\ldots,c}^{(i_1)}(k_1,\ldots,k_c).$$

Let $\Pr(m_n=r) > 0$. If, additionally, the service times $\{B_{n,r}\}$ in stage r themselves are independent r.v.'s and if the sequence $\{B_{n,r}\}$ and the decision that a customer leaves the system immediately after service in stage r or that he goes to stage $r+1$ are independent of each other as well as of the remaining components of the govering sequence, we have

$$(17)\qquad p_{1,\ldots,c}^{(d_r)}(k_1,\ldots,k_c) = \sum_{m=1}^{\tilde{k}_1}\int_0^\infty e^{-\lambda x}\frac{(\lambda x)^{\tilde{k}_1-m}}{(k_1-m)!}\,dB_r(x)\,p_{1,\ldots,c}^{(s_r)}(m,\tilde{k}_2,\ldots,\tilde{k}_c).$$

Remark. In [6], the following problem has been considered which, in certain sense, is an inversion of (16): What can be said about the arriaval epochs in stage 1 assuming that an identity of the form (16) holds ? In particular, under what conditions it can be concluded from an assumption of the form (16) that the arrival epochs in stage 1 form a Poisson process.

Theorem 5. If the service times $\{B_{n\,r}\}$ in stage r are independent exponentially distributed r.v.'s with parameter μ_r, which are independent of the remaining components of the govering sequence, we have

$$(18)\qquad \mu_r \cdot p_{1,\ldots,c}(k_1,\ldots,k_c) = \frac{\Pr(m_1=r)}{EA_1}\, p^{(d_r)}_{1,\ldots,c}(k_1,\ldots,k_{r-1},k_r-1,\ldots,k_c) + \frac{\Pr(m_1>r)}{EA_1}\, p^{(i_{r+1})}_{1,\ldots,c}(k_1,\ldots,k_c)$$

for $k_r = 1,2,\ldots$ and $k_j = 0,1,\ldots$ with $j \neq r$.

5. Applications

In this section we assume that the govering sequence consists of completely independent r.v.'s. Moreover, let the interarrival times A_n be exponentially distributed with parameter λ and let the decision that a customer leaves the system immediately after his service in stage r or that he goes to stage $r+1$ be independent of the remaining components of the govering sequence for every $r = 1,2,\ldots,c-1$. By p_r we denote the probability $p_r = \Pr(Y_{n,r} = 1) = \Pr(m_n \geqq r+1)$ that a customer after service in stage r goes to stage $r+1$.

5.1 Alternating priorities

In [17], an algorithm has been given to determine the generating functions of the distributions $(p^{(d_r)}_{1,\ldots,c}(k_1,\ldots,k_c))$ for $1 \leqq r \leqq c$ and of $(p^{(i_r)}_{1,\ldots,c}(k_1,\ldots,k_c))$ for $2 \leqq r \leqq c$, respectively, where the following selection rule is considered giving the stages alternating priorities $(c < \infty)$:

$$(19)\qquad u(r;k_1,k_2,\ldots,k_c) = \begin{cases} r & \text{if } k_r > 0\,, \\ r+1 & \text{if } k_r = 0,\ k_{r+1} > 0,\ r < c\,, \\ 1 & \text{if } k_r = 0,\ k_{r+1} = 0,\ r < c\,, \\ 1 & \text{if } k_r = 0,\ r = c\,. \end{cases}$$

A formula for the expectation $\sum_{k=1}^{\infty} k p^{(d_c)}_{(1,\ldots,c)}(k)$ of the toal number of customers in the system immediately after departure epochs from stage c has been established. The special case $c = 2$, $p_1 = 1$ has been solved in [16] and [18], where, in particular, formulas have been obtained for the mean queue lengths $\sum_{k=1}^{\infty} k p_{(1,2)}(k)$ and $\sum_{k=1}^{\infty} k p^{(o_r)}_j(k)$ $(j;\ r = 1,2)$ as well as for moments of waiting time.

The relationships stated in Sect. 4 give the possibility to determine further stationary queue length characteristics in a simple way. For example, (13) can be used to determine the customer-stationary queue length distribution at arbitary termination epochs of service times in stage r starting from the above mentioned algorithm given in [17]. Furthermore, from (10) - (12) we get customer-stationary characteristics seen by some set of stages. From (17) we obtain customer-stationary characteristics at service beginning. Finally, by means of (7) and (9) we get customer-stationary characteristics at arrival epochs including those at stage 1 for the total number of customers in the considered set of stages. If $p_1 = p_2 = \dots = p_{h-1} =1$, the relationships (9) and (16) lead to the time-stationary distribution $(p_{(1,2,\dots,h)}(k))$

In order to illustrate this by concrete results, for simplicity we restrict the attention to mean queue lengths assuming that $c = 2$, $p_1 = 1$. In this case, in [18] the following results have been obtained:

$$\sum_{k=1}^{\infty} kp_1^{(o_1)}(k) = (\rho_1 + Q_M)(1 - \rho_1)(1 - \rho_1 + \rho_2)^{-1},$$

$$\sum_{k=1}^{\infty} kp_1^{(o_2)}(k) = \rho_2(1 + \rho_2 + Q_M)(1 - \rho_1 + \rho_2)^{-1},$$

$$\sum_{k=1}^{\infty} kp_2^{(o_1)}(k) = (1 + \rho_2 + Q_M)(1 - \rho_1 + \rho_2)^{-1},$$

$$\sum_{k=1}^{\infty} kp_2^{(o_2)}(k) = (\rho_1 + Q_M)(1 - \rho_1 + \rho_2)^{-1},$$

where $Q_M = \frac{\lambda^2}{2} \mathrm{E}[(B_{n,1} + B_{n,2})^2](1 - \rho_1 - \rho_2)^{-1}$; $\rho_j = \lambda \mathrm{E} B_{n,j}$. From these formulas we get for example that, by using (10),

$$\sum_{k=1}^{\infty} kp^{(o_1,o_2)}(k) = \frac{1}{2} [(\rho_1 + Q_M)(1 - \rho_1) + \rho_2(1 + \rho_2 + Q_M)](1 - \rho_1 + \rho_2)^{-1},$$

$$\sum_{k=1}^{\infty} kp^{(o_1,o_2)}(k) = \frac{1}{2} (1 + \rho_1 + \rho_2 + 2Q_M)(1 - \rho_1 + \rho_2)^{-1}.$$

Furthermore, from (9) and (16) we get that

$$\sum_{k=1}^{\infty} kp_{(1,2)}(k) = \sum_{k=1}^{\infty} kp_1^{(o_2)}(k) + \sum_{k=1}^{\infty} kp_2^{(o_2)}(k),$$

i.e., for the formula

$$\sum_{k=1}^{\infty} kp_{(1,2)}(k) = (\rho_1 + \rho_2 + \rho_2^2 + Q_M(1 + \rho_2))(1 - \rho_1 + \rho_2)^{-1}$$

derived in [16] a separate proof is not necessary.

5.2 The head-of-the-line priority discipline

For $c = 2$. the selection rule

$$u(r;k_1,k_2) = \begin{cases} 1 & \text{if } k_1 > 0, \\ 2 & \text{if } k_1 = 0,\ k_2 > 0, \end{cases} \tag{20}$$

which dose not depend on the type r of the just finished service time, is considered in [7]. It prefers the customers waiting for recieving their service in stage 1 in face of the customers waiting for recieving their service in stage 2. For this selection rule, an algorithm has been given in [7] to determine the generating functions of the distributions $(p_{1,2}^{(s_r)}(k_1,k_2))$ for r =1,2 and of $(p_{1,2}(k_1,k_2))$, respectively. Note that obviously $p_{1,2}^{(s_2)}(k_1,k_2) = 0$ for $k_1 > 0$. Using (17) we can determine the generating function of $(p_{1,2}^{(d_r)}(k_1,k_2))$ for $r = 1,2$ and by (15) and (16) those of $(p_{1,2}^{(i_r)}(k_1,k_2))$ for $r = 1,2$. Finally, the relationships (10) - (13) can be used in the same manner as in the case of alternating priorities (see Subsec. 5.1). In this way it is possible to extend some results obtained in [12], where the case $p_1 = 1$ and exponentially distributed service times have been considered.

In [3] and [12], the selection rule ($c = 2$)

$$(21) \qquad u(r;k_1,k_2) = \begin{cases} 1 & \text{if } k_1 > 0,\ k_2 = 0 \\ 1 & \text{if } k_2 > 0 \end{cases}$$

is considered, which is a dual to that given in (20). It prefers the customers which after service in stage 1 go to stage 2, i.e., the service of these customers in stage 2 is realized immeadiately after their service in stage 1. Obviously, the relationships stated in Sect. 4 can be applied for this selection rule, too (see[7]). Namely, various customer-stationary characteristics can be obtained starting from the generating function of the time-stationary distribution $(p_{(1,2)}(k))$, which easily follows from the well-known Pollaczek-Khinchine formula for the standard $M/GI/1/\infty$ queue with an appropriately chosen service time distribution.

5.3 Instantaneous Bernoulli feedback

Let $c = \infty$, $p_r = p$ for r =1,2,... and let the service times $\{B_{n,j},\ j = 1, \ldots, m_n\}$ in the consecutive stages be identically distributed. Note that in this case, with respect to the total number of customers in the system, the tandem model described in Sect. 2 is anything but the single-server queue with instantaneous Bernoulli feedback considered, for example, in [1]. Thereby, a customer is of type m if he is fed back exactly $(m-1)$times, where the service time placed at a customer's disposal in stage r is interpreted as the service time he gets after his $(r-1)$th feedback. Because under the above made assumptions the stationary characteristics of the total number of customers in the system do not depend on the selection rule $u(r;k_1,k_2,\ldots)$

it can be shown (see [1]) that, again, the generating function of $(p_{(1,\ldots)}(k))$ easily follows from the Pollaczek-Khinchine formula for the standard $M/GI/1/\infty$ queue and that starting from this various customer-stationary characteristics can be obtained by using relationships of the form as stated in Sect. 4.

Appendix

Simple proof of the relationships stated in Sect. 4 can be provided using the technique of marked point processes as given, for example, in [2] and [5]. To illustrate this we first sketch the proof of Theorem 1.

As shown in [9] (see also [11]) the mixing condition (6) ensures that the limit distributions $(p^{l}_{1,\ldots,c}(k_1,\ldots,k_c))$ considered in Sect. 4 can be represented as follows. Namely, if (6) is satisfied, the probability $p^{l}_{1,\ldots,c}(k_1,\ldots,k_c)$ is equal to the ratio of the time-average mean number

$$\lim_{t\to\infty}\frac{1}{t}\,E\{\mathrm{card}\{n:\ T^{l}_{n}\in[0,t),\ Q^{o}(T^{l}_{n}\pm 0)=(k_1,\ldots,k_c)\}\}$$

of "l-points" per time unit immeadiately after (resp. before) which the system is in the state $(k_1,\ldots,k_c)$ to the time-average mean number

$$\lim_{t\to\infty}\frac{1}{t}\,E\{\mathrm{card}\{n:\ T^{l}_{n}\in[0,t)\}\}$$

of all "l-points" per time unite. Now, in order to prove (7), it suffice to observe that

$$\left|\mathrm{card}\{n:T^{l_1}_{n}\in[0,t),\ \sum_{r=j}^{j+h}Q^{o}_{r}(T^{l_1}_{n}-0)=k\}-\mathrm{card}\{n:T^{l_2}_{n}\in[0,t),\ \sum_{r=j}^{j+h}Q^{o}_{r}(T^{l_2}_{n}+0)=k\}\right|<1$$

for every $t>0$, where $l_1=(\mathrm{i}_j,\mathrm{i}_{j+1},\ldots,\mathrm{i}_{j+h})$ and $l_2=(\mathrm{o}_j,\mathrm{o}_{j+1},\ldots,\mathrm{o}_{j+h})$. Thus,

$$\lim_{t\to\infty}\frac{1}{t}\,E\{\mathrm{card}\{n:\ T^{l_1}_{n}\in[0,t),\ \sum_{r=j}^{j+h}Q^{o}_{r}(T^{l_1}_{n}-0)=k\}\}$$
$$=\lim_{t\to\infty}\frac{1}{t}\,E\{\mathrm{card}\{n:\ T^{l_2}_{n}\in[0,t),\ \sum_{r=j}^{j+h}Q^{o}_{r}(T^{l_2}_{n}+0)=k\}\}$$

and, consequently,

$$\lim_{t\to\infty}\frac{1}{t}\,E\{\mathrm{card}\{n:\ T^{l_1}_{n}\in[0,t)\}\}=\lim_{t\to\infty}\frac{1}{t}\,E\{\mathrm{card}\{n:\ T^{l_2}_{n}\in[0,t)\}\}$$

This gives (7). The relationships (8) and (9) can be proved analogously.

The proof of relationships (10) in Theorem 2 can be provided as follows. Denoting by $\lambda^{l}(k_1,\ldots,k_c)$,

$$\lambda^{l}(k_1,\ldots,k_c)=\lim_{t\to\infty}\frac{1}{t}\,E\{\mathrm{card}\{n:\ T^{l}_{n}\in[0,t),\ Q^{o}(T^{l}_{n}\pm 0)=(k_1,\ldots,k_c)\}\},$$

the above considered time-average mean number of "l-points" per time unit immeadiately after (resp. before) which the system is in the state $(k_1,\dots,k_c)$, and by λ^l,

$$\lambda^l = \lim_{t\to\infty}\frac{1}{t}\,\mathrm{E}\{\mathrm{card}\{n:\ T_n^l\in[0,t)\}\},$$

the time-average mean number of all "l-points", we have

$$p_{1,\dots,c}^{(o_{j_1},\dots,o_{j_r})}(k_1,\dots,k_c) = \frac{\lambda^{(o_{j_1},\dots,o_{j_r})}(k_1,\dots,k_c)}{\lambda^{(o_{j_1},\dots,o_{j_r})}}$$

$$= \sum_{e=1}^{r}\frac{\lambda^{(o_{j_e})}(k_1,\dots,k_c)}{\lambda^{(o_{j_1},\dots,o_{j_r})}} = \sum_{e=1}^{r}\frac{\lambda^{(o_{j_e})}}{\lambda^{(o_{j_1},\dots,o_{j_r})}}\,p_{1,\dots,c}^{(o_{j_e})}(k_1,\dots,k_c).$$

Because $\mathrm{E}\{\mathrm{card}\{n:\ T_n^{(i_j)}\in[0,t)\}\} = \mathrm{E}Q_j^{o}(t) + \mathrm{E}\{\mathrm{card}\{n:\ T_n^{(o_j)}\in[0,t)\}\}$, we get from the proof of Theorem 1 that

$$\lim_{t\to\infty}\frac{1}{t}\,\mathrm{E}Q_j^{o}(t) = 0$$

for every $j = 1,2,\dots,c$ and, consequently,

$$\lim_{t\to\infty}\frac{1}{t}\,\mathrm{E}\{\sum_{j=1}^{e} Q_j^{o}(t)\} = 0$$

for every $e = 1,2,\dots,c$. Thus, we have

$$\lim_{t\to\infty}\frac{1}{t}\,\mathrm{E}\{\mathrm{card}\{n:\ T_n^{i_1}\in[0,t),\ m_n\geqq j_e\}\} = \lambda^{(o_{j_e})}$$

and, consequently,

$$\Pr(m_1\geqq j_e) = \frac{\lambda^{(o_{j_e})}}{\lim_{t\to\infty}\frac{1}{t}\,\mathrm{E}\{\mathrm{card}\{n:\ T_n^{i_1}\in[0,t)\}\}}.$$

This gives (10) taking into consideration that

$$\lambda^{(o_{j_1},\dots,o_{j_r})} = \sum_{f=1}^{r}\lambda^{(o_{j_f})}.$$

In order to prove (13) we can proceed analogously. Namely, for $r<c$ we have

$$p_{1,\dots,c}^{(o_r)}(k_1,\dots,k_c) = \frac{\lambda^{(o_r)}(k_1,\dots,k_c)}{\lambda^{(o_r)}}$$

$$= \frac{1}{\lambda^{(o_r)}}\,[\lambda^{(d_r)}(k_1,\dots,k_c) + \lambda^{(i_{r+1})}(k_1',\dots,k_c')]$$

$$= \frac{\lambda^{(d_r)}}{\lambda^{(o_r)}}\,p_{1,\dots,c}^{(d_r)}(k_1,\dots,k_c) + \frac{\lambda^{(i_{r+1})}}{\lambda^{(o_r)}}\,p_{1,\dots,c}^{(i_{r+1})}(k_1',\dots,k_c')$$

$$= \frac{\Pr(m_1=r)}{\Pr(m_1\geqq r)}\,p_{1,\dots,c}^{(d_r)}(k_1,\dots,k_c) + \frac{\Pr(m_1>r)}{\Pr(m\geqq r)}\,p_{1,\dots,c}^{(i_{r+1})}(k_1',\dots,k_c').$$

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