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STATIONARY WAITING TIME DISTRIBUTION IN A $GI/E_k/m$ QUEUE

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Abstract In this paper, the stationary waiting time distributions $F_q(x)$ and $F(x)$ are explicitly formulated for the $GI/E_k/m$ queue under the first-come first-served discipline. The transition probability matrix and the imbedded probabilities play the important roles in this study. Some numerical results are presented for various systems as $E_\ell/E_k/m$, $U_\ell/E_k/m$, $D/E_k/m$, etc. Further the properties of $F(x)$ are considered.

1. Introduction

The problems of waiting time have been discussed, since Kiefer and Wolfowitz [5] proved that the sequence of waiting times of customers in a GI/G/m queueing system converges in distribution if and only if $\rho<1$. There are researches on the approximate expressions or inequalities for the mean waiting time *EW*, for instance [9] and [11]. We shall treat the function of the stationary waiting time distribution, since it gives us further information. In this field, Lindley [6] has shown the waiting time distribution as an integral equation in a GI/G/1. Kendall [4], Takacs [8] and Tumura [13] have given the formulae in a GI/M/m. Avis [1] has derived the formulae in each $M/E_2/2$, $E_\ell/E_2/2$ and $D/E_2/2$. Hokstad [2] has presented an approximate expression of the distribution function in a M/G/m by extending the formula in a M/G/1. Takahashi [10] has analyzed in a PH/PH/m and Neuts & Takahashi [7] have done in a GI/PH/m, noticing the tail of the distribution function.

In this paper, using the transition probability matrix T and the imbedded probability $\{q_n\}$, we derive exactly $F_q(x)$ and $F(x)$ under the first-come first-served discipline in a $GI/E_k/m$ system. Here $F_q(x)$ and $F(x)$ denote the functions of the stationary waiting time distribution in queue and in the system respectively. $\{q_n\}$ and some parameters which are used in this study can be calculated by the method of [3]. This paper is a sequel to [3]. In section

2, we explain the structure of queueing system and the notation. In section 3, $F_q(x)$ and $F(x)$ are derived exactly. Finally in section 4, some numerical results are expressed in the tables and figures for various inter-arrival distributions E_ℓ, U_ℓ, D, etc. The properties of $F(x)$ are considered.

2. Notation

At first, we shall explain the structure of a $GI/E_k/m$ queueing system and the notation used in this paper. The $GI/E_k/m$ system has arbitrarily distributed inter-arrival times with mean rate λ and an infinite single queue served by m-servers whose service time has the k-stage Erlangian distribution with mean rate μ. Suppose that the traffic intesity $\rho = \lambda/(m\mu)<1$. Let $A(t)$ and $B_k(t)$ be the distribution function of inter-arrival times and service times respectively. We merely assume that the distribution function $A(t)$ is absolutely continuous except for $\{t^\xi\}$ which has not finite cluster values (see [3] and [12]).

The service system consists of m-service channels and each of them is divided into k-phases. The first (or entry) phase is called by *No*.1, the second phase by *No*.2, ... and the last (or exit) phase by *No*.k. Let n be the number of customers in the system (including in service) and n_J be the total number of customers in the J-th phases (summed across all service channels) for J=1,2,...,k. Let $N(\tau)$ denote the state of the system $[n\ ;\ (n_1,n_2,\ldots,n_k)]$ and $t(\tau)$ denote the elapsed time since the last arrival time, at time τ. For each n ($n \geq$ m), the phase-states $\{(n_1,n_2,\ldots,n_k)\}$ are ordered lexicographically in ascending order. The total number of phase-states $\{(n_1,n_2,\ldots,n_k)\}$ is $L = L(\mathrm{k,m}) = \binom{\mathrm{m+k-1}}{\mathrm{m}}$. Letting $(n_1(i),n_2(i),\ldots,n_k(i))$ denote the i-th phase-state ordered above, we briefly use the index notation $[n\ ;\ i]$ instead of $[n\ ;\ (n_1(i),n_2(i),\ldots,n_k(i))]$ for $1 \leq i \leq L$.

According to [3] and [12], there exists the stationary probability density $P_{n;i}(t) = \lim_{\tau\to\infty} \Pr\{t \leq t(\tau)<t+dt$ and $N(\tau)$ is $[n;i]\}/dt$ (in case of an aperiodic function for $A(t)$). The equilibrium equations for the density $\{P_{n;i}(t)\}$ are expressed in the form of difference-differential equations (recurrence formulae) for n≧m,

$$(2.1) \qquad [\frac{d}{dt} + \lambda(t) + \nu]\ \underset{\sim}{P}_n(t) = \nu\ T_1\ \underset{\sim}{P}_n(t) + \nu\ T_2\ \underset{\sim}{P}_{n+1}(t)$$

with initial conditions

$$(2.2) \qquad \underset{\sim}{P}_{n+1}(0) = \int_0^\infty \underset{\sim}{P}_n(t)\ \lambda(t)dt \qquad (n \geq \mathrm{m}),$$

where $\nu = m \cdot k\,\mu, \quad \lambda(t) = \frac{1}{1-A(t)} \cdot \frac{d}{dt} A(t)$

and $\underset{\sim}{P}_n(t) = [P_{n;1}(t), P_{n;2}(t), \ldots, P_{n;L}(t)]'$.

Here the transpose of $(T_1 + T_2)$ is a transition probability matrix. According to a technique for solving difference equations, we assume

(2.3) $\underset{\sim}{P}_{n+1}(t) = \theta \underset{\sim}{P}_n(t) \qquad (n \geqq m, \ |\theta|<1).$

Then the equilibrium equations (2.1) are transformed into the system of linear-differential equations

(2.4) $[\frac{d}{dt} + \lambda(t) + \nu]\, \underset{\sim}{P}_n(t) = \nu\, T(\theta)\, \underset{\sim}{P}_n(t) \qquad (n \geqq m),$

where $T(\theta) = T_1 + \theta\, T_2$.

A general solution of (2.4) is given by

(2.5) $$\begin{aligned}\underset{\sim}{P}_n(t) &= [1 - A(t)]\, e^{-\nu t} \exp\{\nu\, T(\theta) t\}\, \underset{\sim}{H}_n \\ &= [1 - A(t)]\, e^{-\nu t} R D(t) R^{-1} \underset{\sim}{H}_n \qquad (n \geqq m),\end{aligned}$$

where $T(\theta) = R \Lambda_2 R^{-1} \quad (T(\theta)\, \underset{\sim}{r}_j = \beta_j\, \underset{\sim}{r}_j, \ \| \underset{\sim}{r}_j \| = 1),$

$R = [\underset{\sim}{r}_1, \underset{\sim}{r}_2, \ldots, \underset{\sim}{r}_L] = [r_{i,j}], \qquad \Lambda_2 = diag\{\beta_1, \beta_2, \ldots, \beta_L\}$

$(\beta_j = \beta_j(\theta), \ \underset{\sim}{r}_j = \underset{\sim}{r}_j(\theta), \ 1 \leq j \leq L),$

$D(t) = diag\{\exp(\nu\beta_1 t), \exp(\nu\beta_2 t), \ldots, \exp(\nu\beta_L t)\}$

and $\underset{\sim}{H}_n$ is an integration constant vector.

Substituting (2.5) into (2.2) and (2.3), we have

$$\begin{aligned}\underset{\sim}{H}_{n+1} &= R \int_0^\infty e^{-\nu t} D(t)\, dA(t) R^{-1} \underset{\sim}{H}_n \\ &= R \Lambda_1 R^{-1} \underset{\sim}{H}_n \qquad (n \geqq m),\end{aligned}$$

where $\theta_j = \int_0^\infty \exp\{\nu(\beta_j - 1)t\}\, dA(t) \qquad (|\theta_j|<1)$

and $\Lambda_1 = diag\{\theta_1, \theta_2, \ldots, \theta_L\}$.

Allowing us to write $\underset{\sim}{H}_n = \lambda R \Lambda_1^{n-m} \underset{\sim}{h}$, (2.5) becomes

$$\underset{\sim}{P}_n(t) = \lambda [1 - A(t)]\, e^{-\nu t} R D(t) \Lambda_1^{n-m} \underset{\sim}{h} \qquad (n \geqq m),$$

where the integration constant vector $\underset{\sim}{h} = [h_1, h_2, \ldots, h_L]'$ is determined by the system of linear equations for $\{P_{n;i}(t)\}$ $(n=0,1,\ldots,m)$.

Consequently the stationary probabilities $\underset{\sim}{p}_n = [p_{n;1}, p_{n;2}, \ldots, p_{n;L}]'$ and $\underset{\sim}{q}_n = [q_{n;1}, q_{n;2}, \ldots, q_{n;L}]'$ (the imbedded probability, just before arrival time) are formulated as

$$\underset{\sim}{p}_n = \frac{\lambda}{\nu} R\Lambda_1^{n-m} [I - \Lambda_1] [I - \Lambda_2]^{-1} \underset{\sim}{h} \qquad (n \geqq m),$$

$$(2.6) \qquad \underset{\sim}{q}_n = \sum_{j=1}^{L} \theta_j^{n-m+1} h_j \underset{\sim}{r}_j$$

$$= R\Lambda_1^{n-m+1} \underset{\sim}{h} \qquad (n \geqq m),$$

where $\quad p_{n;i} = \int_0^\infty p_{n;i}(t)dt$,

$$q_{n;i} = \Pr\{N(\tau_c -) \text{ is } [n;i]\} \qquad (\text{as } c \to \infty)$$

$$= \frac{1}{\lambda} \int_0^\infty p_{n;i}(t)\lambda(t)dt$$

and $\{\tau_c\}$ indicates the sequence of arrival time of customers.

The parameters (θ_1, β_1), (θ_2, β_2), ... and (θ_L, β_L) are given by the L-sets of solutions of the following simultaneous equations

$$|\beta I - T(\theta)| = 0 \quad \text{and} \quad \theta = \int_0^\infty \exp\{\nu(\beta-1)t\}dA(t)$$

where $\quad |\theta|<1$ and I is the L-dimensional unit matrix.

For further details, see [3] (α^k in [3] has been replaced by θ here).

3. Waiting Time Distribution

In this section, under the first-come first-served discipline, we shall derive an explicit expression for the stationary distribution functions

$$F_q(x) = \Pr\{w_q \leqq x\} \quad \text{and} \quad F(x) = \Pr\{w \leqq x\} ,$$

where w_q and w indicate the waiting-time in queue and in the system respectively.

When some servers are idle, an arriving customer need not wait in queue. So we have $\Pr\{w_q = 0 \mid [n;i]\} = 1$ (for $n=0,1,\ldots,m-1$). Here we use the notation $\Pr\{\cdot|[n;i]\}$ instead of $\Pr\{\cdot|N(\tau_c-) \text{ is } [n;i]\}$.

Thus $\quad Q_0 = \Pr\{w_q = 0\}$

$$= \sum_{n=0}^{m-1} \sum_i \Pr\{w_q = 0 \mid [n;i]\}\, q_{n;i}$$

$$= \sum_{n=0}^{m-1} q_n \quad (= 1 - \sum_{n=m}^{\infty} q_n)$$

$$= 1 - \underset{\sim}{e}' R\Lambda_1 [I - \Lambda_1]^{-1} \underset{\sim}{h} ,$$

where $\underset{\sim}{e}' = [1,1,\ldots,1]$.

On the other hand, when all servers are busy, an arriving customer must wait in queue, and the transition processes in the service-phase will be a Poisson process of rate ν. For a customer in queue, let K denote the number of phase-steps in the service phase to reach a service channel counting from his arrival time. So we have $\Pr\{0 < w_q \leq x \mid [n;i]\}$

$$= \sum_{\ell=1}^{\infty} \Pr\{K=\ell \mid [n;i]\} \int_0^x \frac{(\nu y)^{\ell-1}}{(\ell-1)!} \nu e^{-\nu y} dy \qquad (n \geq m)$$

and $\quad Q_\ell = \Pr\{K=\ell\}$

$$= \sum_{n=m}^{\infty} \sum_{i=1}^{L} \Pr\{K=\ell \mid [n;i]\} q_{n;i}$$

Thus $\quad \Pr\{0 < w_q \leq x\} = \sum_{n=m}^{\infty} \sum_{i=1}^{L} \Pr\{0 < w_q \leq x \mid [n;i]\} q_{n;i}$

$$= \sum_{\ell=1}^{\infty} Q_\ell \int_0^x \frac{(\nu y)^{\ell-1}}{(\ell-1)!} \nu e^{-\nu y} dy .$$

Therefore $F_q(x)$ and $F(x)$ can be formulated as follows;

$$F_q(0) = Q_0,$$

(3.1) $$F_q(x) = Q_0 + \int_0^x f_q(y)dy \qquad (x > 0),$$

$$F(x) = \int_0^x F_q(x-y)dB_k(y) \qquad (x \geq 0),$$

where

(3.2) $$f_q(x) = \nu e^{-\nu x} \sum_{\ell=1}^{\infty} \frac{(\nu x)^{\ell-1}}{(\ell-1)!} Q_\ell \qquad (x > 0),$$

and $$dB_k(x) = \frac{(k\mu)^k}{(k-1)!} x^{k-1} e^{-k\mu x} dx \qquad (x \geq 0).$$

So we need the probabilities Q_ℓ or $\Pr\{K=\ell \mid [n;i]\}$.

In order to derive Proposition concerning Q_ℓ in the general system $GI/E_k/m$ we shall analyze a $GI/E_3/2$. We have $L = L(3, 2) = 6$, $\nu = 6\mu$ and the order of states (n_1,n_2,n_3) are set as follows $(n \geqq 2)$;

(n_1,n_2,n_3)	(0,0,2)	(0,1,1)	(0,2,0)	(1,0,1)	(1,1,0)	(2,0,0)
i	1	2	3	4	5	6

Then from (2.4), the matrix $T(\theta)$ is explicitly given by

$$T(\theta) = \begin{pmatrix} 0 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 0 \\ \theta & 0 & 0 & 0 & 1/2 & 0 \\ 0 & \theta/2 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \theta/2 & 0 & 0 \end{pmatrix}$$

Since the probabilities $\Pr\{K=\ell \mid [n;i]\}$ are put in order as follows;

$\Pr\{K=\ell \mid [n;i]\}$ $n=2,3$ (k=3, m=2)

$n;i$ \ ℓ	1	2	3	4	5	6	7	8
2;1	1							
2;2	1/2	1/2						
2;3		1/2	1/2					
2;4	1/2	1/4	1/4					
2;5		1/4	3/8	3/8				
2;6			1/4	3/8	3/8			
3;1		1/2	1/4	1/4				
3;2			3/8	5/16	5/16			
3;3				3/8	5/16	5/16		
3;4				5/16	11/32	11/32		
3;5					11/32	21/64	21/64	
3;6						11/32	21/64	21/64

Q_ℓ are given by

$$Q_1 = q_{2;1} + \frac{1}{2} q_{2;2} + \frac{1}{2} q_{2;4}$$

$$Q_2 = \frac{1}{2} q_{3;1} + \frac{1}{2} q_{2;2} + \frac{1}{2} q_{2;3} + \frac{1}{4} q_{2;4} + \frac{1}{4} q_{2;5}$$

(3.3) $$Q_3 = \frac{1}{4} q_{3;1} + \frac{3}{8} q_{3;2} + \frac{1}{2} q_{2;3} + \frac{1}{4} q_{2;4} + \frac{3}{8} q_{2;5} + \frac{1}{4} q_{2;6}$$

$$Q_4 = \frac{1}{4} q_{3;1} + \frac{5}{16} q_{3;2} + \frac{3}{8} q_{3;3} + \frac{5}{16} q_{3;4} + \frac{3}{8} q_{2;5} + \frac{3}{8} q_{2;6}$$

$$Q_5 = \frac{5}{16} q_{4;1} + \frac{5}{16} q_{3;2} + \frac{5}{16} q_{3;3} + \frac{11}{32} q_{3;4} + \frac{11}{32} q_{3;5} + \frac{3}{8} q_{2;6}$$

$$\vdots$$

However, it is difficult to obtain all Q_ℓ in this manner.

In order to overcome this difficulty we introduce a linear operator T which defines

(3.4) $\mathsf{T}R = R\Lambda_2$.

Using $T(\theta_j)\, \underset{\sim}{r}_j = \beta_j\, \underset{\sim}{r}_j$, we have

$$R\Lambda_2 = [\underset{\sim}{r}_1, \underset{\sim}{r}_2, \ldots, \underset{\sim}{r}_L]\Lambda_2$$
$$= [\beta_1\, \underset{\sim}{r}_1, \beta_2\, \underset{\sim}{r}_2, \ldots, \beta_L\, \underset{\sim}{r}_L]$$
$$= [T(\theta_1)\, \underset{\sim}{r}_1,\ T(\theta_2)\, \underset{\sim}{r}_2, \ldots,\ T(\theta_L)\, \underset{\sim}{r}_L]\ .$$

Namely, the transformation $\mathsf{T}R$ is corresponding to the product of the matrix $T(\theta_j)$ and the vector $\underset{\sim}{r}_j$ for each j.

Moreover we introduce a L-dimensional row-vector

(3.5) $$\underset{\sim}{d}' = \underset{\sim}{e}'\, T_2$$
$$= [d_1, d_2, \ldots, d_L]$$
$$= \frac{1}{m}\, [n_k(1),\ n_k(2), \ldots,\ n_k(L)]$$

where $[n;i]$ is $[n;\ (n_1(i),\ n_2(i), \ldots,\ n_k(i))]$.

That is, d_i implies the utilization rate of the exit service-phases in the state $[n;\ (n_1(i),\ n_2(i), \ldots,\ n_k(i))]$ for $i=1,2,\ldots,L$.

In case of $GI/E_3/2$, (3.4) and (3.5) becomes

(3.4)' $$T(\theta_j)\, \underset{\sim}{r}_j = T(\theta_j)\ [r_{1,j},\ r_{2,j},\ r_{3,j},\ r_{4,j},\ r_{5,j},\ r_{6,j}]'$$
$$= [\tfrac{1}{2}\, r_{2,j}\ ,\ (r_{3,j} + \tfrac{1}{2}\, r_{4,j}),\ \tfrac{1}{2}\, r_{5,j}\ ,$$
$$(\theta_j\, r_{1,j} + \tfrac{1}{2}\, r_{5,j}),\ (\tfrac{1}{2}\, \theta_j\, r_{2,j} + r_{6,j}),\ \tfrac{1}{2}\, \theta_j\, r_{4,j}]'$$
$$(1 \leqq j \leqq 6)$$

and

(3.5)' $$\underset{\sim}{d}' = [1,\ 1/2,\ 0,\ 1/2,\ 0,\ 0]\ .$$

When we consider a state probability vector

(3.6) $$\underset{\sim}{q}_2 = R\Lambda_1\, \underset{\sim}{h}$$
$$= \sum_{j=1}^{6} \theta_j\, h_j\, \underset{\sim}{r}_j\ ,$$

it follows from (2.6) and (3.4)'

$$(3.7)\qquad \mathrm{T}\underset{\sim}{q}_2 = \sum_{j=1}^{6} \theta_j h_j T(\theta_j) \underset{\sim}{r}_j$$

$$= \sum_{j=1}^{6} \theta_j h_j [\tfrac{1}{2} r_{2,j},\ (r_{3,j} + \tfrac{1}{2} r_{4,j}),\ \tfrac{1}{2} r_{5,j},$$

$$(\theta_j r_{1,j} + \tfrac{1}{2} r_{5,j}),\ (\tfrac{1}{2}\theta_j r_{2,j} + r_{6,j}),\ \tfrac{1}{2}\theta_j r_{4,j}]'$$

$$= [\tfrac{1}{2} q_{2;2},\ (q_{2;3} + \tfrac{1}{2} q_{2;4}),\ \tfrac{1}{2} q_{2;5},$$

$$(q_{3;1} + \tfrac{1}{2} q_{2;5}),\ (\tfrac{1}{2} q_{3;2} + q_{2;6}),\ \tfrac{1}{2} q_{3;4}]',$$

$$(3.8)\qquad \mathrm{T}^2\underset{\sim}{q}_2 = \sum_{j=1}^{6} \theta_j h_j T^2(\theta_j) \underset{\sim}{r}_j$$

$$= \sum_{j=1}^{6} \theta_j h_j [(\tfrac{1}{2} r_{3,j} + \tfrac{1}{4} r_{4,j}),\ (\tfrac{1}{2}\theta_j r_{1,j} + \tfrac{3}{4} r_{5,j}),$$

$$(\tfrac{1}{4}\theta_j r_{2,j} + \tfrac{1}{2} r_{6,j}),\ (\tfrac{3}{4}\theta_j r_{2,j} + \tfrac{1}{2} r_{6,j}),$$

$$(\tfrac{1}{2}\theta_j r_{3,j} + \tfrac{3}{4}\theta_j r_{4,j}),\ (\tfrac{1}{2}\theta_j^2 r_{1,j} + \tfrac{1}{4}\theta_j r_{5,j})]'$$

$$= [(\tfrac{1}{2} q_{2;3} + \tfrac{1}{4} q_{2;4}),\ (\tfrac{1}{2} q_{3;1} + \tfrac{3}{4} q_{2;5}),\ (\tfrac{1}{4} q_{3;2} + \tfrac{1}{2} q_{2;6}),$$

$$(\tfrac{3}{4} q_{3;2} + \tfrac{1}{2} q_{2;6}),\ (\tfrac{1}{2} q_{3;3} + \tfrac{3}{4} q_{3;4}),\ (\tfrac{1}{2} q_{4;1} + \tfrac{1}{4} q_{2;5})]'.$$

From (3.3), (3.4)$'$, (3.5)$'$, (3.6), (3.7) and (3.8), the relation

$$(3.9)\qquad \begin{aligned} Q_1 &= \underset{\sim}{d}'\underset{\sim}{q}_2 \\ Q_2 &= \underset{\sim}{d}'\mathrm{T}\underset{\sim}{q}_2 \\ Q_3 &= \underset{\sim}{d}'\mathrm{T}^2\underset{\sim}{q}_2 \end{aligned}$$

holds.

This implies as follows:

The one-step transition probability Q_1 consists of a state probability vector $\underset{\sim}{q}_2$.

The two-step transition probability Q_2 consists of a $\mathrm{T}\underset{\sim}{q}_2$ which is the image of $\underset{\sim}{q}_2$ by the operator T.

Three-step transition probability Q_3 consists of a $\mathrm{T}^2\underset{\sim}{q}_2$, which is the image of $\mathrm{T}\underset{\sim}{q}_2$ by the operator T.

For the GI/E$_k$/m systems, it is seen that the relation corresponding to (3.9) holds. The operator T yields the product $T(\theta_j)\ \underset{\sim}{r}_j$ $(j=1,2,\ldots,L)$. And the transition probability matrix $T(\theta)$ is produced from the equilibrium equation

(2.1) which is basically made as Kolmogorov's back-ward equations. This means that the operator T makes just one stepped-back state vector from a certain vector. Hence we obtain the following Proposition concerning Q_ℓ.

Proposition: The state probability vectors $\mathbf{Q}_\ell$ and a row vector $\mathbf{d}'$ are defined by

$$\mathbf{Q}_1 = \mathbf{q}_m$$

$$\mathbf{Q}_\ell = \mathrm{T}\mathbf{Q}_{\ell-1} \qquad (\ell \geqq 2)$$

$$\mathbf{d}' = \mathbf{e}'T_2 ,$$

then the ℓ-step transition probabilities Q_ℓ are given

$$Q_\ell = \mathbf{d}'\mathbf{Q}_\ell \qquad (\ell \geqq 1).$$

Accordingly, the relations

$$\mathbf{Q}_\ell = \mathrm{T}^{\ell-1}\mathbf{Q}_1$$

$$= \mathrm{T}^{\ell-1}R\Lambda_1 \mathbf{h}$$

$$= R\Lambda_2^{\ell-1}\Lambda_1 \mathbf{h}$$

and $$Q_\ell = \mathbf{d}'R\Lambda_2^{\ell-1}\Lambda_1 \mathbf{h} \qquad (\ell \geqq 1)$$

are obtained easily. Since the probabilities of

$$\sum_{n=m}^{\infty} q_n = \mathbf{e}'R\,[I - \Lambda_1]^{-1}\,\Lambda_1\,\mathbf{h}$$

and

$$\sum_{\ell=1}^{\infty} Q_\ell = \mathbf{d}'R\,[I - \Lambda_2]^{-1}\,\Lambda_1\,\mathbf{h}$$

equal the probability $1 - Q_0$, we have

$$\mathbf{d}'R = \mathbf{e}'R\,[I - \Lambda_1]^{-1}\,[I - \Lambda_2] .$$

So the probabilities Q_ℓ are rewritten as

$$Q_0 = 1 - \mathbf{e}'R\,[I - \Lambda_1]^{-1}\,\Lambda_1\,\mathbf{h}$$

$$Q_\ell = \mathbf{e}'R\,[I - \Lambda_1]^{-1}\,[I - \Lambda_2]\,\Lambda_2^{\ell-1}\,\Lambda_1\,\mathbf{h} \qquad (\ell \geqq 1).$$

Consequently, (3.2) and (3.1) are explicitly formulated by

$$f_q(x) = \nu \exp\{-\nu x\}\, \underset{\sim}{e}' R\, [I - \Lambda_1]^{-1}\, [I - \Lambda_2]\, \Lambda_1 D(x)\, \underset{\sim}{h}$$

$$= \nu \sum_{j=1}^{L} (1-\beta_j)\, s_j \exp\{-\nu\,(1-\beta_j)\, x\} \qquad (x>0),$$

$$F_q(x) = 1 - \exp\{-\nu x\}\, \underset{\sim}{e}' R\, \Lambda_1\, [I - \Lambda_1]^{-1}\, D(x)\, \underset{\sim}{h}$$

$$= 1 - \sum_{j=1}^{L} s_j \exp\{-\nu\,(1-\beta_j)\, x\} \qquad (x \geqq 0),$$

(3.10)

$$f(x) = [1 - \sum_{j=1}^{L} s_j\, \gamma_j^{-1}]\, \frac{(k\mu)^k}{(k-1)!}\, x^{k-1} \cdot \exp\{-k\mu x\}$$

$$+ \nu \sum_{j=1}^{L} s_j\, \gamma_j^{-k}\, (1-\beta_j)\, [\exp\{-\nu\,(1-\beta_j)\, x\}$$

$$- \exp\{-k\mu x\} \sum_{i=0}^{k-2} \frac{(k\mu x)^i}{i\,!}\, \gamma_j^{\,i}] \qquad (x>0),$$

$$F(x) = B_k(x) - \sum_{j=1}^{L} s_j\, \gamma_j^{-k}\, [\exp\{-\nu\,(1-\beta_j)\, x\}$$

$$- \exp\{-k\mu x\} \sum_{i=0}^{k-1} \frac{(k\mu x)^i}{i\,!}\, \gamma_j^{\,i}] \qquad (x>0),$$

where $s_j = \frac{1}{1-\theta_j} h_j\, \theta_j \cdot \sum_{i=1}^{L} r_{i,j}$ and $\gamma_j = 1 - m(1-\beta_j)$ $(1 \leqq j \leqq L)$.

Remark: If we need the virtual waiting time distribution $\overline{F}_q(x)$ and $\overline{F}(x)$, we may use $\underset{\sim}{p}_m$ instead of $\underset{\sim}{q}_m$ in Q_ℓ;

$$\underset{\sim}{p}_m = \frac{\lambda}{\nu} R\, [I - \Lambda_1]\, [I - \Lambda_2]^{-1}\, \underset{\sim}{h} \quad \text{and} \quad \underset{\sim}{q}_m = R\Lambda_1\, \underset{\sim}{h}\ .$$

Thus we can get

$$\overline{F}_q(x) = 1 - \frac{\lambda}{\nu} \exp\{-\nu x\}\, \underset{\sim}{e}' R\, [I - \Lambda_2]^{-1}\, D(x)\, \underset{\sim}{h} \qquad (x \geqq 0),$$

$$\overline{F}(x) = \int_0^x \overline{F}_q\,(x-y)\, dB_k(y) \qquad (x > 0).$$

4. Numerical examples and Consideration

For the GI/E_k/m, when the inter-arrival distribution $A(t)$ is set, the parameters $\{\theta_j, \beta_j\}$, the eigen vectors $\underset{\sim}{r}_j$ and the integration constant vector $\underset{\sim}{h}$ can be numerically calculated by the method of [3]. By substituting these arguments into (3.10) the numerical results are obtained. We deal with the following four-type inter-arrival distributions with mean rate λ:

E_ℓ; the ℓ-stage Erlangian distribution

(clearly, E_1 indicates the exponential distribution M)

$$dA(t) = \frac{(\ell\lambda)^\ell}{(\ell-1)!} t^{\ell-1} \exp\{-\ell\lambda t\}\, dt \qquad (t>0)$$

$c.v. = \sqrt{1/\ell}$ (a coefficient of variation).

U_ℓ; the uniform distribution

(a typical example of the general distributions)

$$dA(t) = \frac{\lambda}{2\,d_\ell}\, dt \qquad (\frac{1}{\lambda}[1-d_\ell] \leqq t \leqq \frac{1}{\lambda}[1+d_\ell])$$

$c.v. = \sqrt{1/\ell}$, where $d_\ell = \sqrt{3/\ell}$ $(\ell \geqq 3)$.

S_ℓ; the *SINE*-curve distribution

(an example of the unimodal distributions)

$$dA(t) = \frac{\lambda}{2\,d_\ell}\{1 + \sin(\frac{\lambda}{d_\ell}t - \frac{1}{d_\ell} + \frac{1}{2})\pi\}\, dt$$

$(\frac{1}{\lambda}[1-d_\ell] \leqq t \leqq \frac{1}{\lambda}[1+d_\ell])$ $c.v. = \sqrt{1/\ell}$,

where $d_\ell = \pi\sqrt{3/\{\ell(\pi^2-6)\}}$ $(\ell \geqq 3\pi^2/(\pi^2-6))$.

D ; the deterministic distribution

(an example of the periodic distributions)

$$dA(t) = \delta(t - \frac{1}{\lambda})\, dt \qquad c.v. = 0.$$

The numerical experiments are performed on the IBM 370 with double precision. Some results [ρ=0.3, 0.6, 0.9; k=2 (m=2,3,4,5) and k=3 (m=2,3)] are shown in Tables and Figures. We will use the following symbols.

$$EW = \int_0^\infty x\, dF(x). \qquad SW^2 = \int_0^\infty x^2\, dF(x) - EW^2 ,$$

$$EW_q = \int_0^\infty x\, dF_q(x), \qquad SW_q^{\,2} = \int_0^\infty x^2\, dF_q(x) - EW_q^{\,2} ,$$

$a = EW - SW$, $b = EW$, $c = EW + SW$ and $d = EW + 2SW$.

Considering the numerical results, the following properties of $F(x)$ are found.

i) There holds the quasi-constancy of the percent levels in various $GI/E_k/m$ systems,

$F(EW-SW) \sim 0.1$, $F(EW) \sim 0.6$,

$F(EW+SW) \sim 0.85$ and $F(EW+2SW) \sim 0.95$.

It may be considered that there are the effects of Erlang distribution.

ii) For fixed ρ, we consider the two types of inter-arrival distributions in the $GI/E_k/m$ systems. For the two systems which have the same value of *c.v.*, the percent levels of the waiting time distribution are nearly equal. The value of *c.v.* has a great influence on the waiting time distribution $F(x)$ for either a small ρ or a large m. In other words, the second moment plays an important role when ρ is fixed.

Table 1 $GI/E_2/2$

ρ	$A(t)$	μEW	μSW_q	μSW	$F_q(0)$	$F(a)$	$F(b)$	$F(c)$	$F(d)$
.3	M	1.07709	.27074	.75717	.86268	.11958	.59263	.85385	.95342
	U_8	1.00342	.04717	.70868	.98988	.11755	.59379	.85469	.95340
	S_8	1.00391	.05108	.70895	.98885	.11759	.59376	.85468	.95340
	E_8	1.00279	.04235	.70837	.99166	.11750	.59382	.85470	.95340
	U_{12}	1.00196	.03514	.70798	.99394	.11743	.59387	.85472	.95339
	S_{12}	1.00213	.03680	.70806	.99394	.11744	.59386	.85472	.95339
	E_{12}	1.00177	.03334	.70789	.99451	.11741	.59388	.85472	.95339
	D	1.00049	.01704	.70731	.99837	.11729	.59396	.85475	.95338
.6	M	1.42877	.77011	1.04550	.55288	.10854	.59915	.85710	.95311
	U_8	1.09477	.28153	.76109	.80549	.12213	.59082	.85330	.95364
	S_8	1.09515	.28272	.76153	.80570	.12209	.59085	.85330	.95364
	E_8	1.09079	.27351	.75816	.81004	.12213	.59080	.85332	.95365
	U_{12}	1.07990	.25293	.75098	.82605	.12192	.59093	.85341	.95365
	S_{12}	1.08009	.25351	.75118	.82607	.12190	.59094	.85341	.95365
	E_{12}	1.07818	.24933	.74978	.82828	.12190	.59094	.85342	.95365
	D	1.05468	.20072	.73504	.86669	.12112	.59141	.85371	.95363
.9	M	4.20734	3.67633	3.74371	.14883	.04337	.62801	.86450	.95065
	U_8	2.20377	1.55800	1.71096	.26664	.08819	.61048	.86198	.95231
	S_8	2.20368	1.55818	1.71112	.26685	.08815	.61049	.86198	.95231
	E_8	2.19758	1.54982	1.70351	.26637	.08862	.61025	.86193	.95234
	U_{12}	2.10782	1.45193	1.61496	.27820	.09219	.60831	.86142	.95252
	S_{12}	2.10779	1.45202	1.61504	.27830	.09218	.60831	.86143	.92252
	E_{12}	2.10491	1.44808	1.61150	.27811	.09240	.60819	.86140	.95253
	D	1.92114	1.24433	1.43121	.30582	.10054	.06356	.85999	.95297

Table 2 $GI/E_2/3$

ρ	$A(t)$	μEW	μSW_q	μSW	$F_q(0)$	$F(a)$	$F(b)$	$F(c)$	$F(d)$
.3	M	1.02666	.13463	.71981	.93087	.11959	.59241	.85417	.95354
	U_8	1.00057	.01606	.70729	.99756	.11731	.59395	.85475	.95338
	S_8	1.00066	.01747	.70732	.99727	.11732	.59394	.85475	.95338
	E_8	1.00045	.01411	.70725	.99806	.11729	.59396	.85476	.95338
	U_{12}	1.00029	.01125	.70720	.99870	.11728	.59397	.85476	.95338
	S_{12}	1.00032	.01181	.70721	.99861	.11728	.59397	.85476	.95338
	E_{12}	1.00026	.01055	.70719	.99885	.11727	.59397	.85476	.95338
	D	1.00005	.00471	.70712	.99974	.11725	.59399	.85476	.95338
.6	M	1.22785	.47558	.85216	.64919	.12309	.59013	.85308	.95374
	U_8	1.04297	.15955	.72488	.87130	.12130	.59114	.85390	.95372
	S_8	1.04315	.16019	.72502	.87134	.12130	.59114	.85389	.95372
	E_8	1.04111	.15487	.72387	.87456	.12117	.59123	.85394	.95371
	U_{12}	1.03566	.14209	.72124	.88693	.12073	.59153	.85406	.95368
	S_{12}	1.03574	.14239	.72130	.88691	.12074	.59153	.85406	.95368
	E_{12}	1.03486	.14001	.72083	.88849	.12067	.58157	.85408	.95367
	D	1.02354	.11056	.71570	.91671	.11966	.59227	.85432	.95359
.9	M	3.05336	2.43905	2.53948	.18549	.06595	.62112	.86393	.95131
	U_8	1.75578	1.02700	1.24689	.31434	.11305	.59581	.85746	.95371
	S_8	1.75574	1.02711	1.24698	.31448	.11303	.59583	.85746	.95371
	E_8	1.75205	1.02167	1.24250	.31405	.11337	.59560	.85738	.95373
	U_{12}	1.69461	.95654	1.18953	.32647	.11615	.59380	.85660	.95388
	S_{12}	1.69459	.95660	1.18957	.32654	.11614	.59380	.85660	.95388
	E_{12}	1.69285	.95404	1.18752	.32636	.11630	.58370	.85656	.95389
	D	1.57590	.81875	1.08183	.35503	.12194	.58994	.85491	.95419

Table 3 $GI/E_2/4$

ρ	$A(t)$	μEW	μSW_q	μSW	$F_q(0)$	$F(a)$	$F(b)$	$F(c)$	$F(d)$
.3	M	1.01078	.07575	.71115	.96352	.11841	.59317	.85455	.95348
	U_8	1.00011	.00621	.70713	.99938	.11725	.59399	.85476	.95338
	S_8	1.00013	.00681	.70714	.99930	.11726	.59398	.85476	.95338
	E_8	1.00008	.00534	.70713	.99953	.11725	.59399	.85476	.95338
	U_{12}	1.00005	.00408	.70712	.99971	.11725	.59399	.85476	.95338
	S_{12}	1.00005	.00430	.70712	.99969	.11725	.59399	.85476	.95338
	E_{12}	1.00004	.00378	.70712	.99975	.11725	.59399	.85476	.95338
	D	1.00001	.00146	.70711	.99996	.11724	.59399	.85476	.95338
.6	M	1.13950	.33082	.78067	.71718	.12561	.58824	.85254	.95397
	U_8	1.02259	.10229	.71447	.91167	.11962	.59227	.85439	.95360
	S_8	1.02268	.10270	.71453	.91164	.11963	.59227	.85439	.95360
	E_8	1.02156	.09917	.71403	.91416	.11952	.59235	.85441	.95359
	U_{12}	1.01842	.09027	.71285	.92383	.11919	.59258	.85448	.95356
	S_{12}	1.01846	.09046	.71287	.92379	.11920	.59258	.85448	.95356
	E_{12}	1.01799	.08889	.71267	.92499	.11915	.59261	.85449	.95356
	D	1.01168	.06878	.71044	.94622	.11847	.53910	.85460	.95350
.9	M	2.48727	1.82030	1.95282	.21565	.08366	.61309	.86274	.95208
	U_8	1.53840	.76185	1.03943	.35306	.12514	.58760	.85390	.95441
	S_8	1.53802	.76193	1.03949	.35317	.12513	.58761	.85391	.95441
	E_8	1.53543	.75794	1.03657	.35279	.12535	.58744	.85384	.95442
	U_{12}	1.49382	.70921	1.00149	.36567	.12705	.58625	.85323	.95451
	S_{12}	1.49381	.70925	1.00151	.36573	.12704	.58625	.85323	.95451
	E_{12}	1.49260	.70737	1.00019	.36557	.12714	.58618	.85320	.95452
	D	1.40820	.60630	.93145	.39510	.13012	.58401	.85210	.95468

Table 4 $GI/E_2/5$

ρ	$A(t)$	μEW	μSW_q	μSW	$F_q(0)$	$F(a)$	$F(b)$	$F(c)$	$F(d)$
.3	M	1.00476	.04569	.70858	.98020	.11777	.59361	.85469	.95343
	U_8	1.00002	.00258	.70711	.99984	.11725	.59399	.85476	.95338
	S_8	1.00003	.00285	.70711	.99981	.11725	.59399	.85476	.95338
	E_8	1.00002	.00217	.70711	.99988	.11725	.59399	.85476	.95338
	U_{12}	1.00001	.00159	.70711	.99993	.11724	.59399	.85476	.95338
	S_{12}	1.00001	.00168	.70711	.99993	.11724	.59399	.85476	.95338
	E_{12}	1.00001	.00145	.70711	.99994	.11724	.59399	.85476	.95338
	D	1.00000	.00048	.70711	.99999	.11724	.59399	.85476	.95338
.6	M	1.09250	.24580	.74861	.76800	.12480	.58863	.85301	.95400
	U_8	1.01290	.07023	.71059	.93797	.11858	.59301	.85460	.95351
	S_8	1.01296	.07051	.71061	.93791	.11859	.59301	.85460	.95351
	E_8	1.01228	.06800	.71037	.93990	.11851	.59306	.85461	.95350
	U_{12}	1.01034	.06140	.70977	.94750	.11830	.59322	.85464	.95348
	S_{12}	1.01036	.06153	.70978	.94742	.11830	.59321	.85464	.95348
	E_{12}	1.01008	.06042	.70968	.94838	.11827	.59324	.85465	.95348
	D	1.00629	.04578	.70859	.96451	.11786	.59354	.85470	.95344
.9	M	2.15331	1.44905	1.61238	.24160	.09724	.60556	.86103	.95282
	U_8	1.41069	.60300	.92930	.38591	.13059	.58357	.85201	.95475
	S_8	1.41068	.60305	.92934	.38600	.13058	.58357	.85201	.95475
	E_8	1.40871	.59992	.92731	.38569	.13071	.58347	.85197	.95476
	U_{12}	1.37644	.56104	.90264	.39893	.13151	.58286	.85167	.95481
	S_{12}	1.37644	.56107	.90266	.39898	.13151	.58286	.85167	.95481
	E_{12}	1.37551	.55959	.90174	.39885	.13156	.58282	.85165	.95481
	D	1.31026	.47907	.85411	.42908	.13257	.58200	.85129	.95489

Table 5 $GI/E_3/2$

ρ	$A(t)$	μEW	μSW_q	μSW	$F_q(0)$	$F(a)$	$F(b)$	$F(c)$	$F(d)$
.3	M	1.06987	.24006	.69514	.86336	.13394	.57852	.85129	.95553
	U_3	1.02253	.12744	.59125	.94700	.13559	.57692	.85077	.95576
	U_4	1.00990	.07759	.58254	.97097	.13584	.57660	.85076	.95583
	U_5	1.00530	.05434	.57990	.98254	.13576	.57664	.85081	.95583
	U_6	1.00328	.04165	.57885	.98843	.13571	.57669	.85084	.95583
	U_7	1.00225	.03389	.57834	.99172	.13567	.57672	.85085	.95582
	U_8	1.00166	.02873	.57806	.99371	.13565	.57674	.85086	.95582
	U_9	1.00129	.02509	.57790	.99501	.13563	.57675	.85087	.95582
	U_{10}	1.00104	.02241	.57778	.99589	.13562	.57676	.85087	.95582
	U_{11}	1.00087	.02035	.57771	.99652	.13561	.57677	.85088	.95582
	U_{12}	1.00074	.01873	.57765	.99700	.13561	.57678	.85088	.95581
	D	1.00008	.00572	.57738	.99965	.13558	.57681	.85089	.95581
.6	M	1.38416	.67848	.89088	.55453	.11391	.59404	.85795	.95406
	U_3	1.15424	.36817	.68475	.72834	.13132	.58078	.85228	.95531
	U_4	1.11389	.29768	.64957	.76610	.13456	.57799	.85106	.95561
	U_5	1.09211	.25777	.63228	.79155	.13578	.57687	.85061	.95575
	U_6	1.07877	.23239	.62236	.80955	.13632	.57634	.85043	.95583
	U_7	1.06988	.21493	.61606	.82283	.13657	.57607	.85035	.95587
	U_8	1.06357	.20223	.61174	.83296	.13671	.57592	.85032	.95589
	U_9	1.05888	.19260	.60863	.84091	.13678	.57583	.85031	.95591
	U_{10}	1.05528	.18506	.60628	.84730	.13682	.57578	.85031	.95592
	U_{11}	1.05242	.17899	.60446	.85254	.13684	.57575	.85031	.95593
	U_{12}	1.05011	.17401	.60300	.85691	.13685	.57573	.85032	.95593
	D	1.02866	.12384	.59048	.90378	.13666	.57581	.85047	.95595
.9	M	3.85530	3.25308	3.30391	.14966	.04251	.62864	.86457	.95061
	U_3	2.36356	1.69279	1.78854	.23691	.07506	.61772	.86373	.95168
	U_4	2.16330	1.47004	1.57935	.25392	.08398	.61317	.86315	.95213
	U_5	2.04499	1.33773	1.45700	.26621	.08990	.60981	.86261	.95248
	U_6	1.96709	1.25026	1.37713	.27553	.09409	.60728	.86214	.95274
	U_7	1.91200	1.18822	1.32106	.28283	.09718	.60533	.86173	.95295
	U_8	1.87102	1.14196	1.27961	.28870	.09956	.60379	.86139	.95311
	S_8	1.87082	1.14225	1.27987	.28906	.09949	.60383	.86140	.95310
	E_8	1.86333	1.13110	1.26993	.28788	.10034	.60328	.86128	.95316
	U_9	1.83938	1.10616	1.24777	.29353	.10145	.60254	.86109	.95324
	U_{10}	1.81421	1.07765	1.22256	.29756	.10297	.60152	.86084	.95335
	U_{11}	1.79372	1.05441	1.20213	.30099	.10423	.60066	.86062	.95344
	U_{12}	1.77673	1.03510	1.18523	.30393	.10528	.59994	.86042	.95352
	S_{12}	1.77664	1.03526	1.18537	.30412	.10525	.59996	.86042	.95352
	E_{12}	1.77303	1.02989	1.18068	.30359	.10566	.59967	.86036	.95355
	D	1.59528	.82750	1.00901	.34257	.11706	.59142	.85774	.95444

Table 6 $GI/E_3/3$

ρ	$A(t)$	μEW	μSW_q	μSW	$F_q(0)$	$F(a)$	$F(b)$	$F(c)$	$F(d)$
.3	M	1.02447	.12130	.58995	.93139	.13634	.57619	.85053	.95588
	U_3	1.00568	.05428	.57990	.98083	.13584	.57656	.85079	.95584
	U_4	1.00211	.03037	.57815	.99123	.13569	.57669	.85085	.95583
	U_5	1.00100	.01993	.57769	.99537	.13563	.57675	.85087	.95582
	U_6	1.00056	.01451	.57753	.99723	.13560	.57678	.85088	.95582
	U_7	1.00035	.01134	.57746	.99817	.13559	.57679	.85088	.95581
	U_8	1.00024	.00930	.57743	.99870	.13559	.57679	.85089	.95581
	U_9	1.00018	.00790	.57740	.99902	.13558	.57680	.85089	.95581
	U_{10}	1.00014	.00689	.57739	.99923	.13558	.57680	.85089	.95581
	U_{11}	1.00011	.00614	.57738	.99937	.13558	.57680	.85089	.95581
	U_{12}	1.00009	.00555	.57738	.99947	.13558	.57680	.85089	.95581
	D	1.00006	.00135	.57735	.99996	.13557	.57681	.85089	.95581
.6	M	1.20524	.42174	.71498	.65141	.13170	.58040	.85224	.95540
	U_3	1.07412	.21661	.61665	.80950	.13721	.57546	.85014	.95596
	U_4	1.05330	.17287	.60268	.84143	.13747	.57509	.85013	.95604
	U_5	1.04234	.14870	.59611	.86187	.13737	.57511	.85023	.95605
	U_6	1.03575	.13290	.59245	.87585	.13722	.57521	.85031	.95605
	U_7	1.03140	.12232	.59017	.88592	.13710	.57532	.85037	.95603
	U_8	1.02835	.11466	.58863	.89347	.13699	.57540	.85042	.95602
	U_9	1.02610	.10887	.58753	.89931	.13691	.57548	.85046	.95601
	U_{10}	1.02438	.10434	.58670	.90395	.13684	.57554	.85049	.95601
	U_{11}	1.02303	.10071	.58607	.90773	.13678	.57559	.85051	.95600
	U_{12}	1.02193	.09774	.58556	.91085	.13673	.57563	.85053	.95599
	D	1.01202	.06810	.58135	.94308	.13624	.57610	.85071	.95592
.9	M	2.82982	2.15953	2.23538	.18691	.06418	.62253	.86422	.95122
	U_3	1.86094	1.11900	1.25916	.28292	.10197	.60222	.86111	.95328
	U_4	1.73287	.97115	1.12981	.30169	.11026	.59641	.85956	.95392
	U_5	1.65746	.88341	1.05534	.31503	.11538	.59262	.85837	.95433
	U_6	1.60792	.82546	1.00733	.32500	.11881	.59000	.85746	.95462
	U_7	1.57294	.78437	.97395	.33275	.12124	.58810	.85675	.95483
	U_8	1.54696	.75376	.94946	.33895	.12305	.58667	.85619	.95499
	S_8	1.54688	.75394	.94961	.33917	.12300	.58671	.85620	.95498
	E_8	1.54238	.74675	.94391	.33820	.12359	.58623	.85603	.95504
	U_9	1.52691	.73007	.93077	.34401	.12443	.58556	.85575	.95511
	U_{10}	1.51099	.71122	.91606	.34822	.12552	.58468	.85538	.95520
	U_{11}	1.49803	.69585	.90418	.35179	.12641	.58396	.85508	.95528
	U_{12}	1.48729	.68309	.89440	.35484	.12714	.58336	.85483	.95535
	S_{12}	1.48725	.68319	.89447	.35496	.12712	.58338	.85483	.95535
	E_{12}	1.48509	.67974	.89184	.35453	.12739	.58315	.85475	.95531
	D	1.37298	.54611	.79472	.39427	.13439	.57721	.85203	.95601

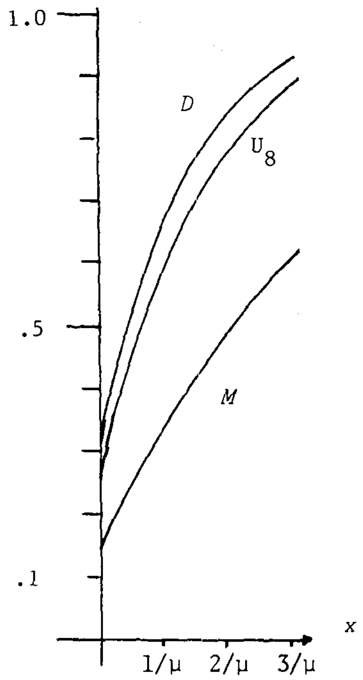

Fig. 1A ρ=0.9 $GI/E_2/2$ $F_q(x)$

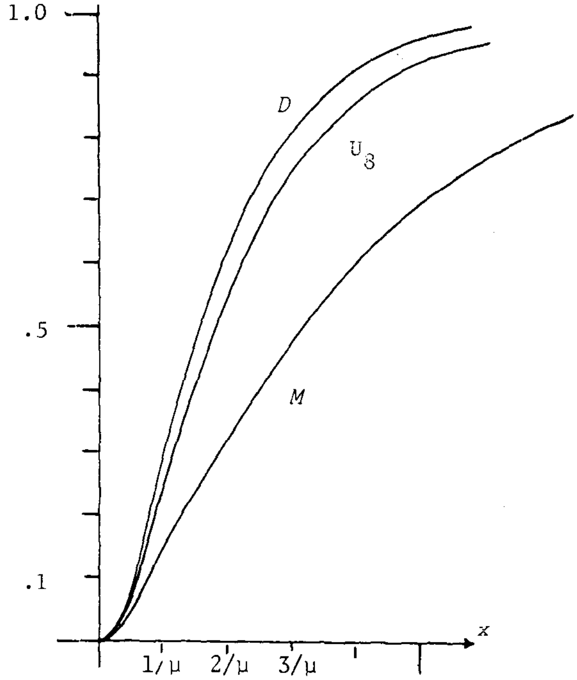

Fig. 1B ρ=0.9 $GI/E_2/2$ $F(x)$

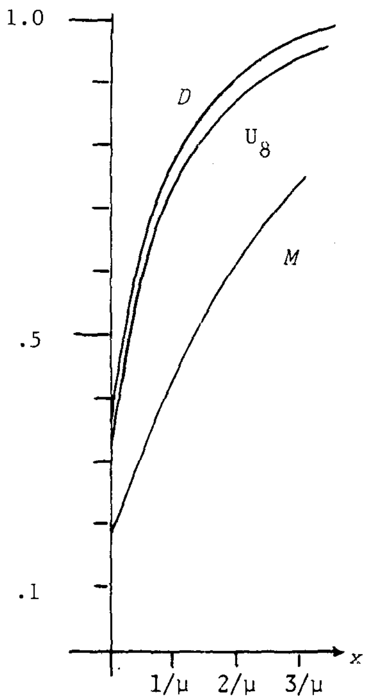

Fig. 2A ρ=0.9 $GI/E_2/3$ $E_q(x)$

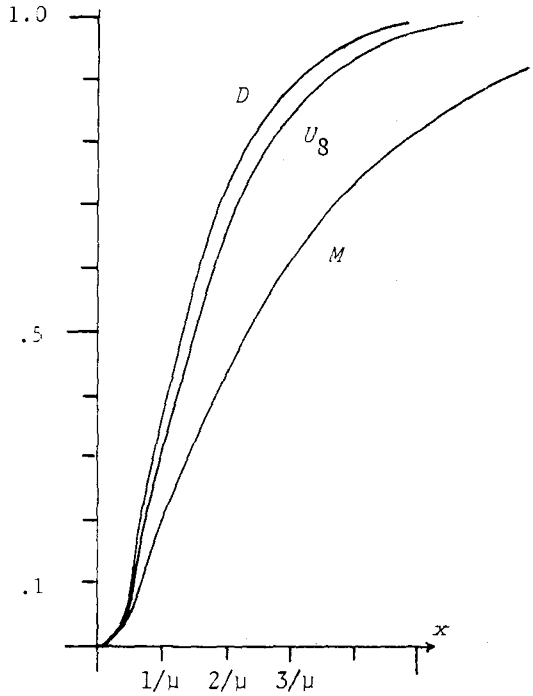

Fig. 2B ρ=0.9 $GI/E_2/3$ $F(x)$

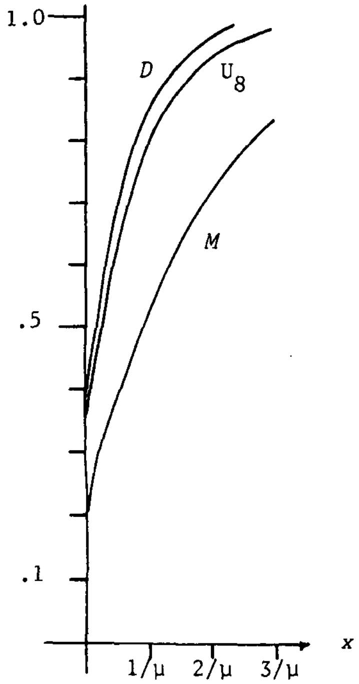

Fig. 3A ρ=0.9 GI/E_2/4 $F_q(x)$

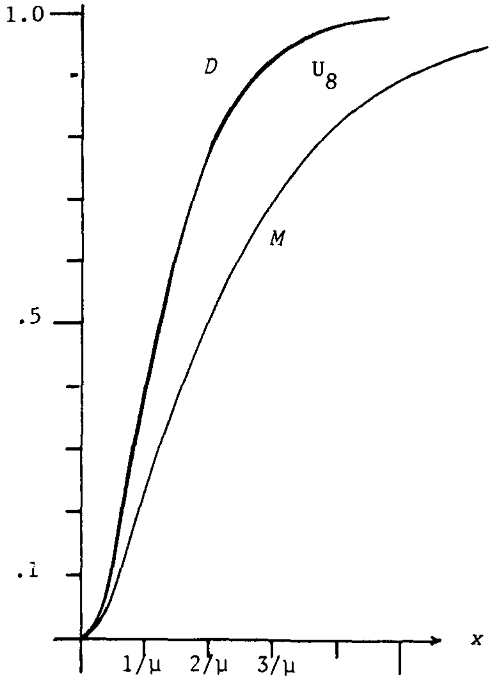

Fig. 3B ρ=0.9 GI/E_2/4 $F(x)$

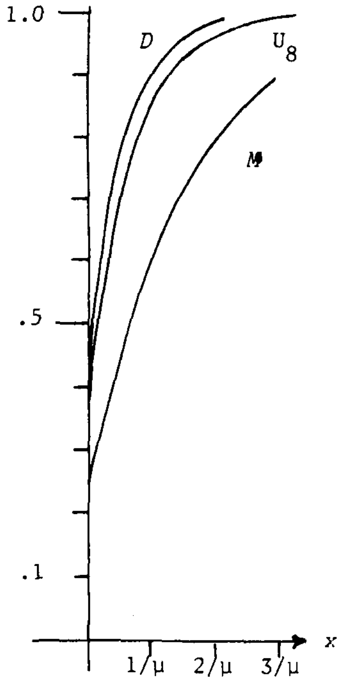

Fig. 4A ρ=0.9 GI/E_2/5 $F_q(x)$

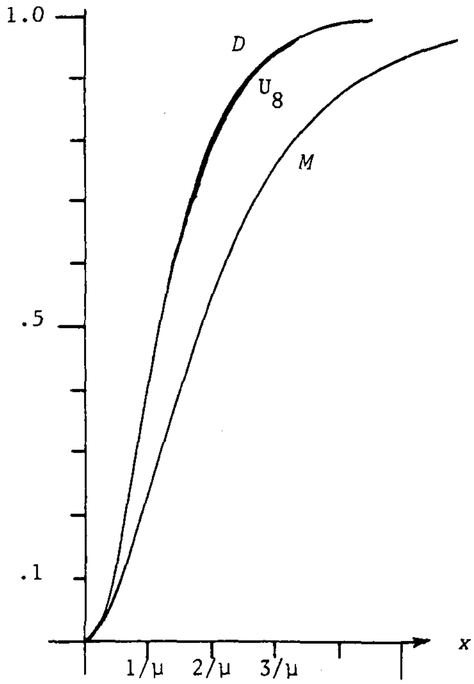

Fig. 4B ρ=0.9 GI/E_2/5 $F(x)$

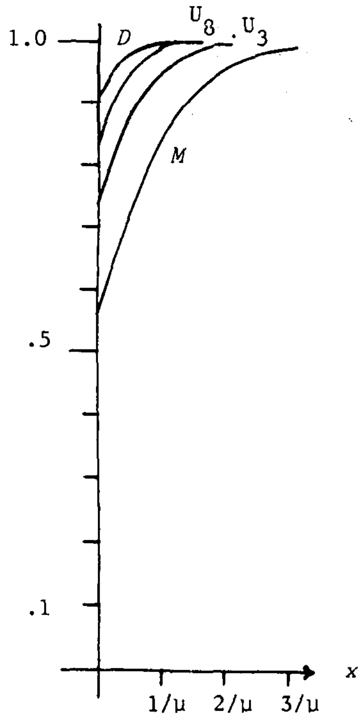

Fig. 5A $\rho=0.6$ $GI/E_3/2$ $F_q(x)$

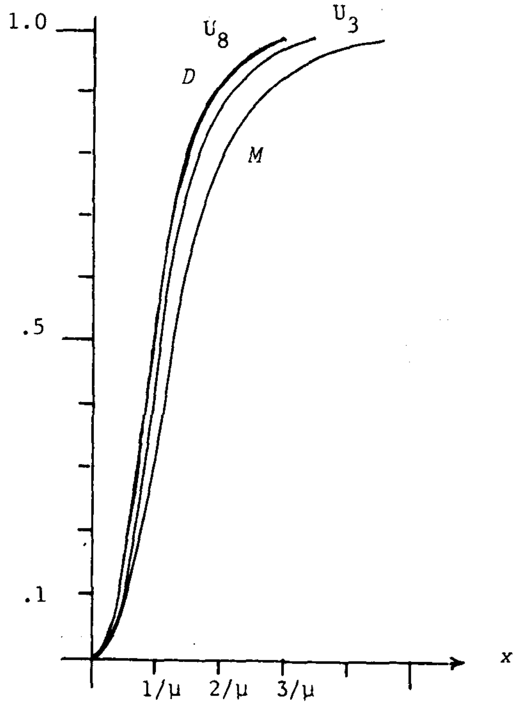

Fig. 5B $\rho=0.6$ $GI/E_3/2$ $F(x)$

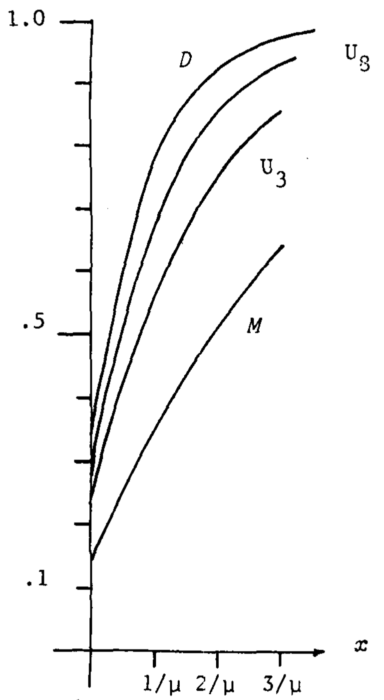

Fig. 6A $\rho=0.9$ $GI/E_3/2$ $E_q(x)$

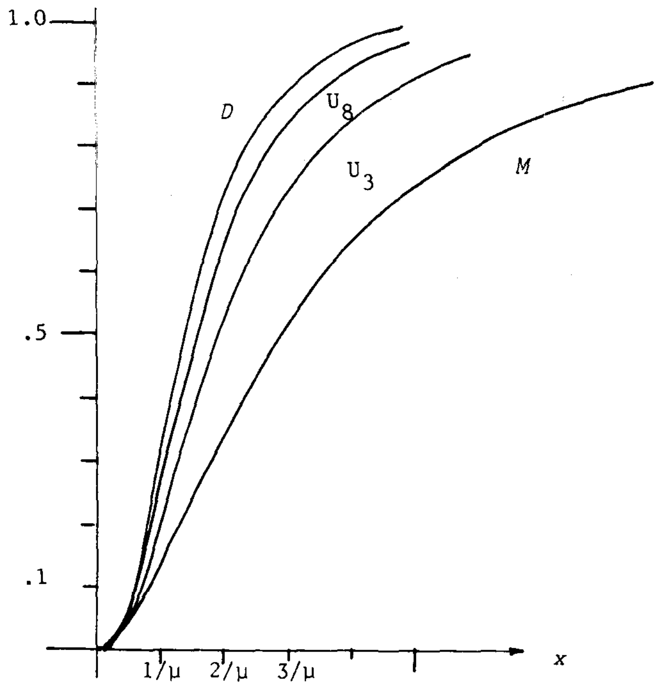

Fig. 6B $\rho=0.9$ $GI/E_3/2$ $F(x)$

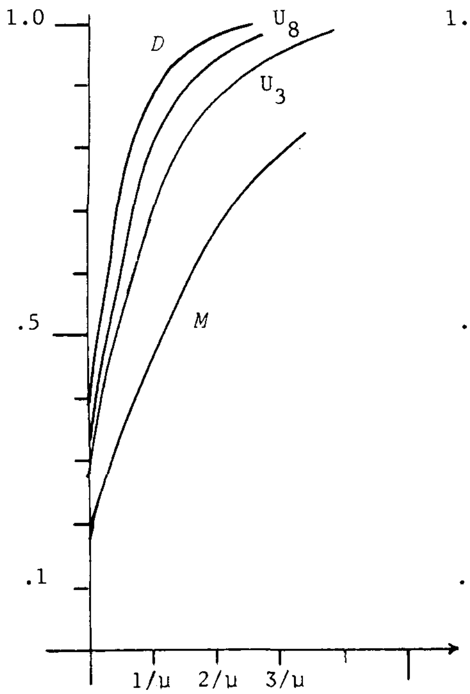

Fig. 7A $\rho=0.9$ $GI/E_3/3$ $F_q(x)$

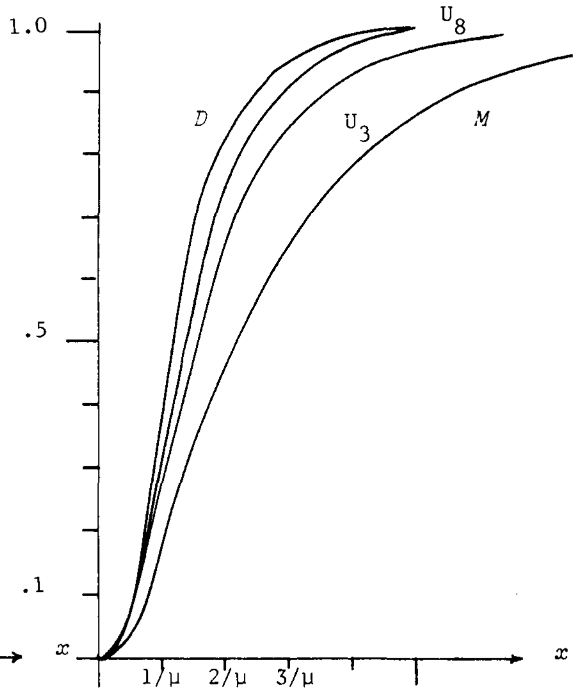

Fig. 7B $\rho=0.9$ $GI/E_3/3$ $F(x)$

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