Journal of the Operations Research
Society of Japan
Vol. 27, No. 2, June 1984

A CHARACTERIZATION OF FACES OF THE BASE POLYHEDRON ASSOCIATED WITH A SUBMODULAR SYSTEM*

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(Received May 21, 1983: Final January 23, 1984)

Abstract For a distributive lattice $\mathcal{D} \subsetneqq 2^E$ and a submodular function f on $\mathcal{D}$ with $\phi \in \mathcal{D}$ and $f(\phi) = 0$, the pair $(\mathcal{D}, f)$ is called a submodular system and, when $E \in \mathcal{D}$, the polyhedron given by

$$B(f) = \{ x \mid x \in R^E, \forall X \in \mathcal{D}: x(X) \leqq f(X), x(E) = f(E) \}$$

is called the base polyhedron associated with $(\mathcal{D}, f)$. We examine the structure of the base polyhedron $B(f)$ and give a characterization of all the faces of $B(f)$. Faces of $B(f)$ are made correspond one-to-one to certain sublattices of $\mathcal{D}$, so that the collection $\boldsymbol{D}$ of all such sublattices of $\mathcal{D}$ is anti-isomorphic with the collection $\boldsymbol{F}$ of all the non-empty faces of $B(f)$. Here, $\boldsymbol{D}$ and $\boldsymbol{F}$ are considered as posets relative to set inclusion. The incidence relation among faces, dimensions of faces, and extreme points and extreme rays of faces are given based on the structure of the sublattices in $\boldsymbol{D}$. These include as special cases recent results on (1) a poset structure of a polymatroid extreme point and connected components by Bixby, Cunningham and Topkis, (2) extreme rays of a cone determined by a distributive lattice by Tomizawa, and (3) adjacency for polymatroid extreme points by Topkis. Moreover, given a sublattice $\mathcal{D}_1$ of $\mathcal{D}$ on which f is modular,

$$F(\mathcal{D}_1) = \{ x \mid x \in B(f), \forall X \in \mathcal{D}_1 : x(X) = f(X) \}$$

is a nonempty face of $B(f)$ and there uniquely exists a sublattice $\mathcal{D}_2$ in $\boldsymbol{D}$ which corresponds to the face $F(\mathcal{D}_1)$. We show a theorem which characterizes the relationship between $\mathcal{D}_1$ and $\mathcal{D}_2$. $\mathcal{D}_2$ is considered as a closure of $\mathcal{D}_1$ and this closure operation is closely related to the concept of maximal skeleton recently considered by Nakamura and Iri. Algorithmic aspects of these characterizations are also discussed.

1. Introduction

Let E be a finite set, R the set of reals, and $\mathcal{D}$ a collection of subsets of E which is closed under set union and intersection, i.e., $\mathcal{D}$ is

*) The present research was carried out when the author was on leave at Institut für Ökonometrie und Operations Research, Universität Bonn, West Germany, and was supported by the Alexander von Humbold Fellowship (1982/83), West Germany.

a distributive lattice with set union and intersection as the lattice operations, join and meet. Also let $f: \mathcal{D} \to R$ be a *submodular function* on $\mathcal{D}$, i.e.,

(1.1) $$f(X) + f(Y) \geqq f(X \cup Y) + f(X \cap Y)$$

for all $X, Y \in \mathcal{D}$. (Here, if (1.1) holds with equality for each $X, Y \in \mathcal{D}$, f is called a *modular function*.) Then the pair $(\mathcal{D}, f)$ is called a *submodular system* [6], [8], where we assume $\emptyset \in \mathcal{D}$ and $f(\emptyset) = 0$. Note that a submodular system is a generalization of the concept of polymatroid [4]. When $E \in \mathcal{D}$, the *base polyhedron* $B(f)$ associated with $(\mathcal{D}, f)$ is given by

(1.2) $$B(f) = \{x \mid x \in R^E, \forall X \in \mathcal{D}: x(X) \leqq f(X), x(E) = f(E)\}.$$

Here, R^E is the set of all real vectors whose coordinates are indexed by E and for a vector $x = (x(e): e \in E) \in R^E$ and a set $X \in \mathcal{D}$ we define

(1.3) $$x(X) = \sum_{e \in X} x(e).$$

Each vector in $B(f)$ is called a *base* of $(\mathcal{D}, f)$.

In the present paper we shall examine and characterize all the faces of the base polyhedron $B(f)$. In Section 2 we give definitions and preliminary basic results on base polyhedra. Main results of the paper are given in Section 3. Faces of $B(f)$ are made correspond one-to-one to certain sublattices of $\mathcal{D}$, so that the collection $\mathbf{D}$ of all such sublattices of $\mathcal{D}$ is anti-isomorphic with the collection $\mathbf{F}$ of all nonempty faces of $B(f)$. Here, $\mathbf{D}$ and $\mathbf{F}$ are considered as posets relative to set inclusion. The incidence relation among faces, dimensions of faces, and extreme points and extreme rays of faces are given based on the structure of sublattices in $\mathbf{D}$. These include as special cases recent results on (1) a poset structure of a polymatroid extreme point and connected components by Bixby, Cunningham and Topkis [3], (2) extreme rays of a cone determined by a distributive lattice by Tomizawa [18], and (3) adjacency for polymatroid extreme points by Topkis [21]. Moreover, given a sublattice $\mathcal{D}_1$ of $\mathcal{D}$ on which f is modular,

(1.4) $$F(\mathcal{D}_1) = \{x \mid x \in B(f), \forall X \in \mathcal{D}_1: x(X) = f(X)\}$$

is a nonempty face of $B(f)$ and there uniquely exists a sublattice $\mathcal{D}_2$ in $\mathbf{D}$ which corresponds to the face $F(\mathcal{D}_1)$. We show a theorem which characterizes the relationship between $\mathcal{D}_1$ and $\mathcal{D}_2$. $\mathcal{D}_2$ is considered as a closure of $\mathcal{D}_1$ and this closure operation is closely related to the concept of maximal skeleton recently considered by Nakamura and Iri [13]. Algorithmic aspects of these characterizations in Section 3 are also discussed in Section 4.

This paper was motivated by the recent work by Bixby, Cunningham and Topkis [3], Nakamura and Iri [13], Tomizawa [18], and Topkis [21]. The importance of submodular functions has now widely been recognized in many combinatorial optimization problems (see, for example, [8], [11], [12], [20]) and the present paper will give a further insight into submodular functions.

2. Definitions and Preliminaries

In this section we give definitions and preliminary results on submodular functions and base polyhedra.

Distributive Lattices and Posets

For a finite set S we denote the cardinality of S by $|S|$ and the set of all the subsets of S by 2^S. Let $\mathcal{D}$ be a distributive lattice formed by subsets of a finite set E with set union and intersection as the lattice operations. We assume $\emptyset, E \in \mathcal{D}$ throughout the present paper. For such a distributive lattice $\mathcal{D}$ there uniquely exists a partially ordered set (poset) $P(\mathcal{D}) = (\Pi(\mathcal{D}), \preceq_{\mathcal{D}})$, where $\Pi(\mathcal{D})$ is a partition of E and $\preceq_{\mathcal{D}}$ is a partial order on $\Pi(\mathcal{D})$ such that

(2.1) $\mathcal{D} = \{\tilde{I} \mid I$ is an ideal of the poset $(\Pi(\mathcal{D}), \preceq_{\mathcal{D}})\}$.

Here, $I \subseteq \Pi(\mathcal{D})$ is an *ideal* of $(\Pi(\mathcal{D}), \preceq_{\mathcal{D}})$ if $T_1 \preceq_{\mathcal{D}} T_2 \in I$ implies $T_1 \in I$ for all $T_1, T_2 \in \Pi(\mathcal{D})$, and we define

(2.2) $\tilde{I} = \cup\{T \mid T \in I\}$

(cf. [2]). (For the construction of the poset $P(\mathcal{D})$, see [6] and [9].) We call $\Pi(\mathcal{D})$ the partition of E *induced by* $\mathcal{D}$. The distributive lattice $\mathcal{D}$ is called *simple* if $\Pi(\mathcal{D})$ is the partition of E into singletons.

For each subset A of E and a partition P' of E we say A is *compatible with* P' if for each $T \in P'$ either $T \subseteq A$ or $T \subseteq E - A$. For two partitions P_1 and P_2 of E we say P_1 is a *refinement* of P_2 if for each $T_1 \in P_1$ there exists $T_2 \in P_2$ such that $T_1 \subseteq T_2$.

For two posets $P_i = (E_i, \preceq_i)$ $(i=1,2)$ we say P_2 is a *homomorphic image* of P_1 if there exists a mapping ψ from E_1 onto E_2 such that $e \preceq_1 e'$ implies $\psi(e) \preceq_2 \psi(e')$ for all $e, e' \in E_1$.

Operations on Submodular Systems

Consider a submodular system $(\mathcal{D}, f)$. For a sublattice $\mathcal{D}_0$ of $\mathcal{D}$ we say f is *modular on* $\mathcal{D}_0$ if f restricted to $\mathcal{D}_0$ is a modular function on $\mathcal{D}_0$. For any $S \in \mathcal{D}$ the *reduction* $(\mathcal{D}, f) \cdot S$ *of* $(\mathcal{D}, f)$ *to* S is the submodular system $(\mathcal{D}^S, f^S)$ defined by

(2.3) $\mathcal{D}^S = \{X \mid X \in \mathcal{D},\ X \subseteq S\}$,

(2.4) $f^S(X) = f(X) \quad (X \in \mathcal{D}^S)$.

Also the *contraction* $(\mathcal{D}, f)/S$ *of* $(\mathcal{D}, f)$ *by* S is the submodular system $(\mathcal{D}_S, f_S)$ defined by

(2.5) $\mathcal{D}_S = \{X - S \mid S \subseteq X \in \mathcal{D}\}$, (2.6) $\quad f_S(X-S) = f(X) - f(S) \qquad (S \subseteq X \in \mathcal{D}).$

Any submodular system obtained by repeated reductions and contractions of $(\mathcal{D},f)$ is called a *minor* of $(\mathcal{D},f)$.

Let $\Pi(\mathcal{D})$ be the partition of E induced by $\mathcal{D}$ and T be an element of $\Pi(\mathcal{D})$. Also let α_T be a new element not in E. Then, define a mapping $\psi: E \to E'' \equiv (E-T) \cup \{\alpha_T\}$ by

(2.7) $$\psi(e) = \begin{cases} e & (e \in E-T), \\ \alpha_T & (e \in T). \end{cases}$$

Then ψ naturally induces a distributive lattice $\mathcal{D}'' \subseteq 2^{E''}$ and a submodular function $f'': \mathcal{D}'' \to R$, respectively, corresponding to $\mathcal{D}$ and f. We call $(\mathcal{D}'',f'')$ the *shrinking of* $(\mathcal{D},f)$ *by* T. When a partition P' of E is a refinement of $\Pi(\mathcal{D})$, we define the *shrinking of* $(\mathcal{D},f)$ *by* P' as the submodular system obtained by repeated shrinkings of $(\mathcal{D},f)$ by elements of P'. If $P' = \Pi(\mathcal{D})$, the shrinking of $(\mathcal{D},f)$ by P' is called the *simplification* of $(\mathcal{D},f)$.

For a subset A of E define

(2.8) $\quad \mathcal{D}//A = \{X \mid X \in \mathcal{D}, \text{ either } A \cap X = \emptyset \text{ or } A \subseteq X\}$

and let $f//A$ be the restriction of f to $\mathcal{D}//A$. We call $(\mathcal{D}//A, f//A)$ (or $\mathcal{D}//A$) the *aggregation of* $(\mathcal{D},f)$ (or $\mathcal{D}$) *by* A. For a partition P' of E the aggregation of $(\mathcal{D},f)$ by P' is the submodular system obtained by repeated aggregations of $(\mathcal{D},f)$ by elements of P'.

We say a submodular system $(\mathcal{D},f)$ is *connected* if for all nonempty complementary $X, Y \in \mathcal{D}$ (i.e., $X \neq \emptyset \neq Y$, $X = E - Y$ and $X, Y \in \mathcal{D}$) we have

(2.9) $\quad f(X) + f(Y) > f(E).$

If $(\mathcal{D},f)$ is not connected, there uniquely exists a partition $P^* = \{T_1, T_2, \ldots, T_k\}$ $(k \geq 2)$ of E such that $T_i \in \mathcal{D}$ $(i=1,2,\ldots,k)$, each reduction $(\mathcal{D}^{T_i}, f^{T_i}) \equiv (\mathcal{D},f)\cdot T_i$ $(i=1,2,\ldots,k)$ is connected, and

(2.10) $\quad f(X) = f^{T_1}(T_1 \cap X) + f^{T_2}(T_2 \cap X) + \cdots + f^{T_k}(T_k \cap X)$

for all $X \in \mathcal{D}$. Note that

(2.11) $\quad \mathcal{B} = \{X \mid X \in \mathcal{D},\ E-X \in \mathcal{D},\ f(X) + f(E-X) = f(E)\}$

is a Boolean sublattice of $\mathcal{D}$ and $P^* = \Pi(\mathcal{B})$. Each $(\mathcal{D}^{T_i}, f^{T_i})$ $(i=1,2,\ldots,k)$ is called a *connected component* of $(\mathcal{D},f)$.

Preliminary Results

Let $(\mathcal{D},f)$ be a submodular system. Based on the general theory of convex polyhedra, we have the following lemma (e.g. [17, p. 67]). For the terminology concerning convex polyhedra we almost follow [17], [14] and [1].

Lemma 2.1. The base polyhedron $B(f)$ can be decomposed as

(2.12) $\quad B(f) = (Q(f) + C(f)) \oplus L(f),$

where $Q(f)$ is a bounded polyhedron, $C(f)$ a pointed cone, and $L(f)$ a linear subspace. Here, $+$ denotes a vector sum and $\oplus$ a direct vector sum.

Note that in Lemma 2.1 $C(f)$ is unique if and only if $B(f)$ is pointed, i.e., $L(f) = \{0\}$ and that $Q(f)$ is unique if and only if $B(f)$ is bounded, i.e., $L(f) = C(f) = \{0\}$.

Since

(2.13) $$L(f) = \{x \mid x \in R^E, \forall X \in \mathcal{D}: x(X) = 0\},$$

we can easily show

Lemma 2.2. Suppose that $\{T_i \mid i \in I\}$ is the partition $\Pi(\mathcal{D})$ of E induced by $\mathcal{D}$, and define for each $i \in I$

(2.14) $$L_i(f) = \{x \mid x \in R^E, x(T_i) = 0, \forall e \in E - T_i: x(e) = 0\}.$$

Then the linear subspace $L(f)$ in Lemma 2.1 is expressed as

(2.15) $$L(f) = \bigoplus_{i \in I} L_i(f).$$

Note that $L_i(f) = \{0\}$ if and only if $|T_i| = 1$. Therefore, $L(f) = \{0\}$, i.e. $B(f)$ is pointed, if and only if $\mathcal{D}$ is simple. The simplification of $(\mathcal{D},f)$ determines the structure of $Q(f)$ and $C(f)$ in Lemma 2.1, since one possible $Q(f) + C(f)$ is given by

(2.16) $$Q(f) + C(f) = \{x \mid x \in B(f), \forall e \in E - E^*: x(e) = 0\},$$

where $E^* = \{e_i \mid i \in I\}$ with $e_i \in T_i$ $(i \in I)$, i.e., E^* is a set of representatives of T_i $(i \in I)$ in $\Pi(\mathcal{D})$.

When $\mathcal{D}$ is simple, $C(f)$ is unique and one possible $Q(f)$ is given by the convex hull of all the extreme points of $B(f)$. The following theorem characterizes the extreme points of $B(f)$.

Theorem 2.3 [9] (when $\mathcal{D} = 2^E$ see [4], [12], [16]). When $\mathcal{D}$ is simple, a base $b \in B(f)$ is an extreme point of $B(f)$ if and only if for a maximal chain

(2.17) $$C: \emptyset = S_0 \subsetneq S_1 \subsetneq \cdots \subsetneq S_n = E$$

in $\mathcal{D}$ we have

(2.18) $$b(S_i - S_{i-1}) = f(S_i) - f(S_{i-1}) \quad (i=1,2,\ldots,n).$$

Denote by $H(P(\mathcal{D})) = (\Pi(\mathcal{D}), A^*(\mathcal{D}))$ the Hasse diagram representing the poset $P(\mathcal{D}) = (\Pi(\mathcal{D}), \preceq_{\mathcal{D}})$. Here, $H(P(\mathcal{D}))$ is a directed graph with a vertex set $\Pi(\mathcal{D})$ and an arc set $A^*(\mathcal{D})$ such that $a \in A^*(\mathcal{D})$ if and only if $\partial^- a \prec_{\mathcal{D}} \partial^+ a$ and there is no element $T \in \Pi(\mathcal{D})$ with $\partial^- a \prec_{\mathcal{D}} T \prec_{\mathcal{D}} \partial^+ a$, where $\partial^- a$ $(\partial^+ a) \in \Pi(\mathcal{D})$ is the terminal (initial) end-vertex of arc a. When $\mathcal{D}$ is simple, we express $P(\mathcal{D}) = (\Pi(\mathcal{D}), \preceq_{\mathcal{D}})$ and $H(P(\mathcal{D})) = (\Pi(\mathcal{D}), A^*(\mathcal{D}))$, respectively, as $P(\mathcal{D}) = (E, \preceq_{\mathcal{D}})$ and $H(P(\mathcal{D})) = (E, A^*(\mathcal{D}))$ as well, since $\Pi(\mathcal{D})$

is the partition of E into singletons.

Now, from the general theory [14], [17], the *recession cone* $C(f)$ of $B(f)$ is expressed as

(2.19) $C(f) = \{x \mid x \in R^E,\ \forall X \in \mathcal{D}\colon x(X) \leqq 0,\ x(E) = 0\}$.

(The term, characteristic cone, is used in [17] instead of recession cone.) The following theorem determines the extreme rays of $C(f)$.

Theorem 2.4 [18]. Suppose $\mathcal{D}$ is simple. Then the set of all the extreme rays of $C(f)$ is given by

(2.20) $\{\zeta(a) \mid a \in A^*(\mathcal{D})\}$,

where $A^*(\mathcal{D})$ is the arc set of the Hasse diagram $H(P(\mathcal{D})) = (E, A^*(\mathcal{D}))$ and $\zeta(a)$ is the vector in R^E defined by

(2.21) $$\zeta(a)(e) = \begin{cases} 1 & (e = \partial^+ a) \\ -1 & (e = \partial^- a) \\ 0 & (\text{otherwise}) \end{cases} \qquad (e \in E).$$

In the next section we shall give a characterization of all the faces of $B(f)$ when $\mathcal{D}$ is simple.

3. A Characterization of Faces of the Base Polyhedron

We assume that $\mathcal{D} \subseteq 2^E$ is a simple distributive lattice with $\emptyset, E \in \mathcal{D}$ and that $f\colon \mathcal{D} \to R$ is a submodular function with $f(\emptyset) = 0$.

For any $\mathcal{F} \subseteq \mathcal{D}$ define

(3.1) $F(\mathcal{F}) = \{x \mid x \in R^E,\ \forall X \in \mathcal{F}\colon x(X) = f(X),\ \forall X \in \mathcal{D} - \mathcal{F}\colon x(X) \leqq f(X)\}$,

(3.2) $F^\circ(\mathcal{F}) = \{x \mid x \in R^E,\ \forall X \in \mathcal{F}\colon x(X) = f(X),\ \forall X \in \mathcal{D} - \mathcal{F}\colon x(X) < f(X)\}$.

Moreover, define

(3.3) $\mathbf{D} = \{\mathcal{D}_0 \mid \mathcal{D}_0$ is a sublattice of $\mathcal{D}$ with $\emptyset, E \in \mathcal{D}_0,\ F^\circ(\mathcal{D}_0) \neq \emptyset\}$.

Lemma 3.1. The collection $\mathbf{D}$ of sublattices of $\mathcal{D}$ defined by (3.3) is given by

(3.4) $\mathbf{D} = \{\mathcal{D}(x) \mid x \in B(f)\}$,

where for each $x \in B(f)$

(3.5) $\mathcal{D}(x) = \{X \mid X \in \mathcal{D},\ x(X) = f(X)\}$.

Proof: If $\mathcal{D}_0 \in \mathbf{D}$, then for some $x \in F^\circ(\mathcal{D}_0)$ we must have $\mathcal{D}_0 = \mathcal{D}(x)$ from (3.2). Conversely, it is easily seen that for any $x \in B(f)$ $\mathcal{D}(x)$ is a sublattice of $\mathcal{D}$ with $\emptyset, E \in \mathcal{D}(x)$ and $x \in F^\circ(\mathcal{D}(x))$, and we have $\mathcal{D}(x) \in \mathbf{D}$. Q.E.D.

It may be noted that the poset induced by $\mathcal{D}(x)$ in (3.5) is what is called the principal structure of $(\mathcal{D}, f - x)$ in [6], where considered is the case when $\mathcal{D} = 2^E$. It should also be noted that for $\mathcal{D}_0 \in \mathbf{D}$ $F^\circ(\mathcal{D}_0)$ is the relative interior of the face $F(\mathcal{D}_0)$.

From Lemma 3.1 $\mathbf{D}$ is the collection of "equality sets", each given by (3.5) (cf. [1]), for $B(f)$ expressed by (1.2). Therefore, from the general theory of convex polyhedra [1], [17] we have the following

Theorem 3.2. The collection $\mathbf{F}$ of all the nonempty faces of $B(f)$ is given by $\{F(\mathcal{D}_0) \mid \mathcal{D}_0 \in \mathbf{D}\}$, and we have

(i) If $\mathcal{D}_1, \mathcal{D}_2 \in \mathbf{D}$ and $\mathcal{D}_1 \neq \mathcal{D}_2$, then $F(\mathcal{D}_1) \neq F(\mathcal{D}_2)$.

(ii) For any $\mathcal{D}_1, \mathcal{D}_2 \in \mathbf{D}$, $\mathcal{D}_1 \subseteqq \mathcal{D}_2$ if and only if $F(\mathcal{D}_2) \subseteqq F(\mathcal{D}_1)$.

In other words, F in (3.1) determines an anti-isomorphism from $\mathbf{D}$ onto $\mathbf{F}$, where $\mathbf{D}$ and $\mathbf{F}$ are considered as posets relative to set inclusion.

$\mathbf{F}$ (or $\mathbf{F} \cup \{\emptyset\}$ when $\mathbf{F}$ does not have a unique minimal element) is called the *face lattice* of $B(f)$ (see [1], [17]).

Lemma 3.3. For any distributive lattices $\mathcal{D}_i$ $(i=1,2)$ with $\emptyset, E \in \mathcal{D}_i$, we have $\mathcal{D}_1 \subseteqq \mathcal{D}_2$ if and only if $\Pi(\mathcal{D}_2)$ is a refinement of $\Pi(\mathcal{D}_1)$ and $P(\mathcal{D}_1) = (\Pi(\mathcal{D}_1), \preceq_{\mathcal{D}_1})$ is a homomorphic image of $P(\mathcal{D}_2) = (\Pi(\mathcal{D}_2), \preceq_{\mathcal{D}_2})$ under the natural mapping (i.e., $T_2 \in \Pi(\mathcal{D}_2)$ is made correspond to $T_1 \in \Pi(\mathcal{D}_1)$ if $T_2 \subseteqq T_1$).

Proof: The lemma follows from the following:

(1) $\Pi(\mathcal{D}_1)$ is the collection of equivalence classes with respect to the equivalence relation $\sim_1$ such that $e \sim_1 e'$ if and only if there exists no $X \in \mathcal{D}_1$ with $e \in X \not\ni e'$ or $e \notin X \ni e'$, and

(2) $T \preceq_{\mathcal{D}_1} T'$ if and only if $T' \subseteqq X \in \mathcal{D}_1$ implies $T \subseteqq X$ for all $X \in \mathcal{D}_1$, where $T, T' \in \Pi(\mathcal{D}_1)$. Q.E.D.

Theorem 3.4. For $\mathcal{D}_0 \in \mathbf{D}$ we have

(3.6) $$\dim F(\mathcal{D}_0) = |E| - |\Pi(\mathcal{D}_0)|,$$

where $\dim F(\mathcal{D}_0)$ denotes the dimension of the face $F(\mathcal{D}_0)$.

Proof: The dimension of the face $F(\mathcal{D}_0)$ is equal to that of the affine set

(3.7) $$M(\mathcal{D}_0) = \{x \mid x \in R^E, \forall X \in \mathcal{D}_0 : x(X) = f(X)\}.$$

Since the rank of the coefficient matrix in the right-hand side of (3.7) is equal to $|\Pi(\mathcal{D}_0)|$, we have (3.6). Q.E.D.

It may be noted that Theorems 2.3 and 2.4 easily follow from Theorems 3.2 and 3.4 and Lemma 3.3.

Lemma 3.5. Suppose $\mathcal{D}_0 \in \mathbf{D}$ and let

(3.8) $$C_0 : \emptyset = S_0 \subsetneqq S_1 \subsetneqq \cdots \subsetneqq S_k = E$$

be a maximal chain in $\mathcal{D}_0$. Then, (3.9) $F(C_0) = F(\mathcal{D}_0)$.

Proof: It suffices to prove that $F(C_0) \subseteq F(\mathcal{D}_0)$. For any $a \in F(C_0)$ and $b \in F(\mathcal{D}_0)$ we have

(3.10) $a(S_i - S_{i-1}) = b(S_i - S_{i-1}) \quad (i=1,2,\ldots,k)$,

(3.11) $\{S_i - S_{i-1} \mid i=1,2,\ldots,k\} = \Pi(\mathcal{D}_0)$,

since C_0 is a maximal chain in $\mathcal{D}_0$. Therefore, for any $X \in \mathcal{D}_0$

(3.12) $a(X) = b(X) = f(X)$,

i.e.,

$$a \in F(\mathcal{D}_0).$$ Q.E.D.

The following theorem characterizes the extreme points of a face of $B(f)$.

Theorem 3.6. Suppose $\mathcal{D}_0 \in \mathbf{D}$. Then a base $b \in B(f)$ is an extreme point of the face $F(\mathcal{D}_0)$ if and only if, for a maximal chain

(3.13) $C\colon \emptyset = S_0 \subsetneqq S_1 \subsetneqq \cdots \subsetneqq S_n = E$

in $\mathcal{D}$ which contains a maximal chain in $\mathcal{D}_0$ as a subchain, b is given by

(3.14) $b(S_i - S_{i-1}) = f(S_i) - f(S_{i-1}) \quad (i=1,2,\ldots,n)$.

Proof: *"If" part.* Let C_0 be a maximal chain in $\mathcal{D}_0$ contained in (3.13). Then from (3.13) and (3.14) we have $b \in F(C_0)$. It follows from Lemma 3.5 that $b \in F(\mathcal{D}_0)$. Moreover, from Theorem 2.3 b is an extreme point of $B(f)$. Consequently, b must be an extreme point of $F(\mathcal{D}_0)$.

"Only if" part. For any extreme point b of $F(\mathcal{D}_0)$, define $\mathcal{D}(b)$ by (3.5). Then, $\mathcal{D}_0 \subseteq \mathcal{D}(b) \in \mathbf{D}$. Since b is also an extreme point of $B(f)$, i.e., $\{b\}$ is a zero-dimensional face $F(\mathcal{D}(b))$ of $B(f)$, a maximal chain in $\mathcal{D}(b)$ is a maximal chain in $\mathcal{D}$ due to Theorem 3.4 (or Theorem 2.3) and there exists a maximal chain in $\mathcal{D}(b)$ which contains a maximal chain in $\mathcal{D}_0$ as a subchain. Q.E.D.

Extreme rays of a face of $B(f)$ are characterized by the following

Theorem 3.7. Suppose $\mathcal{D}_0 \in \mathbf{D}$ and $\Pi(\mathcal{D}_0) = \{T_i \mid i \in I\}$. Then the set of all the extreme rays of the recession cone of the face $F(\mathcal{D}_0)$ is given by

(3.15) $\{\zeta(a) \mid a \in A^*(\mathcal{D}),\ \exists\, i \in I\colon \{\partial^+ a, \partial^- a\} \subseteq T_i\}$,

where $A^*(\mathcal{D})$ is the arc set of the Hasse diagram $H(P(\mathcal{D})) = (E, A^*(\mathcal{D}))$ representing the poset $P(\mathcal{D}) = (E, \preceq_{\mathcal{D}})$ and $\zeta(a)$ is defined by (2.21).

Proof: The present theorem easily follows from Theorem 2.4 and the fact that the recession cone $C(\mathcal{D}_0)$ of the face $F(\mathcal{D}_0)$ is given by

(3.16) $C(\mathcal{D}_0) = \{x \mid x \in R^E,\ \forall X \in \mathcal{D}_0\colon x(X) = 0,\ \forall X \in \mathcal{D} - \mathcal{D}_0\colon x(X) \leqq 0\}$.

Q.E.D.

Now, let us consider the problem of finding an element $\mathcal{D}_0 \in \mathbf{D}$ for which $B(f) = F(\mathcal{D}_0)$.

Theorem 3.8. A submodular system $(\mathcal{D},f)$ is connected if and only if $\{\emptyset,E\} \in \mathbf{D}$, i.e., there exists a base $b \in B(f)$ such that

(3.17) $\quad b(X) < f(X)$

for any $X \in \mathcal{D}$ with $\emptyset \neq X \neq E$.

Proof: *"If" part.* Easy. (Note that $b(E) = f(E)$ for $b \in B(f)$.)

"Only if" part. Suppose $(\mathcal{D},f)$ is connected and $B(f) = F(\mathcal{D}_0)$ for some $\mathcal{D}_0 \in \mathbf{D}$. We show $\mathcal{D}_0 = \{\emptyset,E\}$. Suppose, on the contrary, that there were $Y_0 \in \mathcal{D}_0$ with $\emptyset \neq Y_0 \neq E$. Then the inequality

(3.18) $\quad - x(Y_0) \leqq - f(Y_0)$

must be implied by the following system of inequalities and an equation:

(3.19) $\quad x(X) \leqq f(X) \qquad (X \in \mathcal{D} - \{E\}),$

(3.20) $\quad x(E) = f(E).$

Therefore, there exist rational $\alpha(X) \geqq 0$ $(X \in \mathcal{D} - \{E\})$ and rational $\alpha(E) < 0$ such that

(3.21) $\quad - \chi(Y_0) = \sum_{X \in \mathcal{D}} \alpha(X)\chi(X),$

(3.22) $\quad - f(Y_0) \geqq \sum_{X \in \mathcal{D}} \alpha(X)f(X),$

where $\chi(X)$ for each $X \in \mathcal{D}$ is the characteristic vector of $X \subseteqq E$. Define

(3.23) $\quad \mathcal{F} = \{X \mid X \in \mathcal{D},\ \alpha(X) > 0\}.$

If $\mathcal{F}$ contains incomparable X_1, X_2 (relative to set inclusion), then for $\varepsilon \equiv \min(\alpha(X_1),\alpha(X_2))$ put

(3.24) $\quad \alpha(X_1) \leftarrow \alpha(X_1) - \varepsilon,$

(3.25) $\quad \alpha(X_2) \leftarrow \alpha(X_2) - \varepsilon,$

(3.26) $\quad \alpha(X_1 \cup X_2) \leftarrow \alpha(X_1 \cup X_2) + \varepsilon,$

(3.27) $\quad \alpha(X_1 \cap X_2) \leftarrow \alpha(X_1 \cap X_2) + \varepsilon.$

Repeat the above (3.24) ~ (3.27) until $\mathcal{F}$ given by (3.23) consists of pair-wise comparable elements of $\mathcal{D}$. After a finite number of steps this process terminates. Note that throughout this process (3.21) and (3.22) hold because of (3.24) ~ (3.27) and the submodularity of f. For the finally obtained $\mathcal{F}$ and α we must have

(3.28) $\quad \mathcal{F} = \{E - Y_0\}$ or $\{\emptyset,\ E - Y_0\},$

(3.29) $\quad \alpha(E - Y_0) = 1, \qquad \alpha(E) = -1$

because of (3.21) with $\mathcal{D}$ replaced by $\mathcal{F} \cup \{E\}$. Therefore,

(3.30) $\quad E - Y_0 \in \mathcal{D}.$

Also we have from (3.22) ~ (3.29)

(3.31) $\quad - f(Y_0) \geqq f(E - Y_0) - f(E),$

or

(3.32) $\quad f(Y_0) + f(E - Y_0) \leqq f(E)$.

Consequently,

(3.33) $\quad f(Y_0) + f(E - Y_0) = f(E)$,

which contradicts the connectedness of $(\mathcal{D}, f)$. Q.E.D.

Theorem 3.9. Suppose $\mathcal{D}_0 \in \boldsymbol{D}$ and $B(f) = F(\mathcal{D}_0)$. Then $\mathcal{D}_0$ is a Boolean sublattice of $\mathcal{D}$ and $\Pi(\mathcal{D}_0)$ is the partition of E given by the decomposition of $(\mathcal{D}, f)$ into connected components. In particular, the number of the connected components is equal to $|\Pi(\mathcal{D}_0)|$.

Proof: It follows from the proof of Theorem 3.8 that if $B(f) = F(\mathcal{D}_0)$ for any $\mathcal{D}_0 \in \boldsymbol{D}$ and $Y_0 \in \mathcal{D}_0$, then we have $E - Y_0 \in \mathcal{D}_0$. Theorefore, $\mathcal{D}_0$ is a complemented distributive lattice, i.e., a Boolean lattice. The rest of the theorem follows from the definition of a connected component. Q.E.D.

From Theorems 3.4 and 3.9 we have

Corollary 3.10. Let n^* be the number of connected components of $(\mathcal{D}, f)$. Then,

(3.34) $\quad \dim B(f) = |E| - n^*$.

The dimension of $B(f)$ was given in [16], [10], [3] (when $\mathcal{D} = 2^E$) and [19]. The connected components of $(\mathcal{D}, f)$ is efficiently found by an algorithm in [3].

Corollary 3.11. Let $\mathcal{D}_1$ be a sublattice of $\mathcal{D}$ with $\emptyset, E \in \mathcal{D}_1$ and suppose f is modular on $\mathcal{D}_1$. Also let

(3.35) $\quad C_1 \colon \emptyset = S_0 \subsetneqq S_1 \subsetneqq \cdots \subsetneqq S_k = E$

be any maximal chain in $\mathcal{D}_1$ and for each $i = 1, 2, \ldots, k$ let n_i^* be the number of connected components of the minor $(\mathcal{D}, f) \cdot S_i / S_{i-1}$. Then the dimension of the face $F(\mathcal{D}_1)$ is given by

(3.36) $$\dim F(\mathcal{D}_1) = |E| - \sum_{i=1}^{k} n_i^*.$$

Proof: Note that since f is modular on $\mathcal{D}_1$, $x \in F(\mathcal{D}_1)$ if and only if for each $i = 1, 2, \ldots, k$ x restricted to $S_i - S_{i-1}$ is a base of $(\mathcal{D}, f) \cdot S_i / S_{i-1}$. Therefore, the present corollary follows from Corollary 3.10. Q.E.D.

Corollary 3.11 when $\mathcal{D} = 2^E$ and $\mathcal{D}_1$ is a chain of length 2 is also shown in [10]. Note that $F(\mathcal{D}_1) \neq \emptyset$ if and only if f is modular on $\mathcal{D}_1$. Also note that in Corollary 3.11 since f is modular on $\mathcal{D}_1$, for each $T \in \Pi(\mathcal{D}_1)$ the minor $(\mathcal{D}, f) \cdot S_i / S_{i-1}$ with $S_i - S_{i-1} = T$ does not depend on the choice of a maximal chain (3.35) in $\mathcal{D}_1$ (cf. [13], [20]).

Theorem 3.12. For any $\mathcal{D}_1, \mathcal{D}_2 \in \mathbf{D}$ we have $\mathcal{D}_1 \cap \mathcal{D}_2 \in \mathbf{D}$ and $F(\mathcal{D}_1 \cap \mathcal{D}_2)$ is the minimal face (relative to set inclusion) which contains both $F(\mathcal{D}_1)$ and $F(\mathcal{D}_2)$.

Proof: Suppose $\mathcal{D}_1, \mathcal{D}_2 \in \mathbf{D}$. Since $\mathcal{D}_1 \cap \mathcal{D}_2$ is a sublattice of $\mathcal{D}$ with $\emptyset, E \in \mathcal{D}_1 \cap \mathcal{D}_2$, it suffices to show $F^\circ(\mathcal{D}_1 \cap \mathcal{D}_2) \neq \emptyset$. By the assumption there exist $b_i \in F^\circ(\mathcal{D}_i)$ $(i=1,2)$. For an arbitrary $\lambda \in R$ with $0 < \lambda < 1$,

(3.37) $\quad \lambda b_1 + (1-\lambda)b_2 \in F^\circ(\mathcal{D}_1 \cap \mathcal{D}_2).$

The minimality of the face follows from Theorem 3.2. Q.E.D.

Theorem 3.12 also follows from the general theory for convex polyhedra (cf. [1]).

If $\mathcal{D}_1, \mathcal{D}_2 \in \mathbf{D}$ and $F(\mathcal{D}_1 \cup \mathcal{D}_2) \neq \emptyset$, then $F(\mathcal{D}_1 \cup \mathcal{D}_2)$ is the unique maximal face of $B(f)$ which is contained in both $F(\mathcal{D}_1)$ and $F(\mathcal{D}_2)$ and there exists $\mathcal{D}_3 \in \mathbf{D}$ such that $F(\mathcal{D}_1 \cup \mathcal{D}_2) = F(\mathcal{D}_3)$. Let $\mathcal{D}_1 \vee \mathcal{D}_2$ be the unique minimal sublattice of $\mathcal{D}$ (minimal relative to set inclusion) which contains both $\mathcal{D}_1$ and $\mathcal{D}_2$. Then $\mathcal{D}_1 \vee \mathcal{D}_2 \subseteqq \mathcal{D}_3$. Note that $\mathcal{D}_3$ is the unique minimal sublattice of $\mathcal{D}$ in $\mathbf{D}$ which contains both $\mathcal{D}_1$ and $\mathcal{D}_2$.

The following theorem gives a necessary and sufficient condition for a sublattice $\mathcal{D}_0$ of $\mathcal{D}$ to be a member of $\mathbf{D}$.

Theorem 3.13. Let $\mathcal{D}_0$ be any sublattice of $\mathcal{D}$ with $\emptyset, E \in \mathcal{D}_0$ and f_0 be the restriction of f to $\mathcal{D}_0$. Then $\mathcal{D}_0 \in \mathbf{D}$ if and only if the following three statements hold:

(i) f_0 is a modular function.

(ii) For any maximal chain

(3.38) $\quad C_0: \emptyset = S_0 \subsetneqq S_1 \subsetneqq \cdots \subsetneqq S_k = E$

in $\mathcal{D}_0$ each minor $(\mathcal{D},f)\cdot S_i/S_{i-1}$ $(i=1,2,\ldots,k)$ is connected.

(iii) Let C_0 be any maximal chain in $\mathcal{D}_0$ as in (3.38) and $\hat{b}$ be any base of $(\mathcal{D}_0, f_0)$ such that

(3.39) $\quad \hat{b}(S_i) = f_0(S_i) \qquad (i=1,2,\ldots,k).$

Then for any $X \in \mathcal{D}$ such that X is compatible with $\Pi(\mathcal{D}_0)$ and $\hat{b}(X) = f(X)$, we have $X \in \mathcal{D}_0$.

Proof: *"If" part.* Suppose (i) ~ (iii) hold. For each $i = 1, 2, \ldots, k$ let b_i^* be a base of $(\mathcal{D}_i, f_i) \equiv (\mathcal{D},f)\cdot S_i/S_{i-1}$ such that for any $X \in \mathcal{D}$ with $S_{i-1} \subsetneqq X \subsetneqq S_i$

(3.40) $\quad b_i^*(X - S_{i-1}) < f(X) - f(S_{i-1}) = f_i(X - S_{i-1}).$

Such a base b_i^* exists due to Theorem 3.8. Define $b^* \in R^E$ by

(3.41) $\quad b^*(e) = b_i^*(e)$

for each $e \in E$ with $e \in S_i - S_{i-1}$ and $i \in \{1,2,\ldots,k\}$. Then $b^* \in B(f)$.

We show $b^* \in F^\circ(\mathcal{D}_0)$, which will complete the proof of the "if" part. Suppose that there were $X_0 \in \mathcal{D}$ such that

(3.42) $b^*(X_0) = f(X_0)$

and for some $T \in \Pi(\mathcal{D}_0)$

(3.43) $X_0 \cap T \neq \emptyset, \quad T - X_0 \neq \emptyset.$

Suppose $T = S_{i_0} - S_{i_0-1}$. By the definition of b^*,

(3.44) $b^*(S_i) = f(S_i) \quad (i=1,2,\ldots,k).$

It follows from (3.42) and (3.44) with $i = i_0-1, i_0$ that for $Y_0 \equiv (X_0 \cup S_{i_0-1}) \cap S_{i_0}$ we have

(3.45) $b^*(Y_0) = f(Y_0),$

since $\mathcal{D}(b^*) = \{X \mid X \in \mathcal{D}, b^*(X) = f(X)\}$ is a sublattice of $\mathcal{D}$. However, because of Theorem 3.8 (3.45) contradicts that $(\mathcal{D}_{i_0}, f_{i_0})$ is connected, since $S_{i_0} \subsetneq Y_0 \subsetneq S_{i_0}$ from (3.43). Therefore, if $b^*(X_0) = f(X_0)$ for any $X_0 \in \mathcal{D}$, X_0 does not satisfy (3.43) for any $T \in \Pi(\mathcal{D}_0)$, i.e., X_0 is compatible with $\Pi(\mathcal{D}_0)$. Then, from (iii) we have $X_0 \in \mathcal{D}_0$. Consequently,

(3.46) $b^*(X) < f(X) \quad (X \in \mathcal{D} - \mathcal{D}_0).$

Furthermore, because of (i) and (3.44)

(3.47) $b^*(X) = f(X) \quad (X \in \mathcal{D}_0).$

It follows from (3.46) and (3.47) that $b^* \in F^\circ(\mathcal{D}_0)$.

"Only if" part. Suppose $\mathcal{D}_0 \in \mathbf{D}$. Then there exists a vector $b' \in F^\circ(\mathcal{D}_0)$ and (i) is immediate. Also (ii) follows from Theorem 3.8. We show (iii). Let $\hat{b}$ be any base of $(\mathcal{D}_0, f_0)$ such that (3.39) holds. Suppose that $X_0 \in \mathcal{D}$ is compatible with $\Pi(\mathcal{D}_0)$ and

(3.48) $\hat{b}(X_0) = f(X_0).$

Since $b' \in F^\circ(\mathcal{D}_0)$ also satisfies

(3.49) $b'(S_i) = f_0(S_i) \quad (i=1,2,\ldots,k),$

we have from (3.39) and (3.49)

(3.50) $\hat{b}(T) = b'(T) \quad (T \in \Pi(\mathcal{D}_0)).$

Since X_0 is compatible with $\Pi(\mathcal{D}_0)$, we have from (3.48) and (3.50)

(3.51) $b'(X_0) = \hat{b}(X_0) = f(X_0).$

Therefore, since $b' \in F^\circ(\mathcal{D}_0)$, we must have $X_0 \in \mathcal{D}_0$. Q.E.D.

From Theorems 3.4 and 3.13 we have the following

Corollary 3.14. Suppose that $B(f)$ is full-dimensional, i.e., $\dim B(f) = |E| - 1$ and let $\mathcal{D}_0$ be a sublattice of $\mathcal{D}$ with $\emptyset, E \in \mathcal{D}_0$. Then, $F(\mathcal{D}_0)$ is a facet of $B(f)$ if and only if $\mathcal{D}_0$ forms a chain $\emptyset = S_0 \subsetneq S_1 \subsetneq S_2 = E$ in $\mathcal{D}$ of length 2 and $(\mathcal{D}, f)\cdot S_i/S_{i-1}$ $(i=1,2)$ are connected.

We also have

Corollary 3.15. Let $\mathcal{D}_1$ be a sublattice of $\mathcal{D}$ with $\emptyset$, $E \in \mathcal{D}_1$ and suppose f is modular on $\mathcal{D}_1$. Also let

(3.52) $\quad C_1: \emptyset = S_0 \subsetneqq S_1 \subsetneqq \cdots \subsetneqq S_k = E$

be any maximal chain in $\mathcal{D}_1$ and for each $i = 1, 2, \ldots, k$ let $P_i = \{T_i^1, T_i^2, \ldots, T_i^{s_i}\}$ be a partition of $S_i - S_{i-1}$ given by the decomposition of $(\mathcal{D},f)\cdot S_i/S_{i-1}$ into connected components. Define $P^* = \cup_{i=1}^k P_i$ and let $\hat{b}$ be a base of $(\mathcal{D},f)$ such that

(3.53) $\quad \hat{b}(S_i) = f(S_i) \qquad (i=1,2,\ldots,k).$

Then $\mathcal{D}_2$ given by

(3.54) $\quad \mathcal{D}_2 = \{X \mid X \in \mathcal{D},\ \hat{b}(X) = f(X),\ X \text{ is compatible with } P^*\}$

belongs to $\mathbf{D}$ and satisfies

(3.55) $\quad F(\mathcal{D}_1) = F(\mathcal{D}_2).$

Proof: It is easy to see that $\mathcal{D}_2$ given by (3.54) satisfies (i) ~ (iii) of Theorem 3.13, and $\mathcal{D}_2 \in \mathbf{D}$. Moreover, for any $b' \in F(\mathcal{D}_1)$ we have

(3.56) $\quad \hat{b}(T_i^j) = b'(T_i^j)$

for each $j = 1, 2, \ldots, s_i$ and $i = 1, 2, \ldots, k$. Consequently,

(3.57) $\quad b'(X) = \hat{b}(X) = f(X)$

for any $X \in \mathcal{D}_2$. Q.E.D.

Note that the operation of getting $\mathcal{D}_2$ from $\mathcal{D}_1$ in Corollary 3.15 is a *closure* operation on sublattices of $\mathcal{D}$ containing $\emptyset$ and E, and $\mathbf{D}$ is the collection of *closed* sublattices of $\mathcal{D}$ containing $\emptyset$ and E.

It should also be noted that Corollary 3.15 is closely related to the concept of *f-skeleton* introduced in [13]. An f-skeleton is a sublattice of $\mathcal{D}$ on which f is modular. A sublattice $\mathcal{D}_1$ of $\mathcal{D}$ in Corollary 3.15 is an f-skeleton and there exists a unique maximal f-skeleton $\mathcal{D}_1{}^*$ (maximal relative to set inclusion) such that $\Pi(\mathcal{D}_1{}^*) = \Pi(\mathcal{D}_1)$, which is called the *maximal* f-skeleton corresponding to $\mathcal{D}_1$ (see [13]). We can easily see that $\mathcal{D}_1{}^*$ may not be closed and $\mathcal{D}_1{}^* \subseteq \mathcal{D}_2$, where $\mathcal{D}_2$ is the closure of $\mathcal{D}_1$ given by (3.54). In fact, $\mathcal{D}_1{}^*$ is the aggregation of $\mathcal{D}_2$ by $\Pi(\mathcal{D}_1)$. Therefore, $\mathcal{D}_2$ is a finer structure than the maximal f-skeleton $\mathcal{D}_1{}^*$. Corollary 3.15 gives a polyhedral interpretation of skeletons.

4. Remarks on Algorithmic Aspects and Related Topics

Based on Theorem 3.13 we can solve the following problem in polynomial time (with respect to $|E|$ using an oracle for function evaluation of f): Given a sublattice $\mathcal{D}_0$ of $\mathcal{D}$, decide whether $\mathcal{D}_0 \in \mathbf{D}$, where we assume that the corresponding posets $\mathcal{P}(\mathcal{D}_0)$ and $\mathcal{P}(\mathcal{D})$ are given in the form of the Hasse diagrams. (ii) in Theorem 3.13 is recognized by using an algorithm proposed in [3]. (i) and (iii) are recognized as follows. Consider the simplification $(\mathcal{D}_0', f_0')$ of $(\mathcal{D}_0, f_0)$ and find an extreme base b_0 of $(\mathcal{D}_0', f_0')$. Then (i) holds if and only if for all $e \in E' \equiv \Pi(\mathcal{D}_0)$

(4.1) $$b_0(e) = f_0'(D(e)) - f_0'(D(e) - \{e\}),$$

where $D(e)$ is the unique minimal element of $\mathcal{D}_0'$ which contains e. It is easy to check (4.1). (A similar method for recognizing a modular function when $\mathcal{D} = 2^E$ was also communicated by Tomizawa.) Let $(\mathcal{D}_1, f_1)$ be the aggregation of $(\mathcal{D}, f)$ by $\Pi(\mathcal{D}_0)$ and further let $(\mathcal{D}_1', f_1')$ be the simplification of $(\mathcal{D}_1, f_1)$, where $\mathcal{D}_0' \subseteq \mathcal{D}_1'$. Then (iii) holds if and only if

(4.2) $$\mathcal{D}_0' = \{X \mid X \in \mathcal{D}_1', \ b_0(X) = f_1'(X)\}.$$

Relation (4.2) is recognized by the use of the algorithm in [3] for determining the poset associated with $\mathcal{D}_0$.

In a similar way we can find in polynomial time in $|E|$ the sublattice $\mathcal{D}_2$, the closure of $\mathcal{D}_1$, in Corollary 3.15 for a given sublattice $\mathcal{D}_1$ of $\mathcal{D}$.

Given a fixed integer $p \geq 1$, we can decide, in polynomial time in $|E|$, whether a given vector $b \in R^E$ belongs to a face of $B(f)$ whose dimension is less than p. This follows from the fact that if $b \in B(f)$, then $\mathcal{D}(b)$ geven by (3.5) belongs to $\mathbf{D}$, $F(\mathcal{D}(b))$ is the minimal face of $B(f)$ which contains b, and if $\dim F(\mathcal{D}(b)) < p$, we have $|T| \leq p$ for each $T \in \Pi(\mathcal{D}(b))$. A polynomial algorithm is given by a direct adaptation of the algorithm in [3] for determining whether a vector in R^E is an extreme point of $B(f)$.

Based on Theorems 3.4 and 3.13 and Lemma 3.3, the incidence relation between extreme points and edges (one-dimensional faces) of $B(f)$ is given as follows:

> "Let b be an extreme point of $B(f)$ and K be an edge of $B(f)$. Suppose $\{b\} = F(\mathcal{D}_0)$ and $K = F(\mathcal{D}_1)$ for some $\mathcal{D}_0, \mathcal{D}_1 \in \mathbf{D}$. Then, $|\Pi(\mathcal{D}_0)| = |E|$ and $|\Pi(\mathcal{D}_1)| = |E| - 1$. Moreover, $b \in K$ if and only if $\mathcal{D}_1 \subseteq \mathcal{D}_0$ and for $T \in \Pi(\mathcal{D}_1)$ with $|T| = 2$ and $S_1, S_2 \in \Pi(\mathcal{D}_1)$ with $S_1 \subsetneq S_2$ and $T = S_2 - S_1$ $(\mathcal{D}, f) \cdot S_2 / S_1$ is connected."

In other words, $b \in K$ if and only if for $\{e, e'\} \in \Pi(\mathcal{D}_1)$ vertices e and e' in the Hasse diagram $H(\mathcal{P}(\mathcal{D}_0)) = (E, A^*(\mathcal{D}_0))$ is connected by an arc in $A^*(\mathcal{D}_0)$ and $H(\mathcal{P}(\mathcal{D}_1))$ is obtained by identifying e with e' in $H(\mathcal{P}(\mathcal{D}_0))$. The incidence relation between extreme points and edges of $B(f)$ can also be derived from Theorem 2.4 (cf. [19]). Consequently, the number of edges incident with a common extreme point is equal to the number of arcs of the Hasse diagram associated with the extreme point, which is bounded by

(4.3) $$|E|^2/4 \ \text{(if } |E| \text{ is even)}, \quad (|E|^2 - 1)/4 \ \text{(if } |E| \text{ is odd)}.$$

Recently, Topkis [21] has given a characterization of adjacency for extreme points of a polymatroid polytope (a polytope of independent vectors of a polymatroid). For a submodular system $(\mathcal{D},f)$ adjacency for extreme points of the base polyhedron $B(f)$ is characterized as follows, based on Theorems 3.4 and 3.13 and Lemma 3.3:

"Let b_i $(i=1,2)$ be distinct extreme points of $B(f)$ and suppose $\{b_i\} = F(\mathcal{D}_i)$ with $\mathcal{D}_i \in \mathbf{D}$ $(i=1,2)$. Then b_1 and b_2 are adjacent in $B(f)$ if and only if the following two statements hold:

(i) There exists a common sublattice $\mathcal{D}_3$ of $\mathcal{D}_1$ and $\mathcal{D}_2$ such that $\emptyset, E \in \mathcal{D}_3$ and $|\Pi(\mathcal{D}_3)| = |E| - 1$.

(ii) For $T \in \Pi(\mathcal{D}_3)$ with $|T| = 2$ and for $S_1, S_2 \in \mathcal{D}_3$ with $S_1 \subsetneq S_2$ and $T = S_2 - S_1$ $(\mathcal{D},f)\cdot S_2/S_1$ is connected."

In other words, b_1 and b_2 are adjacent if and only if there exist vertices e and e' in E connected by an arc in both $H(\mathcal{P}(\mathcal{D}_1))$ and $H(\mathcal{P}(\mathcal{D}_2))$ such that the Hasse diagrams obtained by identifying e with e' in $H(\mathcal{P}(\mathcal{D}_1))$ and $H(\mathcal{P}(\mathcal{D}_2))$, respectively, coincide with each other.

It was shown in [7] (and implicit in [5]) that any generalized polymatroid [5] is isomorphic with some base polyhedron of a submodular system which is obtained by increasing the cardinality of the ground set by one (also see [15]). Furthermore, any polymatroid polytope is a generalized polymatroid [5]. Therefore, the above characterization of adjacency for extreme points of $B(f)$ implies the result in [21]. It should also be noted that the number of adjacent extreme points of a given fixed extreme point of $B(f)$ is at most the number of edges incident with the given extreme point. Therefore, an upper bound of the number of adjacent extreme points of a given extreme point of a polymatroid polytope is given by (4.3) with $|E|$ replaced by $|E'| + 1$, where E' is the ground set of the polymatroid.

Acknowledgment: The author is grateful to one of the two referees for useful comments on the presentation of this paper.

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